

## Geodesics as Equations of Motion

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**Abstract** The empirical description of the evolution of a physical system is the account of observed changes of state of the system in time, the measure of time being that of the Systeme International. To describe the Sun-Mercury system Einstein proposed a model based on the assumption that the evolution of the system, i.e. its relative space-time trajectory describes a geodesic on space-time endowed with the Schwarzschild metric. The evolution parameter of a geodesic is, however, an affine parameter or equivalently the proper time. What this could mean is the subject of this paper.

**Keywords** Sun-Mercury · Geodesic · SI time

### 1 Introduction

Mathematics distinguish between abstract structure and local coordinates mirroring the structure without assigning any particular meaning to the coordinates. In physics, however, coordinates acquire a meaning through the operational definitions applied to measure the coordinates. The Systeme International provides a system of units of measurement the meanings of which, directly or indirectly, are based on a coherent set of operational definitions. To ascertain the correspondence between the predictions of a model defined in a physical theory which comprises the physical constants (mass, charge etc.) that serve to identify the system, and the behavior of the system, the operational definitions must be compatible with the theory, i.e. there must be a one-to-one relation between the coordinates measured by applying the operational definitions and the choice of coordinates emanating from the mathematical structure of the theory used in the computations of predictions of models.

The compatibility between the SI operational definitions of spatial distance and temporal duration for the space-time coordinates of the theory of general relativity is a priori not obvious though it seems that the time coordinate  $t$  is measured using the SI operational definitions. In fact, with respect to the interpretation of the cosmic redshift and the gravitational displacement of spectral lines the time coordinate is taken to be the SI time. A conceptual

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problem appears, however, with the introduction of the additional hypothesis that the motion of a material body, described as a point particle, is a geodesic in space-time. The evolution parameter is then an affine parameter, i.e. a parameter linearly related to the proper time  $s$  which itself is an affine parameter. Any affine time parameter differs from the coordinate SI time  $t$ . This is the problem discussed in the following by considering the description of the Sun-Mercury system and its Newtonian approximation.

## 2 The Sun-Mercury System

The model pictures the Mercury as a body moving along a geodesic in the space-time endowed with the Schwarzschild metric, a solution of the Einstein equation for empty space, interpreted as the gravitational field produced by the Sun. A comparison with the empirical data led Einstein to the conclusion that the model describes the precession of the perihelion of Mercury. The model is defined by [1]

$$L = c^2 \left(1 - \frac{2\mu}{r}\right) \left(\frac{dt}{ds}\right)^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \left(\frac{dr}{ds}\right)^2 - r^2 \left( \left(\frac{d\theta}{ds}\right)^2 + \sin^2 \theta \left(\frac{d\phi}{ds}\right)^2 \right) \quad (1)$$

with the constraint

$$L = c^2 \quad (2)$$

From the Euler-Lagrange equations it follows that the value of  $\theta = \frac{\pi}{2}$  and  $L$  is a constant of motion. we are then left with the following equations

$$\left(1 - \frac{2\mu}{r}\right) \frac{dt}{ds} = k \quad (3)$$

$$c^2 \left(1 - \frac{2\mu}{r}\right) \left(\frac{dt}{ds}\right)^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \left(\frac{dr}{ds}\right)^2 - r^2 \left(\frac{d\phi}{ds}\right)^2 = c^2 \quad (4)$$

$$r^2 \frac{d\phi}{ds} = h \quad (5)$$

where  $k$  and  $h$  are constants. From these equations we get

$$\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2} \left(1 - \frac{2\mu}{r}\right) - \frac{2\mu}{c^2 r} = c^2 (k^2 - 1) \quad (6)$$

$$r^2 \frac{d\phi}{ds} = h \quad (7)$$

or alternatively, using that  $\frac{d}{ds} = k \left(1 - \frac{2\mu}{r}\right)^{-1} \frac{d}{dt}$ ,

$$k^2 \left(1 - \frac{2\mu}{r}\right)^{-2} \left(\frac{dr}{dt}\right)^2 + \frac{h^2}{r^2} \left(1 - \frac{2\mu}{r}\right)^{-1} - \frac{2\mu}{c^2 r} = c^2 (k^2 - 1) \quad (8)$$

$$k \left(1 - \frac{2\mu}{r}\right)^{-1} r^2 \frac{d\phi}{dt} = h \quad (9)$$

where eqs 6 and 7 are equations of motion in the proper time  $s$ , and eqs 8 and 9 are equations of motion in the coordinate time  $t$ . By using that  $\frac{d}{ds} = \frac{h}{r^2} \frac{d}{d\phi}$  and  $\frac{d}{dt} = \frac{h}{kr^2} \left(1 - \frac{2\mu}{r}\right) \frac{d}{d\phi}$  we get in both cases

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{c^2}{h^2} (k^2 - 1) + \frac{2\mu u}{h^2} + \frac{2\mu u^3}{c^2} \quad (10)$$

where  $u = \frac{1}{r}$ , or by differentiating by  $\phi$  we get the well-known equation

$$\frac{d^2 u}{d\phi^2} + u = c^2 \frac{\mu}{h^2} + 3\mu u^2 \quad (11)$$

When  $\mu = \frac{GM}{c^2}$  where  $G$  is the gravitational constant and  $M$  the mass of the Sun, this equation describes the motion relative to the Sun of any planet in the solar system, including the residual perihelion precession, i.e. the precession not caused by the gravitational perturbations from the other planets. The residual perihelion precession is associated with the last term of eq. 11.

### 3 The Motions

The solution of eq 11 gives the relative distance  $r$  between the the Sun and a planet as a function of the azimuthal angle  $\phi$ . This relation is, however, a secondary result of the observation of the relative distance and azimuthal angle over long periods of time which describe the relative motions in space-time parametrized by the coordinate SI time  $t$ . It is a spatial projection of the space-time trajectory. Let  $\tilde{\gamma} : s \mapsto \tilde{\gamma}(s)$  and  $\gamma : t \mapsto \gamma(t)$  be the solutions of eqs 6 and 7, and eqs 8 and 9, respectively. Though they both gives the same spatial projection, they correspond to different motions in space-time. It is well-known that the curve in space-time traced by  $\tilde{\gamma}(t)$  approximates well the empirical curve; however, this choice is not admitted since it follows from the eqs 3 – 5 that  $s(t)$  is a non-trivial function of  $t$ . The question is then if the solution  $\gamma(t) = \tilde{\gamma}(s(t))$  traces the empirical curve; if not, we can conclude that Einstein's hypothesis that the motions of material bodies are geodesics in space-time is untenable.

The answer to this question it is sufficient to consider the Newtonian approximation of the equations of motion and to investigate whether the the curve  $\gamma_N(t)$  corresponds to the empirical Newtonian curve. The Newtonian equations are obtained by replacing  $\left(1 - \frac{2GM}{c^2 r}\right)$  by 1 which is a good approximation since  $\frac{GM}{c^2 r} \approx 26 \times 10^{-6}$  for Mercury. In fact, the relative motion of any planet can be decomposed to a motion in an elliptical path and the precession of the perihelion of the ellipse. For example, in hundred years Mercury makes about 415 turns while the perihelion is precessing 574". Most of the precession is caused by perturbations from the other planets, the residual precession being 44". The measure of time for these empirical results being the SI time measure. From equations 6 – 9 we then get

$$\frac{1}{2}m \left(1 + \frac{E}{mc^2}\right) \left(\frac{dr}{dt}\right)^2 + \frac{h^2}{2mr^2} - \frac{GMm}{r} = E \quad (12)$$

$$\sqrt{1 + \frac{E}{mc^2}} \frac{d\phi}{dt} = \frac{h}{r^2} \quad (13)$$

with the choice  $E = \frac{1}{2}mc^2(k^2 - 1)$  for the Newtonian energy and where  $m$  the mass of Mercury, and  $k = \sqrt{1 + \frac{2E}{mc^2}}$ . By choosing  $t' \left(1 + \frac{E}{mc^2}\right)^{-1/2} t$  we get

$$\frac{1}{2}m \left(\frac{dr}{dt'}\right)^2 + \frac{h^2}{2mr^2} - \frac{GMm}{r} = E \quad (14)$$

$$\frac{d\phi}{dt'} = \frac{h}{r^2} \quad (15)$$

However, since  $t' \neq s$  either the eqs 12 and 13, or eqs 14 and 15 must be taken to describe the evolution of the system in space-time. To choose  $s$  contradicts the empirical results which are recorded in the coordinate SI time. The choice  $t'$ , on the other hand, is related to the coordinate time and we are left to estimate relation between them.

The excentricity of the path is  $\varepsilon = \sqrt{1 + \frac{2El^2}{(GMm)^2 m}}$  [2]; thus,

$$\frac{E}{mc^2} = -\frac{(GMm)^2}{2l^2 c^2} (1 - \varepsilon^2) = -\frac{(GM)^2}{2h^2 c^2} (1 - \varepsilon^2) = -\frac{1}{2} \left( \frac{GM}{c^2 r} \right)^2 \frac{c^2}{v^2} (1 - \varepsilon^2) \quad (16)$$

$$\approx -1,33 \times 10^{-2} \quad (17)$$

since  $l = mh \approx mrv$  and the average orbital speed of Mercury  $v = 47 \times 10^3 m/s$  and

$$\sqrt{1 + \frac{E}{mc^2}} \approx 0,99335 \quad (18)$$

The orbital period of Mercury is 87,9691 days as measured in SI time  $t$ . Since  $\sqrt{1 + \frac{2E}{mc^2}} = 0,99335$  the orbital period predicted eqs 12 and 13 (or 14 and 15) corresponds to 87,3811 in  $t'$ -days. We can therefore conclude that of the eqs 14 and 15 do not describe the correct the Newtonian motion of Mercury around the Sun.

Alternatively, we can choose  $E = \frac{1}{2} m \frac{c^2}{k^2} (k^2 - 1)$  which gives  $k = \left(1 - \frac{E}{mc^2}\right)^{-1/2}$ . Then, for  $t'' = \left(1 - \frac{E}{mc^2}\right)^{1/2} t$  we get the equations

$$\frac{1}{2} m \left( \frac{dr}{dt''} \right)^2 + \frac{h^2}{2mr^2} - \frac{GMm}{r} = \frac{E}{1 - \frac{E}{mc^2}} \quad (19)$$

$$\frac{d\phi}{dt''} = \frac{h}{r^2} \quad (20)$$

for which the eccentricity of the path is  $\varepsilon = \sqrt{1 + \frac{2El^2}{\left(1 - \frac{E}{mc^2}\right) (GMm)^2 m}}$  or

$$\sqrt{1 - \frac{E}{mc^2}} = \frac{1}{\sqrt{1 - \frac{1}{2} \left( \frac{GM}{c^2 r} \right)^2 \frac{c^2}{v^2} (1 - \varepsilon^2)}} \approx \frac{1}{0,9935} \approx 1,0067 \quad (21)$$

Thus, the orbital period is 88,5550  $t''$ -days for the solution of the eqs 19 and 20.

We can therefore conclude that neither of the eqs 14 and 15 nor eqs 19 and 20 describe the correct the Newtonian motion of Mercury around the Sun.

#### 4 Conclusion

The discussion in sect. 3 shows that only the equations 6 and 7 gives the correct space-time motion provided we interpret  $s$  as the SI measure of time, thus, that  $\tilde{\gamma}(t)$  corresponds to the empirical space-time curve; moreover, that the equations 8 and 9 do also fail, they do not predict the correct Newyonian motions. We can therefore conclude that Einstein's hypothesis that the motions of material bodies follows geodesics in space-time is untenable.

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**References**

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