

Research Article

On the Weakly Semi-Fredholm Structure of a Class of Singular Hilbert-Type Operator Equations

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We studied the solution space of a linear singular Hilbert type operator equation

$T(a(z)f(z)) + b(z)f(z) = 0$ **in the Banach space $L_q(\mathbb{C}, d\sigma)$, $q > p > 1$, specified by coefficients**

$a, b \in L_r(\mathbb{C}, d\sigma)$ **on the complex plane $\mathbb{C}$, and the operator action**

$Tf(z) = \frac{1}{\pi} v.p. \int_{\mathbb{C}} \frac{f(\xi) - f(\bar{\xi})}{(z - \xi)^2} d\sigma(\xi, \bar{\xi})$, $f \in L_p(\mathbb{C}, d\sigma)$, **where $1/r + 1/q = 1/p$. Based on the classical**

Coifman-Rochberg-Weiss smoothing and N. Kolmogorov - E. Riesz compactness theorems, we

formulated conditions under which the singular operator equation under regard possesses the weakly semi-Fredholm structure.

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1. Introduction

Singular operator equations on complex domains are widely used ^{[1][2][3][4]} in many investigations of important problems of modern mechanics, radiophysics, solid state physics and other applied sciences. Their theory is closely related ^{[5][2][6]} with partial differential equations, geometry of complex-analytic and quasi-conformal mappings. In particular, the following ^{[1][7]} class of expressions

$$Af(z) := T(a(z)f(z)) + b(z)f(z), \quad (1.1)$$

characterized by its coefficients $a, b \in L_r(\mathbb{C}, d\sigma)$, $q > r > 1$, and acting on complex function $f \in L_q(\mathbb{C}, d\sigma)$, $q > 1$, endowed with the norm

$$\|f\|_q = \left(\int_{\mathbb{C}} |f(\xi)|^q d\sigma(\bar{\xi}, \xi) \right)^{1/q} \quad (1.2)$$

with respect to the Lebesgue measure $d\sigma(\xi, \bar{\xi}) = d\bar{\xi} \wedge d\xi/(2i)$ on the complex plane $\mathbb{C}$, define singular integral operator equations

$$Af(z) = 0, \quad (1.3)$$

specified by the singular Beurling-Ahlfors [\[8\]\[9\]\[10\]\[11\]\[12\]\[13\]](#) transform

$$Tf(z) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|z-\xi|>\varepsilon} \frac{f(z) - f(\xi)}{(z - \xi)^2} d\sigma(\xi, \bar{\xi}), \quad (1.4)$$

which is equivalent to the Calderon-Zygmund [\[9\]](#) mapping $T = \partial_z \partial_{\bar{z}}^{-1} : L_q(\mathbb{C}, d\sigma) \rightarrow L_q(\mathbb{C}, d\sigma)$, $q > 1$, for all $z \in \mathbb{C}$.

Let us introduce the following definition, which reflects [\[7\]\[2\]\[11\]](#) some weaker versions of the classical Fredholm operators.

Definition 1.1. A bounded operator $A : L_q(\mathbb{C}, d\sigma) \rightarrow L_p(\mathbb{C}, d\sigma)$ with closed $\text{Range } A \subset L_p(\mathbb{C}, d\sigma)$ is called semi-Fredholm, if its kernel $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$, or its cokernel $\text{coKer } A = L_p(\mathbb{C}, d\sigma) / \overline{\text{Range } A}$ is finite dimensional. If the image $\text{Range } A \subset L_p(\mathbb{C}, d\sigma)$ is not closed and its kernel $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$ is finite dimensional, the bounded operator $A : L_q(\mathbb{C}, d\sigma) \rightarrow L_p(\mathbb{C}, d\sigma)$ is called weakly semi-Fredholm.

In particular, our special interest lies in proving the weakly semi-Fredholm structure of the operator (1.1), which consists in stating the finite dimension of its kernel $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$ and the notclosedness of its $\text{Range } A \subset L_p(\mathbb{C}, d\sigma)$. Based on the functional analytic properties of the Beurling-Ahlfors singular transform (1.4), as well as on the structural properties of the singular operator mapping (1.4) on the Banach space $L_p(\mathbb{C}, d\sigma)$, we stated our main theorem.

Theorem 1.2. *The bounded operator (1.1) on the Banach space $L_q(\mathbb{C}, d\sigma)$, $q > p > 1$, is weakly semi-Fredholm, that is $\dim \text{Ker } A < \infty$ and the $\text{Range } A \subset L_p(\mathbb{C}, d\sigma)$ is not closed, if a locally continuous coefficient $a \in L_r(\mathbb{C}, d\sigma)$, $r > 1$, a coefficient $b \in L_r(\mathbb{C}, d\sigma)$, $r > 1$, and the chosen factorization $b = b_-^{-1} b_+$ is such, that for some measurable mapping $\omega : \mathbb{C} \rightarrow \mathbb{C}$ there hold conditions: $\omega(z) b_+(z)^{-1} \in L_s(\mathbb{C}, d\sigma)$, $s > 1$, $\omega(z) b_+^{-1} a(z)$, $\omega(z) b_-^{-1}(z) \in L_\infty(\mathbb{C}, d\sigma)$, where $1/m = 1/q + 1/r + 1/s - \mu/2$, $\nu \in (1/r + 1/s - 1/m, 1 - 1/q)$, $\mu < 2$, and $\delta < |\omega(z) b_-(z)^{-1}| - |\omega(z) b_+^{-1} a|$ for some $\delta > 0$ almost everywhere.*

The above statement in its main part depends on the following classical functional analysis N. Kolmogorov and F. Riesz theorems [\[14\]\[15\]\[16\]\[17\]\[18\]\[19\]\[20\]](#), describing linear mappings between Banach spaces:

Theorem 1.3. (N. Kolmogorov) Let $1 \leq p < \infty$ and $\mathcal{F}$ be a subset of $L_p(\mathbb{C}, d\sigma)$. Then the set $\mathcal{F}$ is totally bounded if and only if (i) $\mathcal{F}$ is bounded; (ii) for every $\varepsilon > 0$ there is such $r > 0$, that for every $f \in \mathcal{F}$ the estimation $\int_{\{|z|>r\}} |f(z)|^p d\sigma(z, \bar{z}) < \varepsilon^p$ holds; (iii) for every $\varepsilon > 0$ there is such $\rho > 0$ that for every $f \in \mathcal{F}$ and $y \in \mathbb{C}$ with $|y| < \rho$ the estimation $\int_{\mathbb{C}} |f(\xi + y) - f(\xi)|^p d\sigma(\xi, \bar{\xi}) < \varepsilon^p$ holds.

Theorem 1.4. (F. Riesz) If a bounded operator $A : X \rightarrow Y$ from a Banach space X to a Banach space Y is compact and its image $\text{Im} A \subset Y$ is closed, then $\dim(AX) < \infty$. Moreover, a linear subspace of a normed space X is locally compact if and only if it is finite dimensional.

Below we proceed to studying the functional analytic properties of the singular Hilbert type transform (1.4) and next proving the weakly semi-Fredholm property of the linear singular operator (1.1).

2. Singular Beurling-Ahlfors operator T and its properties

We consider the Cauchy type integral

$$Pf(z) := \frac{1}{2\pi i} \int_{\mathbb{C}} f(\xi) \left(\frac{1}{\xi - z} - \frac{1}{\xi} \right) d\sigma(\xi, \bar{\xi}) \quad (2.1)$$

for a complex variable $z \in \mathbb{C}$ on the dense subspace $C_0^2(\mathbb{C}, d\sigma) \subset L_q(\mathbb{C}, d\sigma)$ for $q > 1$, naturally defining [8][5][4] the following two functions:

$$\partial P f(z) / \partial \bar{z} = f(z), \quad \partial P f(z) / \partial z = T f(z) \quad (2.2)$$

for any point $z \in \mathbb{C}$. It is easy to check that the Beurling-Ahlfors operator $T = \partial_z \partial_{\bar{z}}^{-1} : L_q(\mathbb{C}, d\sigma) \rightarrow L_q(\mathbb{C}, d\sigma)$ for $z \in \mathbb{C}$ and $2 < q < \infty$. Moreover, applying the Holder inequality to the expression (2.1), we obtain

$$|Pf(z)| \leq K_q \|f\|_q |z|^{\frac{-2}{q}+1} \quad (2.3)$$

for arbitrary function $f \in L_q(\mathbb{C}, d\sigma)$ and any $z \in \mathbb{C}$, where

$K_q = \left(\int_{\mathbb{C}} |\xi|^{\frac{-q}{q-1}} |(\xi - 1)|^{\frac{q}{1-q}} d\sigma(\xi) \right)^{\frac{q-1}{q}} < \infty$ for $q > 2$. In addition, from the following identity

$$\begin{aligned} Pf_{z_1}(z_2 - z_1) &= \frac{1}{2\pi i} \int_{\mathbb{C}} f(\xi) \left(\frac{1}{\xi - z_2} - \frac{1}{\xi - z_1} \right) d\sigma(\xi) = \\ &= Pf(z_2) - Pf(z_1) \end{aligned} \quad (2.4)$$

for the shifted function $f_{z_1}(\xi) := f(\xi + z_1)$, $\xi \in \mathbb{C}$, and the estimation (2.3) one derives the inequality

$$|Pf(z_1) - Pf(z_2)| \leq K_q \|f\|_q |z_2 - z_1|^{\frac{-2}{q}+1} \quad (2.5)$$

for arbitrary $z_1, z_2 \in \mathbb{C}$, meaning the Hölder continuity of the mapping (2.1). To formulate our main statement about the solution set of the equation (1.3), we first state that the operator $T : L_q(\mathbb{C}, d\sigma) \rightarrow L_q(\mathbb{C}, d\sigma)$ for $1 < q < \infty$ is bounded. To do this, we first observe this property for the particular case $q = 2$, having calculated the integral

$$\begin{aligned} \int_{\mathbb{C}} |Tf(z)|^2 d\sigma(z, \bar{z}) &= \int_{\mathbb{C}} |Tf(z)|^2 d\sigma(z, \bar{z}) = \\ &= \int_{\mathbb{C}} \partial^2 \overline{Pf(z)} / \partial \bar{z} \partial z Pf(z) d\sigma(z, \bar{z}) = \\ &= \int_{\mathbb{C}} \partial \overline{f(z)} / \partial \bar{z} Pf(z) d\sigma(z, \bar{z}) = \int_{\mathbb{C}} \overline{f(z)} \partial Pf(z) / \partial \bar{z} d\sigma(z, \bar{z}) = \\ &= \int_{\mathbb{C}} \overline{f(z)} f(z) d\sigma(z, \bar{z}) = \int_{\mathbb{C}} |f(z)|^2 d\sigma(z, \bar{z}), \end{aligned} \quad (2.6)$$

where we took into account that the space $C_0^2(\mathbb{C}, d\sigma) \subset L_2(\mathbb{C}, d\sigma)$ is dense in $L_2(\mathbb{C}, d\sigma)$. The result (2.6) evidently means that the mapping $T : L_2(\mathbb{C}, d\sigma) \rightarrow L_2(\mathbb{C}, d\sigma)$ is isometry of the Hilbert space $L_2(\mathbb{C}, d\sigma)$ with the norm $\|T\| = 1$. Moreover, for any $f \in L_q(\mathbb{C}, d\sigma)$ there holds [8][4][13], the following general Zygmund-Kalderon inequality

$$\|Tf\|_q \leq \alpha_q \|f\|_q, \quad (2.7)$$

where $\alpha_q > 0, 1 < q < \infty$, are some bounded constants. The estimation (2.7) in its main part significantly depends on the operator representation $T = -S^2 : L_q(\mathbb{C}, d\sigma) \rightarrow L_q(\mathbb{C}, d\sigma)$, where, by definition,

$$Sf(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f(\xi) - f(z)}{(\xi - z) |\xi - z|} d\sigma(\xi, \bar{\xi}). \quad (2.8)$$

The latter is considered on the dense subspace $C_0^2(\mathbb{C}, d\sigma) \subset L_q(\mathbb{C}, d\sigma)$ for $q \geq 2$ and satisfies the following inequality

$$\|Sf\|_q \leq \sqrt{\alpha_q} \|f\|_q \quad (2.9)$$

for $f \in C_0^2(\mathbb{C}, d\sigma) \subset L_2(\mathbb{C}, d\sigma)$ and some constant $\alpha_q > 0, q \geq 2$, which makes it possible to extend the operator $T = -S^2 : C_0^2(\mathbb{C}, d\sigma) \rightarrow L_q(\mathbb{C}, d\sigma)$ by continuity on the whole space $L_q(\mathbb{C}, d\sigma), q \geq 2$, and to obtain the classical Zygmund-Calderón inequality (2.7) for all $f \in L_q(\mathbb{C}, d\sigma), q \geq 2$. Based on the properties of the adjoint operator $T^* = T : L_{q'}(\mathbb{C}, d\sigma) \rightarrow L_{q'}(\mathbb{C}, d\sigma), 1 < q' \leq 2$, one finally obtains that the operator $T : L_q(\mathbb{C}, d\sigma) \rightarrow L_q(\mathbb{C}, d\sigma)$ is well defined and bounded on the space $L_q(\mathbb{C}, d\sigma)$ for all $1 < q < \infty$. Moreover, the singular operator (2.8), as well as that of (1.4), can be interpolated [21][22][23] as operators from $L_{q_1}(\mathbb{C}, d\sigma)$ to $L_{q_2}(\mathbb{C}, d\sigma)$, where $1 < q_1 < q < q_2$, where they become compact, thus presenting a useful tool for studying functional properties of the operator (1.1). So, proceed now to studying them in detail and proving Theorem 1.2.

3. Operator A and its weakly semi-Fredholm structure

We consider the operator mapping (1.1) and observe that, owing to the inequality (2.7), the following estimation

$$\|Af\|_p \leq \|T(af)\|_p + \|b\|_r \|f\|_q \leq (\alpha_q \|a\|_r + \|b\|_r) \|f\|_q, \quad (3.1)$$

holds for $1 < p < q < \infty$, where $r = pq/(q-p)$, and $f \in L_q(\mathbb{C}, d\sigma)$, thus meaning that the corresponding operator norm $\|A\| \leq (\alpha_q \|a\|_r + \|b\|_r)$, that is bounded.

Now we proceed to analyzing some needed preliminaries of our main theorem 1.2 about the weakly semi-Fredholm structure of the bounded operator (1.1) from the Banach space $L_q(\mathbb{C}, d\sigma)$, $1 < q < \infty$, into the Banach space $L_p(\mathbb{C}, d\sigma)$, $q > p \geq 1$.

First of all, take into account that for $q > p \geq 1$ the image $\text{Range } A \subset L_p(\mathbb{C}, d\sigma)$ can not be closed, as $\dim \text{Range } A = \infty$. Moreover, the image $\text{Range } A^* \subset L_q(\mathbb{C}, d\sigma)'$ of the adjoint singular transformation $A^* = A - \{T, b\} : L_p(\mathbb{C}, d\sigma)' \rightarrow L_q(\mathbb{C}, d\sigma)'$, where $\{T, b\} := T \circ b + b \circ T$, is not also closed, as owing to the Banach closed image theorem one obtains both the equality $\text{codim Range } A^* = \dim \text{Ker } A < \infty$ and $\dim \text{Ker } A^* = \infty$, contradicting the condition $\overline{\text{Range } A} < \infty$. Thus, to state that $\dim \text{Ker } A < \infty$, we will construct a new suitably reduced operator on the kernel subspace $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$, next extend it on the whole space $L_q(\mathbb{C}, d\sigma)$ and state its compactness by means of both the interpolating properties of the singular Beurling-Ahlfors operator (1.4) and the classical N. Kolmogorov Theorem 1.3, formulated in the case of the Banach space $L_q(\mathbb{C}, d\sigma)$. As this compact operator on $L_q(\mathbb{C}, d\sigma)$, if reduced on its closed subspace $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$, persists to be compact and its image is closed, we right away derive, as a consequence from the F. Riesz Theorem 1.4, that $\dim \text{Ker } A < \infty$.

Namely, consider the closed kernel subspace $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$ of the operator (1.1), that is the condition $A \text{Ker } A = 0$, which is equivalently rewritten as the equation

$$\mathcal{R}\omega(z)b_+^{-1}[T, a]f = -\mathcal{R}\omega(z)(b_+^{-1}aT + b_-^{-1})f \quad (3.2)$$

for some nonzero mapping $\omega : \mathbb{C} \rightarrow \mathbb{C}$, where $[\cdot, \cdot]$ denotes the usual commutator and

$$\mathcal{R}(\cdot) := -\frac{w_\nu(z)}{\pi} \int_{\mathbb{C}} \frac{(\cdot)d\sigma(\xi, \bar{\xi})}{|\xi - z|^{2-\mu}} \quad (3.3)$$

is the so called fractional Cauchy-Vekua-Riesz [4] operator with parameters $\nu \in (1/r + 1/s - 1/m, 1 - 1/q)$, $\mu < 2$ and $w_\nu(z) := (1 + |z|^2)^{-\nu}$, $z \in \mathbb{C}$. The lefthand side expression of (3.2) defines a

singular operator $A_T : L_q(\mathbb{C}, d\sigma) \rightarrow W_m^\mu(\mathbb{C}, w_\nu d\sigma)$, $1/m = 1/q + 1/r + 1/s - \mu/2$, where

$$A_T f := \mathcal{R}\omega(z)b_+(z)^{-1}(1+|z|^2)^{-\left(\frac{1}{2r}+\frac{1}{2s}\right)}[T, a]f \quad (3.4)$$

for any $f \in L_q(\mathbb{C}, d\sigma)$ and $\text{Range } A_T \subset W_m^\mu(\mathbb{C}, w_\nu d\sigma)$, if the function $\omega(z)b_+^{-1}(z) \in L_s(\mathbb{C}, d\sigma)$, owing to the classical [24] Hardy-Littlewood theorem. Similarly, the righthand side of (3.2) defines a singular operator $B_T : L_q(\mathbb{C}, d\sigma) \rightarrow W_m^\mu(\mathbb{C}, w_\nu d\sigma)$ as

$$B_T f := -\mathcal{R}\omega(z)(b_+^{-1}aT + b_-^{-1})(1+|z|^2)^{-\left(\frac{1}{2r}+\frac{1}{2s}\right)}f \quad (3.5)$$

for any $f \in L_q(\mathbb{C}, d\sigma)$, where $\omega(z)b_+^{-1}a(z), \omega(z)b_-^{-1}(z) \in L_\infty(\mathbb{C}, d\sigma)$ and $\delta < |\omega(z)b_-(z)^{-1}| - |\omega(z)b_+^{-1}a|$ for some $\delta > 0$ almost everywhere. The operators (3.4) and (3.5) are characterized by the following lemma.

Lemma 3.1. *Let a locally continuous function $a \in L_r(\mathbb{C}, d\sigma)$, $r > 1$, and the function factorization $b(z) := b_-(z)^{-1}b_+(z) \in L_r(\mathbb{C}, d\sigma)$, $1/r = 1/p - 1/q$, satisfy for almost all $z \in \mathbb{C}$ the following conditions: there exists a measurable mapping $\omega : \mathbb{C} \rightarrow \mathbb{C}$, such that $\omega(z)b_+(z)^{-1} \in L_s(\mathbb{C}, d\sigma)$, $s > 1$, $\omega(z)b_+^{-1}a(z), \omega(z)b_-^{-1}(z) \in L_\infty(\mathbb{C}, d\sigma)$, where $1/m = 1/q + 1/r + 1/s - \mu/2$ and $\delta < |\omega(z)b_-(z)^{-1}| - |\omega(z)b_+^{-1}a|$ for some $\delta > 0$ almost everywhere. Then 1) the linear operator (3.4) is bounded and compact; 2) the linear operator (3.5) is both bounded and bounded from below and maps the closed subspaces into closed ones.*

Proof. To state the lemma above we first observe that, owing to the classical Coifman-Rochberg-Weiss

[25][12] smoothing property and the Kolmogorov Theorem 1.3, the operator (3.4) is bounded and compact, if to take into account a locally continuity of a function $a \in L_r(\mathbb{C}, d\sigma)$ and the condition $\omega(z)b_+(z)^{-1} \in L_s(\mathbb{C}, d\sigma)$, $s > 1$, imposed above, giving rise jointly with the smoothing property of the Cauchy-Vekua-Riesz operator (3.3) and the limiting condition

$$\lim_{\varepsilon \rightarrow 0} \left[\int_{|z| > 1/\varepsilon} |A_T f(z)|^m w_\nu(z) d\sigma(z, \bar{z}) + \left(\int_{|\xi| > 1/\varepsilon} \int_{|z| > 1/\varepsilon} \frac{|A_T f(z) - A_T f(\xi)|^m}{|z - \xi|^{2+m\mu}} w_\nu(z) w_\nu(\xi) d\sigma(z, \bar{z}) d\sigma(\xi, \bar{\xi}) \right)^{1/m} \right] = 0$$

for any $f \in L_q(\mathbb{C}, d\sigma)$, $q > 1$, to the compact embedding of the image $\text{Range } A_T \subset W_m^\mu(\mathbb{C}, w_\nu d\sigma)$, if $1/m = 1/r + 1/q + 1/s - \mu/2$. Moreover, owing to the Vecua type multiplier condition $\omega(z)b_+^{-1}a(z), \omega(z)b_-^{-1}(z) \in L_\infty(\mathbb{C}, d\sigma)$, as well as $\delta < |\omega(z)b_-(z)^{-1}a(z)| - |\omega(z)b_+^{-1}a|$ for some $\delta > 0$ almost everywhere, $\nu \in (1/r + 1/s - 1/m, 1 - 1/q)$, $\mu < 2$, the smoothing property of the Cauchy-Vekua-Riesz operator (3.3) $\mathcal{R} : L_m(\mathbb{C}, d\sigma) \rightarrow W_m^\mu(\mathbb{C}, w_\nu d\sigma)$ makes the operator (3.5) both

bounded and bounded from below on $L_q(\mathbb{C}, d\sigma)$. The latter gives rise to its mapping closed subspaces into closed ones. $\square$

Lemma 3.2. *Let $q > p > 1$, $1/r = 1/p - 1/q$, and $a, b \in L_r(\mathbb{C}, d\sigma)$. Then the Range $A \subset L_p(\mathbb{C}, d\sigma)$ of the operator (1.1) is not closed.*

Proof. Suppose that the Range $A \subset L_p(\mathbb{C}, d\sigma)$ is closed. Then operator (1.1) induces a bounded bijection $L_q(\mathbb{C}, d\sigma)/\text{Ker } A \rightarrow L_p(\mathbb{C}, d\sigma)$, which is isomorphism by the Banach open mapping Theorem [19]. Hence, there exists a bounded constant $C > 0$, for which

$$\text{dist}_{L_q}(f, \text{Ker } A) \leq C \|Af\|_p \quad (3.6)$$

for all $f \in L_q(\mathbb{C}, d\sigma)$. Fix now $z_0 \in \mathbb{C}$ and for small $\varepsilon > 0$ take $B_\varepsilon := B(z_0; \varepsilon) \subset \mathbb{C}$ and construct for any $f \in L_q(\mathbb{C}, d\sigma)$ the function

$$f_\varepsilon := |B_\varepsilon|^{-1/q} \chi_{B_\varepsilon}, \|f_\varepsilon\|_q = 1. \quad (3.7)$$

By Hölder's inequality with exponents r/p and $(1 - p/r)^{-1}$ one obtains

$$\|af_\varepsilon\|_p^p = |B_\varepsilon|^{-p/q} \int_{B_\varepsilon} |a|^p d\sigma \leq |B_\varepsilon|^{-p/q} \int_{B_\varepsilon} |a|^r d\sigma |B_\varepsilon|^{1-p/r} = \|a\|_{L_q(B_\varepsilon, d\sigma)}^p, \quad (3.8)$$

as $-p/q + 1 - p/r = -p/q + 1 - p(1/p - 1/q) = 0$, canceling the powers of $|B_\varepsilon|$ exactly. Since $a \in L_r(\mathbb{C}, d\sigma)$, absolute continuity of the integral (3.8) gives $\lim_{\varepsilon \rightarrow 0} \|a\|_{L_q(B_\varepsilon, d\sigma)}^p = 0$, and likewise for $b \in L_r(\mathbb{C}, d\sigma)$ too. As the Beurling-Ahlfors operator $T : L_p(\mathbb{C}, d\sigma) \rightarrow L_p(\mathbb{C}, d\sigma)$ is bounded for $p > 1$, we obtain that

$$\|Af_\varepsilon\|_p \leq \alpha_p \|af_\varepsilon\|_p + \|bf_\varepsilon\|_p \leq \alpha_p \|a\|_{L_q(B_\varepsilon, d\sigma)}^p + \|b\|_{L_q(B_\varepsilon, d\sigma)}^p \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (3.9)$$

Moreover, $f_\varepsilon \rightharpoonup 0$ weakly in $L_q(\mathbb{C}, d\sigma)$, as $\varepsilon \rightarrow 0$, since $|\int_{L_q(B_\varepsilon, d\sigma)} f_\varepsilon g d\sigma| \leq \|f_\varepsilon\|_q \|g\|_{L_{q'}(B_\varepsilon, d\sigma)} \xrightarrow{\varepsilon \rightarrow 0} 0$ for any $g \in L_{q'}(\mathbb{C}, d\sigma)$, $1/q + 1/q' = 1$. By (3.6) we can write down $f_\varepsilon = k_\varepsilon + \rho_\varepsilon$ with $k_\varepsilon \in \text{Ker } A$ and $\|\rho_\varepsilon\|_q \rightarrow 0$, as $\varepsilon \rightarrow 0$, so $\|k_\varepsilon\|_q \rightarrow 1$, as $\varepsilon \rightarrow 0$. Now taking into account that the subspace $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$ is finite dimensional, can be chosen a sequence $\varepsilon_j \rightarrow 0$, as $j \rightarrow \infty$, such that $k_{\varepsilon_j} \rightarrow k \in L_q(\mathbb{C}, d\sigma)$, as $j \rightarrow \infty$, and $\|k\|_q = 1$. But, $k_{\varepsilon_j} = f_{\varepsilon_j} - \rho_{\varepsilon_j} \rightharpoonup 0$ weakly in $L_q(\mathbb{C}, d\sigma)$, as $j \rightarrow \infty$, so $k = 0$, contradicting with the condition $\|k\|_q = 1$. $\square$

Proof. (Proof of the main Theorem 1.2) As the operator $A_T : L_q(\mathbb{C}, d\sigma) \rightarrow W_m^\mu(\mathbb{C}, w_\nu d\sigma)$ is compact, its reduction $A_T|_{\text{Ker } A} : \text{Ker } A \rightarrow W_m^\mu(\mathbb{C}, w_\nu d\sigma)$ on the closed subspace $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$ is also compact. Since there holds equality $A_T \text{Ker } A = B_T \text{Ker } A$, the image $B_T \text{Ker } A \subset W_m^\mu(\mathbb{C}, w_\nu d\sigma)$ is also closed. Then, based on the F. Riesz compactness Theorem 1.4, we derive the boundedness of the dim Range

$A_T|_{\text{Ker } A}$ and, as a consequence, the boundedness $\dim \text{Ker } A < \infty$. Really, if $(f_n)_{n \in \mathbb{N}}$ is a bounded sequence in $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$, by the compactness of the operator $A_T : L_q(\mathbb{C}, d\sigma) \rightarrow W_m^\mu(\mathbb{C}, w_\nu d\sigma)$ we obtain the existence of some subsequence $(f_{n_j})_{j \in \mathbb{N}}$ in $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$, for which the sequence $(A_T f_{n_j})_{j \in \mathbb{N}}$ converges in $W_m^\mu(\mathbb{C}, w_\nu d\sigma)$, hence it is Cauchy. Taking now the above equality $A_T \text{Ker } A = B_T \text{Ker } A$, we can easily derive that

$$\| (A_T f_{n_j})_{j \in \mathbb{N}} \|_{W_m^\mu(\mathbb{C}, w_\nu d\sigma)} \leq \| (B_T f_{n_j})_{j \in \mathbb{N}} \|_{W_m^\mu(\mathbb{C}, w_\nu d\sigma)} = \| (A_T f_{n_j})_{j \in \mathbb{N}} \|_{W_m^\mu(\mathbb{C}, w_\nu d\sigma)} \rightarrow 0.$$

So, the subsequence $(f_{n_j})_{j \in \mathbb{N}}$ in $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$ is Cauchy too, thus converging to some element of $\text{Ker } A$, which is closed. The latter means that the bounded sequences in $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$ have convergent subsequences, that gives rise, owing to the F. Riesz Theorem 1.4, the boundedness of the $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$. Now, taking into account Lemma 3.2, we derive that the image $\text{Range } A \subset L_p(\mathbb{C}, d\sigma)$ is not closed, thus proving Theorem 1.2. $\square$

Remark 3.3. Taking into account the properties (2.2) of the P - operator (2.1), the above equation (3.2) can be reformulated as the following partial differential equation

$$b_+ \frac{\partial P f(z)}{\partial \bar{z}} + b_- \frac{\partial P (af)(z)}{\partial z} = 0, \quad (3.10)$$

generating quasi-conformal mappings of the complex plane $\mathbb{C}$, and whose solutions naturally generate the kernel $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$ of the operator (1.1). This observation provides an alternative viewpoint on equation (3.2) and a more detailed analysis of the subspace $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$ makes it possible to establish Theorem 1.2 in a somewhat more general form. It is also worth to mention that similar to (3.10) equations were extensively studied in [26], and analytical techniques there devised can be useful for studying problems, treated in the present work.

4. Conclusion

Based on the functional analytic properties of the singular Calderon-Zygmund type integral operator

$$Tf(z) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|z-\xi|>\varepsilon} \frac{f(z) - f(\xi)}{(z - \xi)^2} d\sigma(\xi, \bar{\xi}),$$

on the Banach space $L_q(\mathbb{C}, d\sigma)$, we have studied the kernel $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$ of the linear singular operator expression

$$Af(z) := T(a(z)f(z)) + b(z)f(z) \quad (4.1)$$

with a locally continuous coefficient $a \in L_r(\mathbb{C}, d\sigma)$, a coefficient $b \in L_r(\mathbb{C}, d\sigma)$, $r > 1$, whose factorization $b = b_+^{-1}b_-$ is chosen such that for some measurable mapping $\omega : \mathbb{C} \rightarrow \mathbb{C}$ there hold the embeddings $\omega(z)b_+(z)^{-1} \in L_s(\mathbb{C}, d\sigma)$, $s > 1$, and $\omega(z)b_+^{-1}a(z), \omega(z)b_-^{-1}(z) \in L_\infty(\mathbb{C}, d\sigma)$, where $1/m = 1/q + 1/r + 1/s - \mu/2$, $\nu \in (1/r + 1/s - 1/m, 1 - 1/q)$, $\mu < 2$, and $\delta < |\omega(z)b_-(z)^{-1}a(z)| - |\omega(z)b_+^{-1}a|$ for some $\delta > 0$ almost everywhere. Making use of the reduced operator approach and classical Coifman-Rochberg-Weiss smoothing property and the Kolmogorov-F. Riesz theorems, we succeeded in stating the weakly semi-Fredholm property of the above operator expression (4.1).

Notes

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Data Availability

All available data are contained in the manuscript.

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