

Research Article

On the Weakly Semi-Fredholm Structure of a Class of Singular Beurling-Ahlfors-Type Operator Equations

Anatolij Prykarpatsky^{1,2,3}, Volodymyr M. Dilnyi²

1. Department of Computer Science and Mathematics, Cracow University of Technology, Poland; 2. Lviv Polytechnic National University, Ukraine; 3. Pedagogical University, Maputo, Mozambique

We studied the solution space of a linear singular Calderon-Zygmund type operator equation

$T(a(z)f(z)) + b(z)f(z) = 0$ **in the Banach space $L_q(K, d\sigma)$, $q > p > 1$, specified by coefficients**

$a, b \in L_r(K, d\sigma)$ **and the operator action $Tf(z) = \frac{1}{\pi} v \cdot p \cdot \int_{\mathbb{C}} \frac{f(\xi) - f(z)}{(z - \xi)^2} d\sigma(\xi, \bar{\xi})$, $f \in L_p(K, d\sigma)$,**

$r = pq/(q - p)$. **Based on classical functional analysis interpolation theory of operators, N.**

Kolmogorov and E. Riesz compactness theorems, we formulated conditions under which the singular operator equation under regard possesses the weakly semi-Fredholm structure.

Correspondence: papers@team.qeios.com — Qeios will forward to the authors

1. Introduction

Singular operator equations on complex domains are widely used ^{[1][2][3][4]} in many investigations of important problems of modern mechanics, radiophysics, solid state physics and other applied sciences. Their theory is closely related ^{[5][2][6]} with partial differential equations, geometry of complex-analytic and quasi-conformal mappings. In particular, the following ^{[1][7]} class of expressions

$$Af(z) := T(a(z)f(z)) + b(z)f(z), \quad (1.1)$$

characterized by its coefficients $a, b \in L_r(\mathbb{C}, d\sigma)$, $q > r > 1$, and acting on complex function $f \in L_q(\mathbb{C}, d\sigma)$, $q > 1$, endowed with the norm

$$\|f\|_q = \left(\int_{\mathbb{C}} |f(\xi)|^q d\sigma(\bar{\xi}, \xi) \right)^{1/q} \quad (1.2)$$

with respect to the Lebesgue measure $d\sigma(\xi, \bar{\xi}) = d\bar{\xi} \wedge d\xi/(2i)$ on the complex plane $\mathbb{C}$, define singular integral operator equations

$$Af(z) = 0, \quad (1.3)$$

specified by the singular Beurling-Ahlfors [\[8\]\[9\]\[10\]\[11\]](#) mapping

$$(Tf)(z) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|z-\xi|>\varepsilon} \frac{f(z) - f(\xi)}{(z - \xi)^2} d\sigma(\xi, \bar{\xi}), \quad (1.4)$$

which is equivalent to the singular Calderon-Zygmund transform $T = \partial_z \partial_{\bar{z}}^{-1} : L_q(\mathbb{C}, d\sigma) \rightarrow L_q(\mathbb{C}, d\sigma)$, $q > 1$, for all $z \in \mathbb{C}$. Similarly the expression (1.4) defines the corresponding singular Beurling-Ahlfors operator mapping $T : L_q(\mathbb{C}, d\sigma) \rightarrow L_q(\mathbb{C}, d\sigma)$, $q > 1$. The related linear singular operator equation (1.3) makes it possible to pose a problem of studying the functional analytic and geometric properties of the corresponding operator

$$A : L_q(\mathbb{C}, d\sigma) \rightarrow L_p(\mathbb{C}, d\sigma) \quad (1.5)$$

on $L_q(\mathbb{C}, d\sigma)$, and whose properties are of the most interest for applications.

Let us introduce the following definition, which reflects [\[7\]\[2\]\[6\]\[10\]](#) some weaker versions of the classical Fredholm operators.

Definition 1.1. A bounded operator $A : L_q(\mathbb{C}, d\sigma) \rightarrow L_p(\mathbb{C}, d\sigma)$ with closed $\text{Range } A \subset L_p(\mathbb{C}, d\sigma)$ is called semi-Fredholm, if its kernel $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$, or its cokernel $\text{coKer } A = L_p(\mathbb{C}, d\sigma) / \overline{\text{Range } A}$ is finite dimensional. If the image $\text{Range } A \subset L_p(\mathbb{C}, d\sigma)$ is neither closed nor its cokernel $L_p(\mathbb{C}, d\sigma) / \overline{\text{Range } A}$ is finite dimensional, the bounded operator $A : L_q(\mathbb{C}, d\sigma) \rightarrow L_p(\mathbb{C}, d\sigma)$ is called weakly semi-Fredholm.

In particular, our special interest lies in proving the weakly semi-Fredholm structure of the operator (1.5), which consists in stating the finite dimension of its kernel $\text{Cer } A \subset L_q(\mathbb{C}, d\sigma)$ and the infinite dimension of its coimage $L_p(\mathbb{C}, d\sigma) / \overline{\text{Range } A}$. Based on the functional analytic properties of the Hilbert type singular transform (1.4) as well as on the structural properties of the singular operator mapping (1.4) on the Banach space $L_p(\mathbb{C}, d\sigma)$, we stated our main theorem.

Theorem 1.2. *The bounded operator (1.5) on the Banach space $L_q(\mathbb{C}, d\sigma)$, $q > p > 1$, is weakly semi-Fredholm, that is $\dim \text{Cer } A < \infty$ and $\text{Range } A \subset L_p(\mathbb{C}, d\sigma)$ is not closed, if the coefficients $0 \neq a, b \in L_r(\mathbb{C}, d\sigma)$ and the chosen factorization $b = b_-^{-1} b_+$ is such, that for some mapping $\omega : \mathbb{C} \rightarrow \mathbb{C}$ there hold the embeddings $b_+(z)\omega(z) \in L_{\bar{q}}(\mathbb{C}, d\sigma)$, where $1/\bar{q} < \min(1/p - 2/r, 1 - 1/p)$, and $b_-(z)^{-1}\omega(z) \in L_\infty(\mathbb{C}, d\sigma)$, as well as the condition $|\omega b_-^{-1}| \geq \delta$ holds for some $\delta > 0$ almost everywhere.*

The above statement in its main part depends on the following classical functional analysis N. Kolmogorov and F. Riesz theorems [\[12\]\[13\]\[14\]\[15\]\[16\]\[17\]](#), describing linear mappings between Banach spaces:

Theorem 1.3. (N. Kolmogorov) Let $1 < p < \infty$ and $\mathcal{F}$ be a subset of $L_p(\mathbb{C}, d\sigma)$. Then the set $\mathcal{F}$ is totally bounded if and only if (i) $\mathcal{F}$ is bounded; (ii) for every $\varepsilon > 0$ there is such $r > 0$, that for every $f \in \mathcal{F}$ the estimation $\int_{\{|z|>r\}} |f(z)|^p d\sigma < \varepsilon^p$ holds; (iii) for every $\varepsilon > 0$ there is such $\rho > 0$ that for every $f \in \mathcal{F}$ and $y \in \mathbb{C}$ with $|y| < \rho$ the estimation $\int_{\mathbb{C}} |f(z + \xi) - f(x)|^p d\sigma < \varepsilon^p$ holds.

Theorem 1.4. (F. Riesz) If a bounded operator $A : X \rightarrow Y$ from a Banach space X to a Banach space Y is compact and its image $\text{Im} A \subset Y$ is closed, then $\dim(AX) < \infty$. Moreover, a linear subspace of a normed space X is locally compact if and only if it is finite dimensional.

Below we proceed to studying the functional analytic properties of the singular Hilbert type transform (1.4) and next proving the weakly semi-Fredholm property of the linear singular operator (1.1).

2. Singular Beurling-Ahlfors operator T and its properties

We consider the Cauchy type integral

$$Pf(z) := \frac{1}{2\pi i} \int_{\mathbb{C}} f(\xi) \left(\frac{1}{\xi - z} - \frac{1}{\xi} \right) d\sigma(\xi, \bar{\xi}) \quad (2.1)$$

for a complex variable $z \in \mathbb{C}$ on the dense subspace $C_0^2(\mathbb{C}, d\sigma) \subset L_q(\mathbb{C}, d\sigma)$ for $q > 1$, naturally defining [\[8\]\[5\]\[4\]](#) the following two functions:

$$\partial Pf(z)/\partial \bar{z} = f(z), \quad \partial Pf(z)/\partial z = Tf(z) \quad (2.2)$$

for any point $z \in \mathbb{C}$. It is easy to check that the Calderon-Zygmund operator $T = \partial_z \partial_{\bar{z}}^{-1} : L_q(\mathbb{C}, d\sigma) \rightarrow L_q(\mathbb{C}, d\sigma)$ for $z \in \mathbb{C}$ and $1 < q < \infty$. Moreover, applying the Holder inequality to the expression (2.1), we obtain

$$|Pf(z)| \leq K_q \|f\|_q |z|^{\frac{-2}{q}+1} \quad (2.3)$$

for arbitrary function $f \in L_q(\mathbb{C}, d\sigma)$ and any $z \in \mathbb{C}$, where

$K_q = \left(\int_{\mathbb{C}} |\xi|^{\frac{-q}{q-1}} |(\xi - 1)|^{\frac{q}{1-q}} d\sigma(\xi) \right)^{\frac{q-1}{q}} < \infty$. In addition, from the following identity

$$\begin{aligned} Pf_{z_1}(z_2 - z_1) &= \frac{1}{2\pi i} \int_{\mathbb{C}} f(\xi) \left(\frac{1}{\xi - z_2} - \frac{1}{\xi - z_1} \right) d\sigma(\xi) = \\ &= Pf(z_2) - Pf(z_1) \end{aligned} \quad (2.4)$$

for the shifted function $f_{z_1}(\xi) := f(\xi + z_1)$, $\xi \in \mathbb{C}$, and the estimation (2.3) one derives the inequality

$$|Pf(z_1) - Pf(z_2)| \leq K_q \|f\|_q |z_2 - z_1|^{\frac{-2}{q}+1} \quad (2.5)$$

for arbitrary $z_1, z_2 \in \mathbb{C}$, meaning the Hölder continuity of the mapping (2.1).

Before formulating our main statement about the solution set of the equation (1.3), we first state that the before defined reduced operator $T : L_q(\mathbb{C}, d\sigma) \rightarrow L_q(\mathbb{C}, d\sigma)$ for $1 < q < \infty$ is bounded. To do this, we first observe this property for the particular case $q = 2$, having calculated the integral

$$\begin{aligned} \int_{\mathbb{C}} |Tf(z)|^2 d\sigma(z, \bar{z}) &= \int_{\mathbb{C}} |Tf(z)|^2 d\sigma(z, \bar{z}) = \\ &= \int_{\mathbb{C}} \partial^2 \overline{Pf(z)} / \partial \bar{z} \partial z Pf(z) d\sigma(z, \bar{z}) = \\ &= \int_{\mathbb{C}} \overline{\partial f(z)} / \partial \bar{z} \partial z Pf(z) d\sigma(z, \bar{z}) = \int_{\mathbb{C}} \overline{f(z)} \partial Pf(z) / \partial \bar{z} d\sigma(z, \bar{z}) = \\ &= \int_{\mathbb{C}} \overline{f(z)} f(z) d\sigma(z, \bar{z}) = \int_{\mathbb{C}} |f(z)|^2 d\sigma(z, \bar{z}), \end{aligned} \quad (2.6)$$

where we took into account that the space $C_0^2(\mathbb{C}, d\sigma) \subset L_2(\mathbb{C}, d\sigma)$ is dense in $L_2(\mathbb{C}, d\sigma)$. The result (2.6) evidently means that the mapping $T : L_2(\mathbb{C}, d\sigma) \rightarrow L_2(\mathbb{C}, d\sigma)$ is isometry of the Hilbert space $L_2(\mathbb{C}, d\sigma)$ with the norm $\|T\| = 1$. Moreover, for any $f \in L_q(\mathbb{C}, d\sigma)$ there holds [8][4][11], the following general Zigmund-Kalderon inequality

$$\|Tf\|_q \leq \alpha_q \|f\|_q, \quad (2.7)$$

where $\alpha_q > 0, 1 < q < \infty$, are some bounded constants. The estimation (2.7) in its main part significantly depends on the operator representation $T = -S^2 : L_q(\mathbb{C}, d\sigma) \rightarrow L_q(\mathbb{C}, d\sigma)$, where, by definition,

$$Sf(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f(\xi) - f(z)}{(\xi - z) |\xi - z|} d\sigma(\xi, \bar{\xi}). \quad (2.8)$$

The latter is considered on the dense subspace $C_0^2(\mathbb{C}, d\sigma) \subset L_q(\mathbb{C}, d\sigma)$ for $q \geq 2$ and satisfies the following inequality

$$\|Sf\|_q \leq \sqrt{\alpha_q} \|f\|_q \quad (2.9)$$

for $f \in C_0^2(\mathbb{C}, d\sigma) \subset L_2(\mathbb{C}, d\sigma)$ and some constant $\alpha_q > 0, q \geq 2$, which makes it possible to extend the operator $T = -S^2 : C_0^2(\mathbb{C}, d\sigma) \rightarrow L_q(\mathbb{C}, d\sigma)$ by continuity on the whole space $L_q(\mathbb{C}, d\sigma)$, $q \geq 2$, and to obtain the classical Zygmund-Calderón inequality (2.7) for all $f \in L_q(\mathbb{C}, d\sigma)$, $q \geq 2$. Based on the properties of the adjoint operator $T^* = T : L_{q'}(\mathbb{C}, d\sigma) \rightarrow L_{q'}(\mathbb{C}, d\sigma)$, $1 < q' \leq 2$, one finally obtains that the operator $T : L_q(\mathbb{C}, d\sigma) \rightarrow L_q(\mathbb{C}, d\sigma)$ is well defined and bounded on the space $L_q(\mathbb{C}, d\sigma)$ for all $1 < q < \infty$. Moreover, the singular operator (2.8), as well as that of (1.4), can be interpolated [18][19][20] as operators from $L_{q_1}(\mathbb{C}, d\sigma)$ to $L_{q_2}(\mathbb{C}, d\sigma)$, where $1 < q_1 < q < q_2$, where they become compact,

thus presenting a useful tool for studying functional properties of the operator (1.5). Proceed now to studying them in detail and proving Theorem 1.2.

3. Operator A and its weakly semi-Fredholm structure

We consider the operator mapping (1.1) and observe that, owing to the inequality (2.7), the following estimation

$$\|Af\|_p \leq \|T(af)\|_p + \|b\|_r \|f\|_q \leq (\alpha_q \|a\|_r + \|b\|_r) \|f\|_q, \quad (3.1)$$

holds for $1 < p < q < \infty$, where $1/r = 1/p - 1/q$, and $f \in L_q(\mathbb{C}, d\sigma)$, thus meaning that the corresponding operator norm $\|A\| \leq (\alpha_q \|a\|_r + \|b\|_r)$, that is bounded.

Now we formulate our main theorem about the weakly semi-Fredholm structure of the bounded operator (1.1) from the Banach space $L_q(\mathbb{C}, d\sigma)$, $1 < q < \infty$, into the Banach space $L_p(\mathbb{C}, d\sigma)$, $q > p \geq 1$.

Theorem 3.1. *The bounded operator (1.5) on the Banach space $L_q(\mathbb{C}, d\sigma)$, $q > p > 1$, is weakly semi-Fredholm, that is $\dim \text{Ker } A < \infty$ and $\text{Range } A \subset L_p(\mathbb{C}, d\sigma)$ is not closed, if the coefficients $0 \neq a, b \in L_r(\mathbb{C}, d\sigma)$ and the chosen factorization $b = b_-^{-1} b_+$ is such, that for some mapping $\omega : \mathbb{C} \rightarrow \mathbb{C}$ there hold the embeddings $b_+(z)\omega(z) \in L_{\bar{q}}(\mathbb{C}, d\sigma)$, where $1/\bar{q} < \min(1/p - 2/r, 1 - 1/p)$, $r > 2p$, and $b_-(z)^{-1}\omega(z) \in L_\infty(\mathbb{C}, d\sigma)$, as well as the condition $|\omega b_-^{-1}| \geq \delta$ holds for some $\delta > 0$ almost everywhere.*

First of all, take into account that for $q > p \geq 1$ the image $\text{Range } A \subset L_p(\mathbb{C}, d\sigma)$ can not be closed, as $\dim \text{Range } A = \infty$. Moreover, the image $\text{Range } A^* \subset L_q(\mathbb{C}, d\sigma)'$ of the adjoint singular transformation $A^* = A - \{T, b\} : L_p(\mathbb{C}, d\sigma)' \rightarrow L_q(\mathbb{C}, d\sigma)'$, where $\{T, b\} := T \circ b + b \circ T$, is not also closed, as owing to the Banach closed image theorem one obtains both the equality $\text{codim } \text{Range } A^* = \dim \text{Ker } A < \infty$ and $\dim \text{Ker } A^* = \infty$, contradicting the condition $\text{codim } \text{Range } A < \infty$. Thus, to state that $\dim \text{Ker } A < \infty$, we will construct a new suitably reduced operator on the kernel subspace $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$, next extend it on the whole space $L_q(\mathbb{C}, d\sigma)$ and state its compactness by means of both the interpolating properties of the singular Beurling-Ahlfors operator (1.4) and the classical N. Kolmogorov Theorem 1.3, formulated in the case of the Banach space $L_q(\mathbb{C}, d\sigma)$. As this compact operator on $L_q(\mathbb{C}, d\sigma)$ persists to be compact and its image is closed, we right away derive, as a consequence from the F. Riesz Theorem 1.4, that $\dim \text{Ker } A < \infty$.

Namely, consider the closed kernel subspace $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$ of the operator (1.5), that is the condition $A \text{Ker } A = 0$, which is equivalently rewritten as the equation

$$R\omega(z)b_+(z)^{-1}Taf = -R\omega(z)b_-(z)^{-1}f, \quad (3.2)$$

where $\omega : \mathbb{C} \rightarrow \mathbb{C}$ is some measurable mapping and

$$\mathcal{R}(\cdot) := -\frac{w(z)}{\pi} \int_{\mathbb{C}} \frac{(\cdot) d\sigma(\xi, \bar{\xi})}{|\xi - z|^{2-\mu}}, \quad (3.3)$$

where $1 < \mu < 2$, $\nu > \mu/2 + 1/m$ and $w_\nu(z) := (1 + |z|^2)^{-\nu}$, is the so called fractional Cauchy-Vekua-Riesz [4] operator. The lefthand side expression of (3.2) defines a singular operator $A_T : L_q(\mathbb{C}, d\sigma) \rightarrow W_m^\mu(\mathbb{C}, w_\nu d\sigma)$, $1/m = 1/r + 1/q + 1/s - \mu/2$, where

$$A_T f := \mathcal{R}\omega(z)b_+(z)^{-1} T a f \quad (3.4)$$

for any $f \in L_q(\mathbb{C}, d\sigma)$ and $\text{Range } A_T \subset W_m^\mu(\mathbb{C}, w_\nu d\sigma)$, if the function $\omega(z)b_+^{-1}(z) \in L_s(\mathbb{C}, d\sigma)$, owing to the classical Hardy-Littlewood theorem. Similarly, the righthand side of (3.2) defines a singular operator $B_T : L_q(\mathbb{C}, d\sigma) \rightarrow W_m^\mu(\mathbb{C}, w_\nu d\sigma)$, where

$$B_T f := -\mathcal{R}\omega(z)b_-(z)^{-1} f \quad (3.5)$$

for any $f \in L_q(\mathbb{C}, d\sigma)$, where $\omega(z)b_-(z)^{-1} \sim (1 + |z|^2)^{\frac{q-m}{2qm} + \nu} \in L_{\frac{qm\mu}{q-m} + \nu, loc}(\mathbb{C}, d\sigma)$ and $\delta < |\omega(z)b_-(z)^{-1}|$ for some $\delta > 0$ almost everywhere. The operators (3.4) and (3.5) are characterized by the following lemma.

Lemma 3.2. Let $a(z) \in L_r(\mathbb{C}, d\sigma)$ and the factorization $b(z) := b_-(z)^{-1}b_+(z) \in L_r(\mathbb{C}, d\sigma)$, $1/r = 1/p - 1/q$, satisfy for almost all $z \in \mathbb{C}$ the following conditions: there exists a measurable mapping $\omega : \mathbb{C} \rightarrow \mathbb{C}$, such that $\omega(z)b_+(z)^{-1} \in L_s(\mathbb{C}, d\sigma)$ and $\omega(z)b_-(z)^{-1} \sim (1 + |z|^2)^{\frac{q-m\mu}{2qm} + \nu} \in L_{\frac{qm\mu}{q-m} + \nu, loc}(\mathbb{C}, d\sigma)$, where $1/m = 1/r + 1/q + 1/s - \mu/2$ and $\delta < |\omega(z)b_-(z)^{-1}|$ for some $\delta > 0$ almost everywhere. Then 1) the linear operator (3.4) is bounded and compact; 2) the linear operator (3.5) maps the closed subspaces into the closed ones.

Proof. To state the lemma above we first observe that, owing to the classical Kolmogorov 1.3 and Riesz theorem 1.4, the operator (3.4) is bounded and compact, taking into account the condition $\omega(z)b_+(z)^{-1} \in L_s(\mathbb{C}, d\sigma)$, imposed above, and the smoothing property of the Cauchy-Vekua-Riesz operator (3.3), giving rise, owing to the limiting condition

$$\lim_{\varepsilon \rightarrow 0} \left[\int_{|z| > 1/\varepsilon} |A_T f(z)|^m w_\nu(z) d\sigma(z, \bar{z}) + \left(\int_{|\xi| > 1/\varepsilon} \int_{|z| > 1/\varepsilon} \frac{|A_T f(z) - A_T f(\xi)|^m}{|z - \xi|^{2+m\mu}} w_\nu(z) w_\nu(\xi) d\sigma(z, \bar{z}) d\sigma(\xi, \bar{\xi}) \right)^{1/m} \right] = 0$$

for any $f \in L_s(\mathbb{C}, d\sigma)$, to the compact embedding of the image $\text{Range } A_T = W_m^\mu(\mathbb{C}, w_\nu d\sigma)$, if $1/m = 1/r + 1/q + 1/s - \mu/2$. Moreover, since the Vekua type multiplier $\omega(z)b_-(z)^{-1} \sim$

$(1 + |z|^2)^{\frac{q-m}{2qm} + \nu} \in L_{\frac{qm}{q-m} + \nu, loc}(\mathbb{C}, d\sigma)$, as well as $\delta < |\omega(z)b_-(z)^{-1}|$ for some $\delta > 0$ almost everywhere, the smoothing property of the Cauchy-Vekua-Riesz operator (3.3) $\mathcal{R} : L_m(\mathbb{C}, w_\nu d\sigma) \rightarrow W_m^\mu(\mathbb{C}, w_\nu d\sigma)$ gives rise to the surjective condition $\text{Range } B_T = W_m^\mu(\mathbb{C}, w_\nu d\sigma)$. $\square$

Proof. (Proof of the main Theorem) We first observe that the reduced operator $A_T : \text{Ker } A \rightarrow W_m^\mu(\mathbb{C}, w_\nu d\sigma)$ is compact too, and whose image $\text{Range } A_T|_{\text{Ker } A} = -\mathcal{R}\omega b_-^{-1} \text{Ker } A \subset W_m^\mu(\mathbb{C}, w_\nu d\sigma)$ is a priori closed, owing its surjectivity of the operator $B_T : L_q(\mathbb{C}, d\sigma) \rightarrow W_m^\mu(\mathbb{C}, w_\nu d\sigma)$. Then, based on the F. Riesz compactness Theorem 1.4, we derive the boundedness of the $\dim \text{Range } A_T$, and as a consequence, the boundedness $\dim \text{Ker } A < \infty$, since the kernel $\text{Ker } \mathcal{R} = \{0\}$ of the Cauchy-Vekua-Riesz operator (3.3) is trivial. The latter means that the operator (1.1) possesses the weakly semi-Fredholm structure, thus proving main theorem. $\square$

Remark 3.3. Taking into account the properties (2.2) of the P - operator (2.1), the above equation (3.2) can be reformulated as the following partial differential equation

$$b_+ \frac{\partial P f(z)}{\partial \bar{z}} + b_- \frac{\partial P (af)(z)}{\partial z} = 0, \quad (3.6)$$

generating quasi-conformal mappings of the complex plane $\mathbb{C}$, and whose solutions naturally generate the kernel $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$ of the operator (1.1). This observation provides an alternative viewpoint on equation (3.2) and a more detailed analysis of the subspace $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$ makes it possible to establish Theorem 1.2 in a somewhat more general form. It is also worth to mention that similar to (3.6) equations were extensively studied in [21], and analytical techniques there devised can be useful for studying problems, treated in the present work.

4. Conclusion

Based on the functional analytic properties of the singular Cauchy type integral operator

$$Pf(z) := \frac{1}{2\pi i} \int_{\mathbb{C}} f(\xi) \left(\frac{1}{\xi - z} - \frac{1}{\xi} \right) d\sigma(\xi, \bar{\xi})$$

on the Banach space $L_q(\mathbb{C}, d\sigma)$, we have studied the kernel $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$ of the linear singular operator expression

$$Af(z) := T(a(z)f(z)) + b(z)f(z) \quad (4.1)$$

with coefficients $a, b \in L_r(\mathbb{C}, d\sigma)$, $b = b_+^{-1}b_-$, where for some measurable mapping $\omega : \mathbb{C} \rightarrow \mathbb{C}$ there hold the embeddings $b_+(z)^{-1}\omega(z) \in L_{\bar{q}}(\mathbb{C}, d\sigma)$ for $1/\bar{q} < \min(1/p - 2/r, 1 - 1/p)$,

$b_-(z)^{-1}\omega(z) \in L_\infty(\mathbb{C}, d\sigma)$, $|b_-(z)^{-1}\omega(z)| \geq \delta$ for some $\delta > 0$ almost everywhere, and the operator $T : L_p(\mathbb{C}, d\sigma) \rightarrow L(\mathbb{C}, d\sigma)$ is defined by the differential Beurling-Ahlfors expression

$$Tf(z) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{\{|z-\xi|>\varepsilon\}} \frac{f(z) - f(\xi)}{(z - \xi)^2} d\sigma(\xi, \bar{\xi}),$$

on the complex plane $\mathbb{C}$. Making use of the reduction operator approach and classical functional analysis results Theorem 1.4 and Lemma 3.2, we succeeded in stating the weakly semi-Fredholm property of the above operator expression (4.1).

Statements and Declarations

Funding

No specific funding was received for this work.

Potential Competing Interests

No potential competing interests to declare.

Data Availability

All available data are contained in the manuscript.

Acknowledgements

Authors are much appreciated to Ya. V. Mykytyuk, J. Koroński, Z. Peradzyński, J. Stochel and V. Mitiuszew for fruitful discussions and instructive comments and remarks.

References

1. ^aBers L (1977). "Quasiconformal Mappings, with Applications to Differential Equations, Function Theory and Topology." *Bull Am Math Soc.* **83**(6):1083–1100.
2. ^a^b^cMikhlin S, Prösdorf S (1987). *Singular Integral Operators*. Springer Science & Business Media.
3. ^ΔSoldatov AP (1991). *Onedimensional Singular Operators and Boundary Value Problems of Theory of Functions*. Moscow: Vyshcha Shkola.
4. ^a^b^c^dVekua IN (1962). *Generalized Analytic Functions*. Oxford: Pergamon Press.

5. ^a ^bBojarski B (2010). "On the Beltrami Equation, Once Again: 54 Years Later." *Ann Acad Sci Fenn Math.* 35:5 9–73.
6. ^a ^bMurphy GJ (1990). *C*-Algebras and Operator Theory*. Academic Press Inc.
7. ^a ^bEstrada R, Kanwal RP (2012). *Singular Integral Equations*. Boston: Birkhäuser.
8. ^a ^b ^cAhlfors LV (1966). *Lectures on Quasiconformal Mappings*. Princeton: Van Nostrand.
9. ^ΔDuren P, Shuster A (2004). *Bergman Spaces*. Providence USA: AMS.
10. ^a ^b ^cTriebel H (1978). *Interpolation Theory, Function Spaces, Differential Operators*. Amsterdam–New York–Oxford: North-Holland Publishing Company.
11. ^a ^bSygmund A (2003). *Trigonometric Series*. Cambridge University Publishing.
12. ^ΔBuldygin VV, Shurenkova AV (1995). "On Criterium of Compactness in L_p Space." *Teor Imovirn Ta Matem S tatystyka [Theory of Probability and Mathematical Statistics]*. 52:32–36.
13. ^ΔIvanishko IA, Krotov VG (2009). "Compactness of Embeddings of Sobolev Type on Metric Measure Spaces." *Math Notes*. 86:775–788. doi:[10.1134/S0001434609110200](https://doi.org/10.1134/S0001434609110200).
14. ^ΔKolmogorov AN (1931). "Über Kompaktheit der Funktionenmengen bei der Konvergenz im Mittel" [On the Compactness of Sets of Functions in the Case of Convergence in the Mean]. *Nachr Ges Wiss Gottingen*. 9:60–63.
15. ^ΔKrotov VG (2012). "Criteria for Compactness in L_p -Spaces, $p > 0$." *Sb Math*. 203(7):1045–1064.
16. ^ΔHanche-Olsen H, Holden H (2016). "The Kolmogorov–Riesz Compactness Theorem." *Expositiones Mathematicae*. 34(2):243–245.
17. ^ΔSudakov VN (1957). "Criteria of Compactness in Function Spaces." *Uspehi Mat Nauk (NS) [Russian Mathematical Surveys]*. 12(3):221–224.
18. ^ΔBennett C, Sharpley R (1988). *Interpolation of Operators*. Academic Press.
19. ^ΔBergh J, Löfström J (1976). *Interpolation Spaces*. Springer.
20. ^ΔKrasnosel'skiĭ MA (1960). "On a Theorem of Riesz–Thorin for Compact Operators." *Soviet Mathematics Doklady*. 131(1):214–217.
21. ^ΔBandura A, Skaskiv O (2019). "Linear Directional Differential Equations in the Unit Ball: Solutions of Bounded L -Index." *Mathematica Slovaca*. 69(5):1089–1098.

Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.