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Geodesics as Equations of Motion

T. Aaberge

Abstract The empirical description of the evolution of a physical system is the account of observed changes of state of the system in time, the measure of time being that of the Systeme International. To describe the Sun-Mercury system Einstein proposed a model based on the assumption that the evolution of the system, i.e. its relative space-time trajectory describes a geodesic on space-time endowed with the Schwartzschild metric. The evolution parameter of a geodesic is, however, an affine parameter or equivalently the proper time. What this could mean is the subject of this paper.

Keywords Sun-Mercury · Geodesic · SI time

1 Introduction

Mathematics distinguish between abstract structure and local coordinates mirroring the structure without assigning any particular meaning to the coordinates. In physics, however, coordinates acquire a meaning through the operational definitions applied to measure the coordinates. The Systeme International provides a system of units of measurement the meanings of which, directly or indirectly, are based on a coherent set of operational definitions. To ascertain the correspondence between the predictions of a model defined in a physical theory which comprises the physical constants (mass, charge etc.) that serve to identify the system, and the behavior of the system, the operational definitions must be compatible with the theory, i.e. there must be a one-to-one relation between the coordinates measured by applying the operational definitions and the choice of coordinates emanating from the mathematical structure of the theory used in the computations of predictions of models.

The compatibility between the SI operational definitions of spatial distance and temporal duration for the space-time coordinates of the theory of general relativity is a priori not obvious. It seems, however, that it is the time coordinate t that is measured using the SI operational definitions. In fact, with respect to the interpretation of the cosmic redshift and the gravitational displacement of spectral lines the time coordinate is taken to be the SI time.

Terje Aaberge
PO Box 216, N-6852 Sogndal, Norway
E-mail: terje@aaberge.no

A conceptual problem appears, however, with the introduction of the additional hypothesis that the motion of a material body, described as a point particle, is a geodesic in space-time. The evolution parameter is then an affine parameter, i.e. a parameter linearly related to the proper time s which itself is an affine parameter. Any affine time parameter differs from the coordinate SI time t. This is the problem discussed in the following by considering the description of the Sun-Mercury system and its Newtonian approximation.

2 The Sun-Mercury System

The model pictures the Mercury as a body moving along a geodesic in the space-time endowed with the Schwarzschild metric, a solution of the Einstein equation for empty space, interpreted as the gravitational field produced by the Sun. A comparison with the empirical data led Einstein to the conclusion that the model describes the precession of the perihelion of Mercury. The model is defined by [1]

$$L = c^2\left(1-\frac{2\mu}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - r^2\left(\left(\frac{d\theta}{ds}\right)^2 + sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right) \quad (1)$$

with the constraint

$$L = c^2 \quad (2)$$

From the Euler-Lagrange equations it follows that the value of $\theta = \frac{\pi}{2}$ and L is a constant of motion. we are then left with the following equations

$$\left(1-\frac{2\mu}{r}\right)\frac{dt}{ds} = k \quad (3)$$

$$c^2\left(1-\frac{2\mu}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\phi}{ds}\right)^2 = c^2 \quad (4)$$

$$r^2\frac{d\phi}{ds} = h \quad (5)$$

where k and h are constants. From these equations we get

$$\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right) - \frac{2\mu}{c^2 r} = c^2\left(k^2-1\right) \quad (6)$$

$$r^2\frac{d\phi}{ds} = h \quad (7)$$

or alternatively, using that $\frac{d}{ds} = k\left(1-\frac{2\mu}{r}\right)^{-1}\frac{d}{dt}$,

$$k^2\left(1-\frac{2\mu}{r}\right)^{-2}\left(\frac{dr}{dt}\right)^2 + \frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right)^{-1} - \frac{2\mu}{c^2 r} = c^2\left(k^2-1\right) \quad (8)$$

$$k\left(1-\frac{2\mu}{r}\right)^{-1} r^2\frac{d\phi}{dt} = h \quad (9)$$

where eqs 6 and 7 are equations of motion in the proper time s, and eqs 8 and 9 are equations of motion in the coordinate time t. By using that $\frac{d}{ds} = \frac{h}{r^2}\frac{d}{d\phi}$ and $\frac{d}{dt} = \frac{h}{kr^2}\left(1-\frac{2\mu}{r}\right)\frac{d}{d\phi}$ we get in both cases

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{c^2}{h^2}\left(k^2-1\right) + \frac{2\mu u}{h^2} + \frac{2\mu u^3}{c^2} \quad (10)$$

where $u = \dfrac{1}{r}$, or by differentiating by ϕ we get the well-known equation

$$\frac{d^2u}{d\phi^2} + u = c^2\frac{\mu}{h^2} + 3\mu u^2 \tag{11}$$

When $\mu = \frac{GM}{c^2}$ where G is the gravitational constant and M the mass of the Sun, this equation describes the motion relative to the Sun of any planet in the solar system, including the residual perihelion precession, i.e. the precession not caused by the gravitational perturbations from the other planets. The residual perihelion precession is associated with the last term of eq. 11.

3 The Motions

The solution of eq. 11 gives the relative distance r between the Sun and a planet as a function of the azimuthal angle ϕ. This relation is, however, a secondary result of the observation of the relative distance and azimuthal angle over long periods of time which describe the relative motions in space-time parametrized by the coordinate SI time t. It is a spatial projection of the space-time trajectory. Let $\tilde{\gamma} : s \mapsto \tilde{\gamma}(s)$ and $\gamma : t \mapsto \gamma(t)$ be the solutions of eqs 6 and 7, and eqs 8 and 9, respectively. It is well-known that the curve in space-time traced by $\tilde{\gamma}(s)$ if s is identified with the SI time approximates well the empirical curve; however, since it follows from the eqs $3-5$ that $s(t)$ is a non-trivial function of t this contradicts the hypothesis that it is the coordinate time t that should be identified with the SI time. The question is then if the solution $\gamma(t) = \tilde{\gamma}(s(t))$ traces the empirical curve; if not, we can conclude that Einstein's hypothesis that the motions of material bodies are geodesics in space-time is untenable.

To answer this question it is sufficient to consider the Newtonian approximation of the equations of motion and to investigate whether the the curve $\gamma_N(t)$ corresponds to the empirical Newtonian curve. The Newtonian equations are obtained by replacing $\left(1 - \frac{2GM}{c^2r}\right)$ by 1 which is a good approximation since $\frac{GM}{c^2r} \approx 2,6 \times 10^{-8}$ for Mercury. In fact, the relative motion of any planet can be decomposed to a motion in an elliptical path and the precession of the perihelion of the ellipse. For example, in hundred years Mercury makes about 415 turns while the perihelion is precessing 574". Most of the precession is caused by perturbations from the other planets, the residual precession being 44". The measure of time for these empirical results being the SI time measure. From equations $6-9$ we then get

$$\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2} - \frac{2\mu}{c^2r} = E \tag{12}$$

$$r^2\frac{d\phi}{ds} = h \tag{13}$$

$$\frac{1}{2}m\left(1 + \frac{E}{mc^2}\right)\left(\frac{dr}{dt}\right)^2 + \frac{h^2}{2mr^2} - \frac{GMm}{r} = E \tag{14}$$

$$\sqrt{1 + \frac{E}{mc^2}}\,\frac{d\phi}{dt} = \frac{h}{r^2} \tag{15}$$

with the choice $E = \frac{1}{2}mc^2\left(k^2-1\right)$ for the Newtonian energy and where m the mass of Mercury, and $k = \sqrt{1+\frac{2E}{mc^2}}$. By choosing $t' = \left(1+\frac{E}{mc^2}\right)^{-1/2} t$ we get

$$\frac{1}{2}m\left(\frac{dr}{dt'}\right)^2 + \frac{h^2}{2mr^2} - \frac{GMm}{r} = E \tag{16}$$

$$\frac{d\phi}{dt'} = \frac{h}{r^2} \tag{17}$$

However, since $t' \neq s$ either the eqs 12 and 13, or eqs 14 and 15 must be taken to describe the evolution of the system in space-time. To choose s contradicts the empirical results which are recorded in the coordinate SI time. The choice t', on the other hand, is related to the coordinate time and we are left to estimate relation between them.

The excentricity of the path is $\varepsilon = \sqrt{1+\frac{2El^2}{(GMm)^2 m}}$ [2]; thus,

$$\frac{E}{mc^2} = -\frac{(GMm)^2}{2l^2c^2}\left(1-\varepsilon^2\right) = -\frac{(GM)^2}{2h^2c^2}\left(1-\varepsilon^2\right) = -\frac{1}{2}\left(\frac{GM}{c^2r}\right)^2\frac{c^2}{v^2}\left(1-\varepsilon^2\right) \tag{18}$$

$$\approx -1,33\times 10^{-8} \tag{19}$$

since $l = mh \approx mrv$ and the average orbital speed of Mercury $v = 47\times 10^3 m/s$ and

$$\sqrt{1+\frac{E}{mc^2}} \approx 1 \tag{20}$$

We can therefore conclude that of the eqs 14 and 15 describe the correct the Newtonian motion of Mercury around the Sun.

Alternatively, we can choose $E = \frac{1}{2}m\frac{c^2}{k^2}\left(k^2-1\right)$ which gives $k = \left(1-\frac{E}{mc^2}\right)^{-1/2}$. Then, for $t'' = \left(1-\frac{E}{mc^2}\right)^{1/2} t$ we get the equations

$$\frac{1}{2}m\left(\frac{dr}{dt''}\right)^2 + \frac{h^2}{2mr^2} - \frac{GMm}{r} = \frac{E}{1-\frac{E}{mc^2}} \tag{21}$$

$$\frac{d\phi}{dt''} = \frac{h}{r^2} \tag{22}$$

for which the eccentricity of the path is $\varepsilon = \sqrt{1+\frac{2El^2}{\left(1-\frac{E}{mc^2}\right)(GMm)^2 m}}$ or

$$\sqrt{1-\frac{E}{mc^2}} = \frac{1}{\sqrt{1-\frac{1}{2}\left(\frac{GM}{c^2r}\right)^2\frac{c^2}{v^2}\left(1-\varepsilon^2\right)}} \approx 1 \tag{23}$$

Thus, t'' can be identified with t in the eqs 19 and 20.

4 Conclusion

$$\left(\frac{dr}{dt}\right)^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{2\mu}{c^2 r} = c^2\left(k^2 - 1\right) \tag{24}$$

$$r^2 \frac{d\phi}{dt} = h \tag{25}$$

and

$$k^2\left(1 - \frac{2\mu}{r}\right)^{-2}\left(\frac{dr}{dt}\right)^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right)^{-1} - \frac{2\mu}{c^2 r} = c^2\left(k^2 - 1\right) \tag{26}$$

$$k\left(1 - \frac{2\mu}{r}\right)^{-1} r^2 \frac{d\phi}{dt} = h \tag{27}$$

where eq. 24 is the eq. 6 with the proper time s is replaced with the coordinate time t. I have shown that though these equations gives different space-time curves, for the Sun-Mercury system they cannot be distinguished. For this system the proper time s can therefore be identified with the coordinate time t itself being identified with the SI time. This conclusion depends on the empirical numbers available to and can be refuted by more refined empirical numbers.

References

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