Review of: "Nonlinearity and Illfoundedness in the Hierarchy of Large Cardinal Consistency Strength"

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In this pre-print Joel Hamkins critically examines what appears to be a wide-spread belief – the claim that it is indeed wide-spread is supported by a number of citations of the expressed views of various respected thinkers – that “natural theories” are linearly ordered or well-ordered by comparison of consistency strength. He seeks to challenge this by offering instances of non-linearity or ill-foundedness in the hierarchy of degrees of consistency strength of certain theories for which he tries to motivate the idea that they are “natural”. He does this after first reviewing the standard examples based on constructions that use the diagonal lemma which are generally considered not to be “natural”.

This sometimes involves looking at relations of comparison of consistency strength between theories which are extensionally the same but have a different intensional presentation (that is, we consider recursive axiomatizations of the theories where the actual extension of the axiom-set is the same in each case but the presentation of the algorithm for determining whether or not a given sentence is an axiom varies between the different axiomatizations). But there are also some examples offered where we get non-linearity in comparisons of consistency strength of different finitely axiomatizable extensions of PA, or or ZFC, where the issue of the distinction between extensions and intensions does not come up. There is also one section where the question is raised whether the empirical phenomenon whereby we frequently observe that pairs of natural theories are provably comparable or equal in consistency strength is an artifact of our methods for constructing “natural” theories or our repertoire of methods for establishing comparisons or equality of consistency strength, and it is asked whether our belief in the empirical support for the phenomenon of linearity and well-foundedness may be a product of “confirmation bias”.

I thought that the claim that the theories in question were indeed “natural” was stronger in the case where theories were considered where the intensional aspect was important, that is, in the section dealing with “cautious enumerations” of the ZFC axioms. For the section where examples were offered of the form, for example, “There are n inaccessible cardinals”, where n is given as the output of some computational process, I found myself having reservations about the extent to which these were “natural” given that the construction of the computational processes themselves seem to involve similar kinds of “tricks from mathematical logic” as with the more usual examples.

A claim is made early on in the piece that directedness or even just convergence in the hierarchy of consistency strength would be sufficient to vindicate the idea that the standard large cardinal axioms and possible future extensions of them are showing us “the one true path upward” as conceived by John Steel in a talk cited in the footnotes. I was unsure how to interpret this. John Steel's idea of “the one true path upward” seems to depend in an essential way on phenomena like the following: past the level of infinitely many Woodin cardinals, any two “natural theories” are comparable by the inclusion relation for their sets of consequences in second-order arithmetic, or similarly for their consequences for the theory of L(R) once we are past the level of infinitely many Woodin cardinals with a measurable cardinal above them. It is phenomena like these which for me are the basis of understanding the notion of “the one true path upward”, and I was unsure how directedness or ""convergence" (I was also unsure exactly what was intended by that) would be enough for this.

For one of the arguments it seemed as though a notation was being used whereby a colon with three dots rather than two after the function symbol f was somehow standing for the idea that f was a partial function with finite domain, at least that was the best I could do by way of making sense of what was intended there, and the argument seemed to come out okay on that interpretation. This wasn't a notation I had seen before so for me that slightly increased the difficulty of reading the argument.

Putting aside all such concerns, the arguments seemed to me to be all mathematically correct, and I thought that the main strength of the article was its success in putting pressure on the concept of “natural” that we try to invoke when talking about linearity or well-foundedness in the hierarchy of consistency strength for “natural” theories. I thought a good job was done here of doing a review of possible candidates for examples where we might want to say we can observe non-linearity or ill-foundedness even for “natural” theories, and good motivation was provided for what seemed to be the key philosophical thesis that we should try to replace vague talk about “naturality” with more precisely formulated notions, to better give content to any claims we want to make about linearity or well-foundedness. I definitely was quite interested in the idea that our feeling that we are observing linearity or well-foundedness amongst the “natural” theories we have encountered may perhaps in some way be a product of confirmation bias arising from the nature of the tools we are using. I suppose it is hard to say too much more about this without embarking on a project of trying to give more precise content to the notion of “natural large cardinal axiom”. I only have very vague ideas about how that might be done at present. One could speculate that if the Ultimate-L Conjecture were to be confirmed, then the fact of the standard order relation of the Borel degrees of universally Baire sets being a well-ordering might have some relevance here.

This article was pleasant to read, provided a good overview of some interesting results, and provided good motivation for closer scrutiny of our notion of “natural extension of ZFC”.