Research Article

On the Weakly Semi-Fredholm Structure of a Class of Singular Beurling-Ahlfors Type Operator Equations

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We studied the solution space of a linear singular Hilbert type operator equation $T\left(a(z)f(z)\right)+b(z)f(z)=0$ **in the Banach space** $L_q(\mathbb{C},d\sigma), q>p>1$, **specified by coefficients** $a,b\in L_r(\mathbb{C},d\sigma)$ **on the complex plane** $\mathbb{C}$, **and the operator action** $Tf(z)=\frac{1}{\pi}v.p.\int_{\mathbb{C}}\frac{f(z)-f(\xi)}{(z-\xi)^2}d\sigma(\xi,\bar{\xi}), f\in L_p(\mathbb{C},d\sigma)$, **where** $1/r+1/q=1/p$. **Based on the classical Coifman-Rochberg-Weiss commutator and Uchiyama inverse transforms together with the F. Riesz local compactness theorem, we stated conditions under which the singular operator equation under regard possesses the weakly semi-Fredholm structure.**

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1. Introduction

Singular operator equations on complex domains are widely used [1][2][3][4] in many investigations of important problems of modern mechanics, radiophysics, solid state physics and other applied sciences. Their theory is closely related [5][2][6] with partial differential equations, geometry of complex-analytic and quasi-conformal mappings. In particular, the following [1][7] class of expressions

$$Af(z):=T\left(a(z)f(z)\right)+b(z)f(z), \tag{1.1}$$

characterized by its coeficients $a,b\in L_r(\mathbb{C},d\sigma)$, $r>1$, and acting on complex function $f\in L_q(\mathbb{C},d\sigma), q>1$, endowed with the norm

$$\|f\|_q=\left(\int_{\mathbb{C}}|f(\xi)|^q d\sigma(\bar{\xi},\xi)\right)^{1/q} \tag{1.2}$$

with respect to the Lebesgue measure $d\sigma(\xi,\bar{\xi}) = \quad d\bar{\xi}\wedge d\xi/(2i)$ on the complex plane $\mathbb{C}$, define singular integral operator equations

$$Af(z) = 0, \tag{1.3}$$

specified by the singular Beurling-Ahlfors [8][9][10][11][12][13] transform

$$Tf(z) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|z-\xi|>\varepsilon}\frac{f(z)-f(\xi)}{(z-\xi)^2}d\sigma(\xi,\bar{\xi}), \tag{1.4}$$

which is equivalent to the Kalderon-Zygmund [9] mapping $T = \partial_z\partial_{\bar{z}}^{-1} : L_q(\mathbb{C},d\sigma)\to L_q(\mathbb{C},d\sigma), q>1$, for all $z\in\mathbb{C}$.

Let us introduce the following definition, which reflects [7][2][11] some weaker versions of the classical Fredholm operators.

Definition 1.1. A bounded operator $A : L_q(\mathbb{C},d\sigma)\to L_p(\mathbb{C},d\sigma)$ with closed $\mathrm{Range}\ A\subset L_p(\mathbb{C},d\sigma)$ is called semi-Fredholm, if its kernel $\mathrm{Ker}\ A\subset L_q(\mathbb{C},d\sigma)$, or its cokernel $\mathrm{coKer\ A} = L_p(\mathbb{C},d\sigma)/\ \overline{\mathrm{Range}A}$ is finite dimensional. If the image $\mathrm{Range}\ A\subset L_p(\mathbb{C},d\sigma)$ is not closed and its kernel $\mathrm{Ker}A\subset L_q(C,d\sigma)$ is finite dimensional, the bounded operator $A : L_q(\mathbb{C},d\sigma)\to L_p(\mathbb{C},d\sigma)$ is called weakly semi-Fredholm.

In particular, our special interest lies in proving the weakly semi-Fredholm structure of the operator (1.1), which consists in stating the finite dimension of its kernel $\mathrm{Ker}\ A\subset L_q(\mathbb{C},d\sigma)$ and the notclosedness of its $\mathrm{Range}\ A\subset L_p(\mathbb{C},d\sigma)$. Based on the functional analytic properties of the Beurling-Ahlfors singular transform (1.4), as well as on the structural properties of the singular operator mapping (1.4) on the Banach space $L_p(\mathbb{C},d\sigma)$, we stated our main theorem.

Theorem 1.2. *The bounded operator (1.1) from the Banach space $L_q(\mathbb{C},d\sigma)$ into the Banach space $L_p(\mathbb{C},d\sigma)$, $q>p>1$, is weakly semi-Fredholm, that is $\dim\mathrm{Ker}A<\infty$ and the $\mathrm{Range}\ A\subset L_p(\mathbb{C},d\sigma)$ is not closed, if its coefficients $a\in L_r(\mathbb{C},d\sigma)\cap W^1_{q(r)}(\mathbb{C},d\lambda), r>1, q(r)>2$, $b\in L_r(\mathbb{C},d\sigma), r>1$, and the factorization $b=b_-^{-1}b_+$ is such that for some measurable mapping $\omega:\mathbb{C}\to\mathbb{C}\backslash\{0\}$ there hold conditions: $\omega b_+^{-1}\in L_s(\mathbb{C},d\sigma), s>1$, $\omega b_-^{-1}\in L_{m(s)}(\mathbb{C},d\sigma), 1/m(s)=1/r+1/s$, as well as $\mathrm{esssup}_{z\in\mathbb{C}}|\alpha_q b_+^{-1}(z)b_-(z)|<1$, where there exists such $m>1$ that $1/m=1/q+1/r+1/s-(\nu+\mu)/2$ and $\nu\in(0,1+\mu/2)$.*

The above statement in its main part depends on the following classical functional analysis N. Kolmogorov and F. Riesz theorems [14][15][16][17][18], describing linear mappings between Banach spaces:

Theorem 1.3. *(F. Riesz) If a bounded operator $A : X\to Y$ from a Banach space X to a Banach space Y is compact and its image $\mathrm{I}mA\subset Y$ is closed, then $\dim(AX)<\infty$. Moreover, a linear subspace of a normed space X is locally compact if and only if it is finite dimensional.*

Below we proceed to studying the functional analytic properties of the singular Hilbert type transform (1.4) and next proving the weakly semi-Fredholm property of the linear singular operator (1.1).

2. Singular Beurling-Ahlfors operator T and its properties

We consider the Cauchy type integral

$$Pf(z) := \frac{1}{2\pi i}\int_{\mathbb{C}} f(\xi)\left(\frac{1}{\xi - z} - \frac{1}{\xi}\right) d\sigma(\xi, \bar{\xi}) \tag{2.1}$$

for a complex variable $z \in \mathbb{C}$ on the dense subspace $C_0^2(\mathbb{C}, d\sigma) \subset L_q(\mathbb{C}, d\sigma)$ for $q > 1$, naturally defining [8][5][4] the following two functions:

$$\partial Pf(z)/\partial \bar{z} = f(z),\ \partial Pf(z)/\partial z = Tf(z) \tag{2.2}$$

for any point $z \in \mathbb{C}$. It is easy to check that the Beurling-Ahlfors operator $T = \partial_z \partial_{\bar{z}}^{-1} : L_q(\mathbb{C}, d\sigma) \to L_q(\mathbb{C}, d\sigma)$ for $z \in \mathbb{C}$ and $2 < q < \infty$. Moreover, applying the Holder inequality to the expression (2.1), we obtain

$$|Pf(z)| \le K_q \|f\|_q |z|^{\frac{-2}{q}+1} \tag{2.3}$$

for arbitrary function $f \in L_q(\mathbb{C}, d\sigma)$ and any $z \in \mathbb{C}$, where $K_q = \left(\int_{\mathbb{C}} |\xi|^{\frac{-q}{q-1}} |(\xi - 1)|^{\frac{q}{1-q}} d\sigma(\xi)\right)^{\frac{q-1}{q}} < \infty$ for $q > 2$. In addition, from the following identity

$$\begin{gathered} Pf_{z_1}(z_2 - z_1) = \tfrac{1}{2\pi i}\textstyle\int_{\mathbb{C}} f(\xi)\left(\frac{1}{\xi - z_2} - \frac{1}{\xi - z_1}\right) d\sigma(\xi) = \\ = Pf(z_2) - Pf(z_1) \end{gathered} \tag{2.4}$$

for the shifted function $f_{z_1}(\xi) := f(\xi + z_1), \xi \in \mathbb{C}$, and the estimation (2.3) one derives the inequality

$$|Pf(z_1) - Pf(z_2)| \ \le K_q \left\|f\right\|_q |z_2 - z_1|^{\frac{-2}{q}+1} \tag{2.5}$$

for arbitrary $z_1, z_2 \in \mathbb{C}$, meaning the Hölder continuity of the mapping (2.1). To formulate our main statement about the solution set of the equation (1.3), we first state that the operator $T : L_q(\mathbb{C}, d\sigma) \to L_q(\mathbb{C}, d\sigma)$ for $1 < q < \infty$ is bounded. To do this, we first observe this property for the particular case $q = 2$, having calculated the integral

$$\begin{gathered} \textstyle\int_{\mathbb{C}} |Tf(z)|^2 d\sigma(z, \bar{z}) = \int_{\mathbb{C}} |Tf(z)|^2 d\sigma(z, \bar{z}) = \\ = \textstyle\int_{\mathbb{C}} \partial^2 \overline{Pf(z)}/\partial\bar{z}\partial z\ Pf(z) d\sigma(z, \bar{z}) = \\ = \textstyle\int_{\mathbb{C}} \partial \overline{f(z)}/\partial\bar{z} Pf(z) d\sigma(z, \bar{z}) = \int_{\mathbb{C}} \overline{f(z)} \partial Pf(z)/\partial\bar{z} d\sigma(z, \bar{z}) = \\ = \textstyle\int_{\mathbb{C}} \overline{f(z)} f(z) d\sigma(z, \bar{z}) = \int_{\mathbb{C}} |f(z)|^2 d\sigma(z, \bar{z}), \end{gathered} \tag{2.6}$$

where we took into account that the space $C_0^2(\mathbb{C},d\sigma)\subset L_2(\mathbb{C},d\sigma)$ is dense in $L_2(\mathbb{C},d\sigma)$. The result (2.6) evidently means that the mapping $T:L_2(\mathbb{C},d\sigma)\to L_2(\ \mathbb{C},d\sigma))$ is isometry of the Hilbert space $L_2(\mathbb{C},d\sigma)$with the norm $\|T\|=1$. Moreover, for any $f\in L_q(\mathbb{C},d\sigma)$ there holds [8][4][13], the following general Zygmund-Kalderon inequality

$$\|Tf\|_q\leq\alpha_q\|f\|_q, \tag{2.7}$$

where $\alpha_q>0, 1<q<\infty,$ are some bounded constants. The estimation (2.7) in its main part significantly depends on the operator representation $T=-S^2:L_q(\mathbb{C},d\sigma)\to L_q(\mathbb{C},d\sigma),$ where, by definition,

$$Sf(z)=\frac{1}{2\pi i}\int_{\mathbb{C}}\frac{f(\xi)-f(z)}{(\xi-z)\,|\xi-z|}d\sigma(\xi,\bar{\xi}). \tag{2.8}$$

The latter is considered on the dense subspace $C_0^2(\mathbb{C},d\sigma)\subset\ L_q(\mathbb{C},d\sigma)$ for $q\geq 2$and satisfies the following inequality

$$\|Sf\|_q\ \leq\sqrt{\alpha_q}\|f\|_q \tag{2.9}$$

for $f\in C_0^2(\mathbb{C},d\sigma)\subset L_2(\mathbb{C},d\sigma)$ and some constant $\alpha_q>0$, $q\geq 2$, which makes it possible to extend the operator $T=-S^2:C_0^2(\mathbb{C},d\sigma)\to L_q(\mathbb{C},d\sigma)$ by continuity on the whole space $L_q(\mathbb{C},d\sigma)$, $q\geq 2$, and to obtain the classical Zygmund-Calderón inequality (2.7) for all $f\in L_q(\mathbb{C},d\sigma),\ q\geq 2$. Based on the properties of the adjoint operator $T^*=T:\ L_{q'}(\mathbb{C},d\sigma)\to L_{q'}(\mathbb{C},d\sigma),\ 1<q'\leq 2,$ one finally obtains that the operator $T:L_q(\mathbb{C},d\sigma)\to L_q(\mathbb{C},d\sigma)$ is well defined and bounded on the space $L_q(\mathbb{C},d\sigma)$ for all $1<q<\infty$. Moreover, the singular operator (2.8), as well as that of (1.4), can be interpolated [19][20][21] [11] as operators from $L_{q_1}(\mathbb{C},d\sigma)$ to $L_{q_2}(\mathbb{C},d\sigma)$, where $1<q_1<q<q_2$, where they become compact, thus presenting a useful tool for studying functional properties of the operator (1.1). So, proceed now to studying them in detail and proving Theorem 1.2.

3. Operator A and its weakly semi-Fredholm structure

We consider the operator mapping (1.1) and observe that, owing to the inequality (2.7), the following estimation

$$\|Af\|_p\leq\|T(af)\|_p+\|b\|_r\|f\|_q\leq(\alpha_q\|a\|_r+\|b\|_r)\|f\|_q, \tag{3.1}$$

holds for $1<p<q<\infty,$ where$1/r=1/p-1/q,$ and $f\in\ L_q(\mathbb{C},d\sigma),$ thus meaning that the corresponding operator norm $\|A\|\leq(\alpha_q\,||a||_r+||b||_r)$, that is bounded.

Now we proceed to analyzing some needed preliminaries of our main theorem 1.2 about the weakly semi-Fredholm structure of the bounded operator (1.1) from the Banach space $L_q(\mathbb{C}, d\sigma)$, into the Banach space $L_p(\mathbb{C}, d\sigma), q > p > 1$. First of all, assume that for $q > p > 1$ the image $\mathrm{Range}\ A \subset L_p(\mathbb{C}, d\sigma)$ is not closed, and $\dim \mathrm{Range}\ A = \infty$. As the image $\mathrm{Range}\ A^* \subset L_q(\mathbb{C}, d\sigma)'$ of the adjoint singular transformation $A^* = A - \{T, b\} : L_p(\mathbb{C}, d\sigma)' \to L_q(\mathbb{C}, d\sigma)'$, where $\{T, b\} := T \circ b + b \circ T$, is not also closed, hence owing to the Banach closed image theorem one obtains both the equality $\mathrm{codim}\ \overline{\mathrm{Range}\ A^*}$ $= \dim \mathrm{Ker}\ A < \infty$ and $\dim \mathrm{Ker}\ A^* = \infty$. For further analysing the weakly semi-Fredholm structure of the bounded operator (1.1) we preliminarily formulate the following lemma, which is of independent interest.

Lemma 3.1. *(bounded below versus compact) Let X and Z be Banach spaces, $M \subseteq X$ a closed subspace, B_T and $C_T : X \to Z$ bounded with: (i) $\|B_T f\|_Z \geq c\|f\|_X$ for all $f \in X$ and some $c > 0$; (ii) C_T is compact; (iii) $B_T f = C_T f$ for all $f \in M$. Then* $\dim M < \infty$.

Proof. Let a sequence $(f_n)_{n\in\mathbb{N}}$ be bounded in M. By (ii) there exists a convergent subsequence $(C_T f_{n_j})_{j\in\mathbb{N}}$ in Z, hence Cauchy; by (iii) and (i) one has $c\|f_{n_j} - f_{n_k}\|_X \leq \|C_T f_{n_j} - C_T f_{n_k}\|_Z \overset{j,k\to\infty}{\to} 0$, so $(f_n)_{n\in\mathbb{N}}$ is Cauchy and converges in the closed subspace $M \subseteq X$. Bounded sequences in M thus have convergent subsequences, thus $\dim M < \infty$ by F. Riesz Theorem 1.3. □

Thus, to state that $\dim \mathrm{Ker} A < \infty$, it is enough, owing to the Lemma 3.1, to construct a new suitably reduced operators C_T and B_T on the kernel subspace $\mathrm{Ker}\ A \subset L_q(\mathbb{C}, d\sigma)$into some Sobolew space $W_m^\mu(\mathbb{C}, w_\nu d\sigma)$, on which there holds the condition $C_T = B_T|_{\mathrm{Ker} A}$,and, next, to extend them on the whole space $L_q(\mathbb{C}, d\sigma)$, on which the first one is compact and the second one maps closed subspaces into closed ones, making use of the classical Coifman-Rochberg-Weiss [22] commutator and Uchiyama [12] inverse transformations approaches. As the compact operator $C_T : L_q(\mathbb{C}, d\sigma) \to W_m^\mu(\mathbb{C}, w_\nu d\sigma)$, reduced on its closed subspace $\mathrm{Ker}\ A \subset L_q(\mathbb{C}, d\sigma)$, persists to be compact and its image is closed, we right away derive, as a consequence from the F. Riesz Theorem 1.3, that $\dim \mathrm{Ker}\ A < \infty$.

Namely, consider the closed kernel subspace $\mathrm{Ker}\ A \subset L_q(\mathbb{C}, d\sigma)$of the operator (1.1), that is the condition $A\ \mathrm{Ker}\ A = 0$, which is equivalently rewritten as the equation

$$\mathscr{R}\omega(z)b_+^{-1}[T, a]f = \quad -\mathscr{R}(\omega(z)b_+^{-1}aT + \omega(z)b_-^{-1})f \tag{3.2}$$

for some nonzero mapping $\omega : \mathbb{C} \to \mathbb{C}\backslash\{0\}$, where $[\cdot, \cdot]$ denotes the usual commutator and

$$\mathscr{R}(\cdot) := -\frac{w_\nu(z)}{\pi}\int_{\mathbb{C}} \frac{(\cdot)d\sigma(\xi, \bar{\xi})}{|\xi - z|^{2-\mu}} \tag{3.3}$$

is the so called fractional Cauchy-Vekua-Riesz [4] operator with paramerters $\nu \in (0, 1+\mu/2)$ and $w_\nu(z) := (1+|z|^2)^{-\nu}, z \in \mathbb{C}$. The lefthand side expression of (3.2) defines a singular operator $A_T : L_q(\mathbb{C}, d\sigma) \to W^\mu_m(\mathbb{C}, w_\nu d\sigma), 1/m = 1/q + 1/r + 1/s - (\mu+\nu)/2$, where

$$A_T f := \quad \mathscr{R}\omega(z) b_+(z)^{-1}[T, a]f \tag{3.4}$$

for any $f \in L_q(\mathbb{C}, d\sigma)$ and Range $A_T \subset W^\mu_m(\mathbb{C}, w_\nu d\sigma)$, if the function $\omega b_+^{-1} \in L_s(\mathbb{C}, d\sigma)$, owing to the classical [23] Hardy-Littlwood theorem. Similarly, the righthand side of (3.2) defines a singular operator $B_T : L_q(\mathbb{C}, d\sigma) \to W^\mu_m(\mathbb{C}, w_\nu d\sigma)$ as

$$B_T f := -\mathscr{R}(\omega b_+^{-1} a T + \omega b_-^{-1}) f \tag{3.5}$$

for any $f \in L_q(\mathbb{C}, d\sigma)$, where $\omega b_+^{-1} \in L_s(\mathbb{C}, d\sigma), \omega b_-^{-1} \in L_{m(s)}(\mathbb{C}, d\sigma), 1/m(s) = 1/r + 1/s$, and $\operatorname{esssup}_{\mathbb{C}} |\alpha_q b_+^{-1}(z) b_-(z)| < 1$. The operators (3.4) and (3.5) are characterized by the following lemma.

Lemma 3.2. *Let a locally continuous function $a \in L_r(\mathbb{C}, d\sigma) \cap W^1_{q(r)}(\mathbb{C}, d\lambda), r > 1, q(r) > 2$, and the function factorization $b := b_-^{-1} b_+ \in L_r(\mathbb{C}, d\sigma)$, $1/r = 1/p - 1/q$, satisfies the following conditions: there exists a measurable mapping $\omega : \mathbb{C} \to \mathbb{C}$, such that $\omega b_+^{-1} \in L_s(\mathbb{C}, d\sigma), s > 1$, $\omega b_-^{-1} \in L_{m(s)}(\mathbb{C}, d\sigma), 1/m(s) = 1/r + 1/s$, where $1/m = 1/q + 1/r + 1/s -$ $(\nu+\mu)/2$ and $\operatorname{esssup}_{z\in\mathbb{C}} |\alpha_q b_+^{-1}(z) b_-(z)| < 1$. Then 1) the linear operator (3.4) is bounded and compact; 2) the linear operator (3.5) is bounded from below and maps the closed subspaces into closed ones.*

Proof. To state the lemma above we first observe that, owing to the classical Coifman-Rochberg-Weiss [22] [12] commutator and Uchiyama [12] inverse transformation teqcniques the operator $A_T : L_q(\mathbb{C}, d\sigma) \to W^\mu_m(\mathbb{C}, w_\nu d\sigma)$ is bounded and compact, if to take into account the condition $a \in L_r(\mathbb{C}, d\sigma) \cap W^1_{q(r)}(\mathbb{C}, d\lambda), r > 1, q(r) > 2$, and the condition $\omega b_+^{-1} \in L_s(\mathbb{C}, d\sigma), s > 1$, emposed above. If a measurable mapping $\omega : \mathbb{C} \to \mathbb{C}$ is chosen such that $\omega b_-^{-1} \in L_{m(s)}(\mathbb{C}, d\sigma)$, $1/m(s) = 1/r + 1/s$, then the operator $B_T : L_q(\mathbb{C}, d\sigma) \to W^\mu_m(\mathbb{C}, w_\nu d\sigma)$, where $1/m = 1/q + 1/r$ $+1/s-$ $(\nu+\mu)/2$ and $\operatorname{esssup}_{z\in\mathbb{C}} |\alpha_q b_+^{-1}(z) b_-(z)| < 1$, maps the closed subspaces into closed ones. Moreover, since it is injective, by the Banach open mapping theorem, it is bounded from below, that is there exists $c > 0$, such that $\|B_T f\|_{W^\mu_m} \geq c\|f\|_p$ for all $f \in L_q(\mathbb{C}, d\sigma)$. $\square$

Lemma 3.3. *Let $q > p > 1, 1/r = 1/p - 1/q$, and $a, b \in L_r(\mathbb{C}, d\sigma)$. Then the Range $A \subset L_p(\mathbb{C}, d\sigma)$ of the operator (1.1) is not closed.*

Proof. Suppose that the Range $A \subset L_p(\mathbb{C}, d\sigma)$ is closed. Then operator (1.1) induces a bounded bijection $L_q(\mathbb{C}, d\sigma)/\operatorname{Ker} A \to \operatorname{Range} A$, which is an isomorphism by the Banach open mapping Theorem [24][17]

[25]. Hence, there exists a bounded constant $c > 0$, for which

$$\operatorname{dist}_{L_q}(f, \operatorname{Ker} A) \leq c\|Af\|_p \tag{3.6}$$

for all $f \in L_q(\mathbb{C}, d\sigma)$. Fix now $z_0 \in \mathbb{C}$ and for small $\varepsilon > 0$ take $B_\varepsilon := B(z_0; \varepsilon) \subset \mathbb{C}$ and construct for any $f \in L_q(\mathbb{C}, d\sigma)$ the function

$$f_\varepsilon := |B_\varepsilon|^{-1/q} \chi_{B_\varepsilon}, \|f_\varepsilon\|_q = 1. \tag{3.7}$$

By Hölder's inequality with exponents r/p and $(1 - p/r)^{-1}$ one obtains

$$\|af_\varepsilon\|_p^p = |B_\varepsilon|^{-p/q} \int_{B_\varepsilon} |a|^p d\sigma \leq |B_\varepsilon|^{-p/q} \int_{B_\varepsilon} |a|^r d\sigma |B_\varepsilon|^{1-p/r} = \|a\|_{L_q(B_\varepsilon, d\sigma)}^p, \tag{3.8}$$

as $-p/q + 1 - p/r = -p/q + 1 - p(1/p - 1/q) = 0$, canceling the powers of $|B_\varepsilon|$ exactly. Since $a \in L_r(\mathbb{C}, d\sigma)$, absolute continuity of the integral (3.8) gives $\lim_{\varepsilon \to 0} \|a\|_{L_q(B_\varepsilon, d\sigma)}^p = 0$, and likewise for $b \in L_r(\mathbb{C}, d\sigma)$ too. As the Beurling-Ahlfors operator $T : L_p(\mathbb{C}, d\sigma) \to L_p(\mathbb{C}, d\sigma)$ is bounded for $p > 1$, we obtain that

$$\|Af_\varepsilon\|_p \leq \alpha_p \|af_\varepsilon\|_p + \|bf_\varepsilon\|_p \leq \alpha_p \|a\|_{L_q(B_\varepsilon, d\sigma)}^p + \|b\|_{L_q(B_\varepsilon, d\sigma)}^p \overset{\varepsilon \to 0}{\to} 0. \tag{3.9}$$

Moreover, $f_\varepsilon \rightharpoonup 0$ weakly in $L_q(\mathbb{C}, d\sigma)$, as $\varepsilon \to 0$, since $|\int_{L_q(B_\varepsilon, d\sigma)} f_\varepsilon g d\sigma| \leq \|f_\varepsilon\|_q \|g\|_{L_{q'}(B_\varepsilon, d\sigma)} \overset{\varepsilon \to 0}{\to} 0$ for any $g \in L_{q'}(\mathbb{C}, d\sigma), 1/q + 1/q' = 1$. By (3.6) we can write down $f_\varepsilon = k_\varepsilon + \rho_\varepsilon$ with $k_\varepsilon \in \operatorname{Ker} A$ and $\|\rho_\varepsilon\|_q \to 0$, as $\varepsilon \to 0$, so $\|k_\varepsilon\|_q \to 1$, as $\varepsilon \to 0$. Now taking into account that the subspace $\operatorname{Ker} A \subset L_q(\mathbb{C}, d\sigma)$ is finite dimensional, can be chosen a sequence $\varepsilon_j \to 0$, as $j \to \infty$, such that $k_{\varepsilon_j} \to k \in L_q(\mathbb{C}, d\sigma)$, as $j \to \infty$, and $\|k\|_q = 1$. But, $k_{\varepsilon_j} = f_{\varepsilon_j} - \rho_{\varepsilon_j} \rightharpoonup 0$ weakly in $L_q(\mathbb{C}, d\sigma)$, as $j \to \infty$, so $k = 0$, contradicting with the condition $\|k\|_q = 1$. $\square$

Proof. (*Proof of the main Theorem* 1.2) As the operator $A_T : L_q(\mathbb{C}, d\sigma) \to W_m^\mu(\mathbb{C}, w_\nu d\sigma)$ is compact, its reduction $A_T|_{\operatorname{Ker} A} : \operatorname{Ker} A \to W_m^\mu(\mathbb{C}, w_\nu d\sigma)$ on the closed subspace $\operatorname{Ker} A \subset L_q(\mathbb{C}, d\sigma)$ is also compact. Since there holds equality $A_T \operatorname{Ker} A = B_T \operatorname{Ker} A$, the image $B_T \operatorname{Ker} A \subset W_m^\mu(\mathbb{C}, w_\nu d\sigma)$ is also closed. Then, based on the F. Riesz compactness Theorem 1.3, we derive the boundedness of the $\dim \operatorname{Range} A_T|_{\operatorname{Ker} A}$ and, as a consequence, the boundedness $\dim \operatorname{Ker} A < \infty$. Really, if $(f_n)_{n \in \mathbb{N}}$ is a bounded sequence in $\operatorname{Ker} A \subset L_q(\mathbb{C}, d\sigma)$, by the compactness of the operator $A_T : L_q(\mathbb{C}, d\sigma) \to W_m^\mu(\mathbb{C}, w_\nu d\sigma)$ we obtain the existence of some subsequence $(f_{n_j})_{j \in \mathbb{N}}$ in $\operatorname{Ker} A \subset L_q(\mathbb{C}, d\sigma)$, for which the sequence $\left(A_T f_{n_j}\right)_{j \in \mathbb{N}}$ converges in $W_m^\mu(\mathbb{C}, w_\nu d\sigma)$, hence it is Cauchy. Taking now the above equality $A_T \operatorname{Ker} A = B_T \operatorname{Ker} A$, we can easily derive that

\[{{\|{f_{n_{j}} - f_{n_{k}}}\|}_{L_{q}} \leq {c{\|{{B_{T}f_{n_{j}}} - {B_{T}f_{n_{k}}}}\|}_{W_{m}^{\mu}}} = {c{\|{{A_{T}f_{n_{j}}} - {A_{T}f_{n_{k}}}}\|}_{W_{m}^{\mu}}\overset{{j,k}\rightarrow\infty}{\rightarrow 0}}.\\]

So, the subsequence $\left(f_{n_j}\right)_{j\in\mathbb{N}}$ in $\mathrm{Ker}\ A\subset L_q(\mathbb{C},d\sigma)$ is Cauchy too, thus converging to some element of $\mathrm{Ker}\ A$, which is closed. The latter means that the bounded sequences in $\mathrm{Ker}\ A\subset L_q(\mathbb{C},d\sigma)$ have convergent subsequences, that gives rise, owing to the F. Riesz Theorem 1.3, the boundedness of the $\mathrm{Ker}\ A\subset L_q(\mathbb{C},d\sigma)$. Now, taking into account Lemma 3.3, we derive that the image $\mathrm{Range}\ A\subset L_p(\mathbb{C},d\sigma)$ is not closed, thus proving Theorem 1.2. □

Remark 3.4. Taking into account the properties (2.2) of the P - operator (2.1), the above equation (3.2) can be reformulated as the following partial differential equation

$$b_{+}\frac{\partial Pf(z)}{\partial\overline{z}}+b_{-}\frac{\partial P\left(af\right)(z)}{\partial z}=0, \tag{3.10}$$

generating quasi-conformal mappings of the complex plane $\mathbb{C}$, and whose solutions naturally generate the kernel $\mathrm{Ker}\ A\subset L_q(\mathbb{C},d\sigma)$ of the operator (1.1). This observation provides an alternative viewpoint on equation (3.2) and a more detailed analysis of the subspace $\mathrm{Ker}\ A\subset L_q(\mathbb{C},d\sigma)$ makes it possible to establish Theorem 1.2 in a somewhat more general form. It is also worth to mention that similar to (3.10) equations were extensively studied in [26], and analytical techniques there devised can be useful for studying problems, treated in the present work.

4. Conclusion

Based on the functional analytic properties of the singular Kalderon-Zygmund type integral operator

$$Tf(z)=\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|z-\xi|>\varepsilon}\frac{f(z)-f(\xi)}{(z-\xi)^2}d\sigma(\xi,\overline{\xi}),$$

on the Banach space $L_q(\mathbb{C},d\sigma)$, we have studied the kernel $\mathrm{Ker}\ A\subset L_q(\mathbb{C},d\sigma)$ of the linear singular operator expression

$$Af(z):=T\left(a(z)f(z)\right)+b(z)f(z) \tag{4.1}$$

with a locally continuos coeficient $a\in L_r(\mathbb{C},d\sigma)\cap W^1_{q(r)}(\mathbb{C},d\lambda), r>1, q(r)>2,$ a coefficient $b\in L_r(\mathbb{C},d\sigma)$, $r>1$, whose factorization $b=b_{+}^{-1}b_{-}$ is chosen such that for some measurable mapping $\omega:\mathbb{C}\to\mathbb{C}$ there hold the embeddings $\omega(z)b_{+}(z)^{-1}\in L_s(\mathbb{C},d\sigma), s>1,$ and $\omega b_{-}^{-1}\in L_{m(s)}(\mathbb{C},d\sigma), 1/m(s)=1/r+1/s,$ where $1/m=1/q+1/r+1/s-(\nu+\mu)/2$ and

$\operatorname{esssup}_{z\in\mathbb{C}} |\alpha_q b_+^{-1}(z) b_-(z)| < 1$. Making use of the reduced operator approach and classical Coifman-Rochberg-Weiss [22] commutator and Uchiyama [12] inverse transformation approaches, we succeeded in stating the weakly semi-Fredholm property of the above operator expression (4.1).

Notes

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Data Availability

All available data are contained in the manuscript.

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