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Geodesics as Equations of Motion

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Abstract The empirical description of the evolution of a physical system is the account of observed changes of state of the system in time, the measure of time being that of the Systeme International. To describe the Sun-Mercury system Einstein proposed a model based on the assumption that the evolution of the system, i.e. its relative space-time trajectory describes a geodesic on space-time endowed with the Schwarzschild metric. The evolution parameter of a geodesic is, however, an affine parameter or equivalently the proper time. However, the geodesic equation can be expressed in coordinate time. It has been shown that the solution of this equation reproduces the empirical evolution. A possiblee interptetation of the proper time is presented.

Keywords Sun-Mercury · Geodesic · SI time

1 Introduction

Mathematics distinguish between abstract structure and local coordinates mirroring the structure without assigning any particular meaning to the coordinates. In physics, however, coordinates acquire a meaning through the operational definitions applied to measure the coordinates. The Systeme International provides a system of units of measurement the meanings of which, directly or indirectly, are based on a coherent set of operational definitions. To ascertain the correspondence between the predictions of a model defined in a physical theory which comprises the physical constants (mass, charge etc.) that serve to identify the system, and the behavior of the system, the operational definitions must be compatible with the theory, i.e. there must be a one-to-one relation between the coordinates measured by applying the operational definitions and the choice of coordinates used when computing the predictions of a model.

Since 1963 the measurement of temporal duration and spatial distance is done by atomics clocks by taking the inverse of the fixed numerical value of the hyperfine transition frequency $\Delta\nu_{C_s}$ of caesium as the basic unit of time. A distance is measured by the time Δt

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a light ray needs to covered by the distance multiplied by the velocity c of light. Both of these definitions are referred to and are valid in any inertial systems of reference. In special relativity time t measured by a clock is the by definition proper time is of the clock for an observer in the same inertial frame of reference, though when he looks at a clock moving with respect to himself, he observes a time delation. In general relativity there are no inertial frames of reference, however, the time t measured by a clock is by definition the proper time of the clock for an observer sitting with the clock, but when he looks at other clocks he will observe time delations that are not only due to the relative motions between the two clocks, but also due to gravitational effects. An example is the gravitational time dilation between clocks on the GPS satellites and the clocks on the Earth which has to be taken into account for the system to work properly. The cosmic redshift and the gravitational displacement of spectral lines the time coordinate is another example. The time referred to is the SI time which is used as the coordinate time. The SI operational of time is therefore compatible with general relativity, i.e. the time t measured by an observer can be identified with the coordinate time he uses to compute predictions of a model formulated in general relativity.

With the introduction of the additional hypothesis to general relativity that the motion of a material body, described as a point particle, is a geodesic in space-time the evolution parameter is an affine parameter, i.e. a parameter linearly related to the proper time s which itself is an affine parameter. This hypothesis has been used to formulate a model of the Sun-Mercury system assuming that relative trajectory of Mercury follows a geodesic in the Schwarzschild space-time determined by the Sun. The empirical trajectory of Mercury has been recorded over centuries in SI time. Now any affine time parameter differs from the coordinate SI time t, but by assuming that s can be identified with t there is a high degree of accordance between the observed and predicted trajectories. This is the problem discussed in the following by considering the description of the Sun-Mercury system and its Newtonian approximation.

2 The Sun-Mercury System

The model pictures the Mercury as a body moving along a geodesic in the space-time endowed with the Schwarzschild metric, a solution of the Einstein equation for empty space, interpreted as the gravitational field produced by the Sun. A comparison with the empirical data led Einstein to the conclusion that the model describes the precession of the perihelion of Mercury. The model is defined by [1]

$$L = c^2\left(1-\frac{2\mu}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - r^2\left(\left(\frac{d\theta}{ds}\right)^2 + sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right) \quad (1)$$

with the constraint

$$L = c^2 \quad (2)$$

From the Euler-Lagrange equations it follows that the value of $\theta = \frac{\pi}{2}$ and L is a constant of motion. we are then left with the following equations

$$\left(1-\frac{2\mu}{r}\right)\frac{dt}{ds} = k \quad (3)$$

$$c^2\left(1-\frac{2\mu}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\phi}{ds}\right)^2 = c^2 \quad (4)$$

$$r^2\frac{d\phi}{ds} = h \quad (5)$$

where k and h are constants. From these equations we get

$$\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right) - \frac{2\mu}{c^2 r} = c^2\left(k^2-1\right) \quad (6)$$

$$r^2\frac{d\phi}{ds} = h \quad (7)$$

or alternatively, using that $\frac{d}{ds} = k\left(1-\frac{2\mu}{r}\right)^{-1}\frac{d}{dt}$,

$$k^2\left(1-\frac{2\mu}{r}\right)^{-2}\left(\frac{dr}{dt}\right)^2 + \frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right)^{-1} - \frac{2\mu}{c^2 r} = c^2\left(k^2-1\right) \quad (8)$$

$$k\left(1-\frac{2\mu}{r}\right)^{-1} r^2\frac{d\phi}{dt} = h \quad (9)$$

where eqs 6 and 7 are equations of motion in the proper time s, and eqs 8 and 9 are equations of motion in the coordinate time t. By using that $\frac{d}{ds} = \frac{h}{r^2}\frac{d}{d\phi}$ and $\frac{d}{dt} = \frac{h}{kr^2}\left(1-\frac{2\mu}{r}\right)\frac{d}{d\phi}$ we get in both cases

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{c^2}{h^2}\left(k^2-1\right) + \frac{2\mu u}{h^2} + \frac{2\mu u^3}{c^2} \quad (10)$$

where $u = \dfrac{1}{r}$, or by differentiating by ϕ we get the well-known equation

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{h^2} + \frac{3\mu}{c^2}u^2 \quad (11)$$

When $\mu = \frac{GM}{c^2}$ where G is the gravitational constant and M the mass of the Sun, this equation describes the motion relative to the Sun of any planet in the solar system, including the residual perihelion precession, i.e. the precession not caused by the gravitational perturbations from the other planets. The residual perihelion precession is associated with the last term of eq. 11.

3 The Motions

The solution of eq. 11 gives the relative distance r between the Sun and a planet as a function of the azimuthal angle ϕ. This relation is, however, a secondary result of the observation of the relative distance and azimuthal angle over long periods of time. Thus, the orbit of Mercury has a period of 88 days and in hundred years Mercury makes about 415 turns while the perihelion is precessing 574". Most of the precession is caused by perturbations from the other planets, the residual precession being 43". the measure of time for these empirical results being the SI time measure. $\tilde{\gamma} : s \mapsto \tilde{\gamma}(s)$ and $\gamma : t \mapsto \gamma(t)$ be the solutions of eqs 6 and 7, and eqs 8 and 9, respectively and let $s(t)$ be a solution of eqs 3-5, then $\gamma(t) = \tilde{\gamma}(s(t))$. It is well-known that the curve in space-time traced by $\tilde{\gamma}(s)$ if s is identified with the SI time t approximates well the empirical curve. The question is then if the solution $\gamma(t)$ traces the empirical curve; if not, we can conclude that Einstein's model of the Sun-Mercury system is untenable.

To answer this question, it is sufficient to consider the Newtonian approximation of the equations of motion and to investigate whether the curve $\gamma_N(t)$ corresponds to the empirical Newtonian curve. The Newtonian equations are obtained by replacing $\left(1-\frac{2GM}{c^2 r}\right)$ by 1 which is a good approximation since $\frac{GM}{c^2 r} \approx 2,6 \times 10^{-8}$ for Mercury. In fact, the relative motion of any planet can be decomposed to a motion in an elliptical path and the precession of the perihelion of the ellipse. From equations $6-9$ we then get

$$\frac{1}{2}m\left(1+\frac{E}{mc^2}\right)\left(\frac{dr}{dt}\right)^2 + \frac{h^2}{2mr^2} - \frac{GMm}{r} = E \tag{12}$$

$$\sqrt{1+\frac{E}{mc^2}}\frac{d\phi}{dt} = \frac{h}{r^2} \tag{13}$$

with the choice $E = \frac{1}{2}mc^2\left(k^2-1\right)$ for the Newtonian energy and where m the mass of Mercury, and $k = \sqrt{1+\frac{2E}{mc^2}}$. By choosing $t' = \left(1+\frac{E}{mc^2}\right)^{-1/2} t$ we get

$$\frac{1}{2}m\left(\frac{dr}{dt'}\right)^2 + \frac{h^2}{2mr^2} - \frac{GMm}{r} = E \tag{14}$$

$$\frac{d\phi}{dt'} = \frac{h}{r^2} \tag{15}$$

The excentricity of the path is $\varepsilon = \sqrt{1+\frac{2El^2}{(GMm)^2 m}}$ [2]; thus,

$$\frac{E}{mc^2} = -\frac{(GMm)^2}{2l^2c^2}\left(1-\varepsilon^2\right) = -\frac{(GM)^2}{2h^2c^2}\left(1-\varepsilon^2\right) = -\frac{1}{2}\left(\frac{GM}{c^2 r}\right)^2\frac{c^2}{v^2}\left(1-\varepsilon^2\right) \tag{16}$$

$$\approx -1,33 \times 10^{-8} \tag{17}$$

since $l = mh \approx mrv$ and the average orbital speed of Mercury $v = 47 \times 10^3 m/s$ and

$$\sqrt{1+\frac{E}{mc^2}} \approx 1 \tag{18}$$

Thus, $t' \approx 1$ and we can therefore conclude that of the eqs 12 and 13 describe the correct the Newtonian motion of Mercury around the Sun.

Alternatively, we can choose $E = \frac{1}{2}m\frac{c^2}{k^2}\left(k^2-1\right)$ which gives $k = \left(1-\frac{E}{mc^2}\right)^{-1/2}$. Then, for $t'' = \left(1-\frac{E}{mc^2}\right)^{1/2} t$ we get the equations

$$\frac{1}{2}m\left(\frac{dr}{dt''}\right)^2 + \frac{h^2}{2mr^2} - \frac{GMm}{r} = \frac{E}{1-\frac{E}{mc^2}} \tag{19}$$

$$\frac{d\phi}{dt''} = \frac{h}{r^2} \tag{20}$$

for which the eccentricity of the path is $\varepsilon = \sqrt{1+\frac{2El^2}{\left(1-\frac{E}{mc^2}\right)(GMm)^2 m}}$ or

$$\sqrt{1-\frac{E}{mc^2}} = \frac{1}{\sqrt{1-\frac{1}{2}\left(\frac{GM}{c^2 r}\right)^2\frac{c^2}{v^2}\left(1-\varepsilon^2\right)}} \approx 1 \tag{21}$$

Thus, t'' can be identified with t in the eqs 19 and 20.

4 Conclusion

I have shown that $\gamma(t) = \tilde{\gamma}(s(t))$ reproduces the empirical space-time curve of the relative motion of Sun and Mercury to a high degree of accuracy, thus, justifying Einstein's model of the system. Though it is reasonable to assume that $\tilde{\gamma}(s(t)) \approx t$ to a high degree of accuracy, it is still left to show. In any case, it is difficult to maintain Einstein's hypothesis that material bodies follow a geodesic is generally valid.

In special relativity the world lines of all inertial frames of reference relative to a given observer is of the form

$$\left(\frac{c}{\sqrt{(1-v^2/c^2)}}, \frac{v^i}{\sqrt{(1-v^2/c^2)}}\right) t + \left(x^0, x^i\right) \tag{22}$$

and the proper time of all of them are satisfying $ds = dt$, i.e. if all clocks are correlated then $s = t$. In a general relativistic setting we must assume that the clocks for all observers are correlated and show the SI time, as long as the C_s atoms are not affected by gravitational fields, however, we cannot assume that the SI time is their proper time. In the particular case of the Mercury system we may therefore assume that observer on Mercury reads off the SI time on his clock, but that the proper time $s(t)$ is the time other observers read off his clock.

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