

Research Article

On the Weakly Semi-Fredholm Structure of a Class of Singular Hilbert-Type Operator Equations

Anatolij Prykarpatsky^{1,2,3}, Volodymyr M. Dilnyi²

1. Department of Computer Science and Mathematics, Cracow University of Technology, Poland; 2. Lviv Polytechnic National University, Ukraine; 3. Pedagogical University, Maputo, Mozambique

We studied the solution space of a linear singular Hilbert type operator equation

$T(a(z)f(z)) + b(z)f(z) = 0$ **in the Banach space $L_q(K, d\sigma)$, $q > p > 1$, specified by coefficients**

$a, b \in L_r(K, d\sigma)$ **and the operator action $Tf(z) = \frac{1}{\pi} v \cdot p \cdot \int_{\mathbb{C}} \frac{f(\xi) - f(z)}{(z - \xi)^2} d\sigma(\xi, \bar{\xi})$, $f \in L_p(K, d\sigma)$,**

$r = pq/(q - p)$. **Based on classical functional analysis interpolation theory of operators, N.**

Kolmogorov and E. Riesz compactness theorems, we formulated conditions under which the the singular operator equation under regard possesses the weakly semi-Fredholm structure.

Correspondence: papers@team.qeios.com — Qeios will forward to the authors

1. Introduction

Singular operator equations on complex domains are widely used ^{[1][2][3][4]} in many investigations of important problems of modern mechanics, radiophysics, solid state physics and other applied sciences. Their theory is closely related ^{[5][2][6]} with partial differential equations, geometry of complex-analytic and quasi-conformal mappings. In particular, the following ^{[1][7]} class of expressions

$$Af(z) := T(a(z)f(z)) + b(z)f(z), \quad (1.1)$$

characterized by its coefficients $a, b \in L_r(\mathbb{C}, d\sigma)$, $q > r > 1$, and acting on complex function $f \in L_q(\mathbb{C}, d\sigma)$, $q > 1$, endowed with the norm

$$\|f\|_q = \left(\int_{\mathbb{C}} |f(\xi)|^q d\sigma(\bar{\xi}, \xi) \right)^{1/q} \quad (1.2)$$

with respect to the Lebesgue measure $d\sigma(\xi, \bar{\xi}) = d\bar{\xi} \wedge d\xi/(2i)$ on the complex plane $\mathbb{C}$, define singular integral operator equations

$$Af(z) = 0, \quad (1.3)$$

specified by the singular Bergman-Beltrami [8][9][10][11] transform

$$Tf(z) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|z-\xi|>\varepsilon} \frac{f(z) - f(\xi)}{(z - \xi)^2} d\sigma(\xi, \bar{\xi}), \quad (1.4)$$

which is equivalent to the mapping $T = \partial_z \partial_{\bar{z}}^{-1} : L_q(\mathbb{C}, d\sigma) \rightarrow L_q(\mathbb{C}, d\sigma)$, $q > 1$, for all $z \in \mathbb{C}$.

The linear singular operator equation (1.3) makes it possible to pose a related problem of studying the functional analytic and geometric properties of the corresponding reduced operator

$$A : L_q(K, d\sigma) \rightarrow L_p(K, d\sigma), \quad (1.5)$$

defined on the closed space of functions $f \in L_q(K, d\sigma)$ with a compact support $K \subset \mathbb{C}$, and, respectively, its solution space $\text{Ker } A \subset L_q(K, d\sigma)$, whose properties are of the most interest for applications.

Let us introduce the following definition, which reflects [7][2][6][10] some weaker versions of the classical Fredholm operators.

Definition 1.1. A bounded operator $A : L_q(K, d\sigma) \rightarrow L_p(K, d\sigma)$ with closed $\text{Range } A \subset L_p(K, d\sigma)$ is called semi-Fredholm, if its kernel $\text{Ker } A \subset L_q(K, d\sigma)$, or its cokernel $\text{coKer } A = L_p(K, d\sigma) / \overline{\text{Range } A}$ is finite dimensional. If the image $\text{Range } A \subset L_p(K, d\sigma)$ is neither closed nor its cokernel $L_p(K, d\sigma) / \overline{\text{Range } A}$ is finite dimensional, the bounded operator $A : L_q(K, d\sigma) \rightarrow L_p(K, d\sigma)$ is called weakly semi-Fredholm.

In particular, our special interest lies in proving the weakly semi-Fredholm structure of the operator (1.5), which consists in stating the finite dimension of its kernel $\text{Ker } A \subset L_q(K, d\sigma)$ and the infinite dimension of its coimage $L_p(K, d\sigma) / \overline{\text{Range } A}$. Based on the functional analytic properties of the Hilbert type singular transform (1.4) as well as on the structural properties of the singular operator mapping (1.4) on the Banach space $L_p(K, d\sigma)$, we stated our main theorem.

Theorem 1.2. *The bounded operator (1.5) on the Banach space $L_q(K, d\sigma)$, $q > p > 1$, is weakly semi-Fredholm, that is $\dim \text{Ker } A < \infty$ and $\text{Range } A \subset L_p(K, d\sigma)$ is not closed, if the coefficients $0 \neq a, b \in L_r(K, d\sigma)$ and the chosen factorization $b = b_-^{-1} b_+$ is such, that for some mapping $\omega : K \rightarrow \mathbb{C}$ there hold the embeddings $b_+(z)\omega(z) \in L_{\bar{q}}(K, d\sigma)$, where $1/\bar{q} < \min(1/p - 2/r, 1 - 1/p)$, and $b_-(z)^{-1}\omega(z) \in L_\infty(\mathbb{C}, d\sigma)$, as well as the condition $|\omega b_-^{-1}| \geq \delta$ holds for some $\delta > 0$ almost everywhere.*

The above statement in its main part depends on the following classical functional analysis N. Kolmogorov and F. Riesz theorems [\[12\]\[13\]\[14\]\[15\]\[16\]\[17\]](#), describing linear mappings between Banach spaces:

Theorem 1.3. (N. Kolmogorov) Let $1 \leq p < \infty$ and $\mathcal{F}$ be a subset of $L_p(K)$, $K \subseteq \mathbb{R}^n$. Then the set $\mathcal{F}$ is totally bounded if and only if (i) $\mathcal{F}$ is bounded; (ii) for every $\varepsilon > 0$ there is such $r > 0$, that for every $f \in \mathcal{F}$ the estimation $\int_{K \cap \{|x| > r\}} |f(x)|^p dx < \varepsilon^p$ holds; (iii) for every $\varepsilon > 0$ there is such $\rho > 0$ that for every $f \in \mathcal{F}$ and $y \in K$ with $|y| < \rho$ the estimation $\int_K |f(x+y) - f(x)|^p dx < \varepsilon^p$ holds.

Theorem 1.4. (F. Riesz) If a bounded operator $A : X \rightarrow Y$ from a Banach space X to a Banach space Y is compact and its image $\text{Im} A \subset Y$ is closed, then $\dim(AX) < \infty$. Moreover, a linear subspace of a normed space X is locally compact if and only if it is finite dimensional.

Below we proceed to studying the functional analytic properties of the singular Hilbert type transform (1.4) and next proving the weakly semi-Fredholm property of the linear singular operator (1.1).

2. Singular Bergman-Beltrami operator T and its properties

We consider the Cauchy type integral

$$Pf(z) := \frac{1}{2\pi i} \int_K f(\xi) \left(\frac{1}{\xi - z} - \frac{1}{\xi} \right) d\sigma(\xi, \bar{\xi}) \quad (2.1)$$

for a complex variable $z \in K \subset \mathbb{C}$ on the dense subspace $C_0^2(K, d\sigma) \subset L_q(K, d\sigma)$ for $q > 1$, naturally defining [\[8\]\[5\]\[4\]](#) the following two functions:

$$\partial Pf(z)/\partial \bar{z} = f(z), \quad \partial Pf(z)/\partial z = Tf(z) \quad (2.2)$$

for any point $z \in K$. It is easy to check that the operator $T = \partial_z \partial_{\bar{z}}^{-1} : L_q(K, d\sigma) \rightarrow L_q(K, d\sigma)$ for $z \in K$ and $1 < q < \infty$. Moreover, applying the Holder inequality to the expression (2.1), we obtain

$$|Pf(z)| \leq K_q \|f\|_q |z|^{\frac{-2}{q}+1} \quad (2.3)$$

for arbitrary function $f \in L_q(K, d\sigma)$ and any $z \in K$, where

$K_q = \left(\int_K |\xi|^{\frac{-q}{q-1}} |(\xi-1)|^{\frac{q}{1-q}} d\sigma(\xi) \right)^{\frac{q-1}{q}} < \infty$. In addition, from the following identity

$$\begin{aligned} Pf_{z_1}(z_2 - z_1) &= \frac{1}{2\pi i} \int_K f(\xi) \left(\frac{1}{\xi - z_2} - \frac{1}{\xi - z_1} \right) d\sigma(\xi) = \\ &= Pf(z_2) - Pf(z_1) \end{aligned} \quad (2.4)$$

for the shifted function $f_{z_1}(\xi) := f(\xi + z_1)$, $\xi \in K$, and the estimation (2.3) one derives the inequality

$$|Pf(z_1) - Pf(z_2)| \leq K_q \|f\|_q |z_2 - z_1|^{\frac{-2}{q}+1} \quad (2.5)$$

for arbitrary $z_1, z_2 \in K$, meaning the Hölder continuity of the mapping (2.1). To formulate our main statement about the solution set of the equation (1.3), we first state that the operator $T : L_q(K, d\sigma) \rightarrow L_q(K, d\sigma)$ for $1 < q < \infty$ is bounded. To do this, we first observe this property for the particular case $q = 2$, having calculated the integral

$$\begin{aligned} \int_K |Tf(z)|^2 d\sigma(z, \bar{z}) &= \int_K |Tf(z)|^2 d\sigma(z, \bar{z}) = \\ &= \int_K \partial^2 \overline{Pf(z)} / \partial \bar{z} \partial z Pf(z) d\sigma(z, \bar{z}) = \\ &= \int_K \overline{\partial f(z)} / \partial \bar{z} Pf(z) d\sigma(z, \bar{z}) = \int_K \overline{f(z)} \partial Pf(z) / \partial \bar{z} d\sigma(z, \bar{z}) = \\ &= \int_K \overline{f(z)} f(z) d\sigma(z, \bar{z}) = \int_K |f(z)|^2 d\sigma(z, \bar{z}), \end{aligned} \quad (2.6)$$

where we took into account that the space $C_0^2(K, d\sigma) \subset L_2(K, d\sigma)$ is dense in $L_2(K, d\sigma)$. The result (2.6) evidently means that the mapping $T : L_2(K, d\sigma) \rightarrow L_2(K, d\sigma)$ is isometry of the Hilbert space $L_2(K, d\sigma)$ with the norm $\|T\| = 1$. Moreover, for any $f \in L_q(K, d\sigma)$ there holds [\[8\]\[4\]\[11\]](#), the following general Zigmund-Kalderon inequality

$$\|Tf\|_q \leq \alpha_q \|f\|_q, \quad (2.7)$$

where $\alpha_q > 0, 1 < q < \infty$, are some bounded constants. The estimation (2.7) in its main part significantly depends on the operator representation $T = -S^2 : L_q(K, d\sigma) \rightarrow L_q(K, d\sigma)$, where, by definition,

$$Sf(z) = \frac{1}{2\pi i} \int_K f(z + \xi) \frac{d\sigma(\xi, \bar{\xi})}{\xi |\xi|}. \quad (2.8)$$

The latter is considered on the dense subspace $C_0^2(K, d\sigma) \subset L_q(K, d\sigma)$ for $q \geq 2$ and satisfies the following inequality

$$\|Sf\|_q \leq \sqrt{\alpha_q} \|f\|_q \quad (2.9)$$

for $f \in C_0^2(K, d\sigma) \subset L_2(K, d\sigma)$ and some constant $\alpha_q > 0, q \geq 2$, which makes it possible to extend the operator $T = -S^2 : C_0^2(K, d\sigma) \rightarrow L_q(K, d\sigma)$ by continuity on the whole space $L_q(K, d\sigma), q \geq 2$, and to obtain the classical Zygmund-Calderón inequality (2.7) for all $f \in L_q(K, d\sigma), q \geq 2$. Based on the properties of the adjoint operator $T^* = T : L_{q'}(K, d\sigma) \rightarrow L_{q'}(K, d\sigma), 1 < q' \leq 2$, one finally obtains that the operator $T : L_q(K, d\sigma) \rightarrow L_q(K, d\sigma)$ is well defined and bounded on the space $L_q(K, d\sigma)$ for all $1 < q < \infty$. Moreover, the singular operator (2.8), as well as that of (1.4), can be interpolated [\[18\]\[19\]\[20\]](#) [\[10\]](#) as operators from $L_{q_1}(K, d\sigma)$ to $L_{q_2}(K, d\sigma)$, where $1 < q_1 < q < q_2$, where they become compact,

thus presenting a useful tool for studying functional properties of the operator (1.5). So, proceed now to studying them in detail and proving Theorem 1.2.

3. Operator A and its weakly semi-Fredholm structure

We consider the operator mapping (1.1) and observe that, owing to the inequality (2.7), the following estimation

$$\|Af\|_p \leq \|T(af)\|_p + \|b\|_r \|f\|_q \leq (\alpha_q \|a\|_r + \|b\|_r) \|f\|_q, \quad (3.1)$$

holds for $1 < p < q < \infty$, where $r = pq/(q-p)$, and $f \in L_q(K, d\sigma)$, thus meaning that the corresponding operator norm $\|A\| \leq (\alpha_q \|a\|_r + \|b\|_r)$, that is bounded.

Now we formulate our main theorem about the weakly semi-Fredholm structure of the bounded operator (1.1) from the Banach space $L_q(K, d\sigma)$, $1 < q < \infty$, into the Banach space $L_p(K, d\sigma)$, $q > p \geq 1$.

Theorem 3.1. *The bounded operator (1.5) on the Banach space $L_q(K, d\sigma)$, $q > p > 1$, is weakly semi-Fredholm, that is $\dim \text{Ker } A < \infty$ and $\text{Range } A \subset L_p(K, d\sigma)$ is not closed, if the coefficients $0 \neq a, b \in L_r(K, d\sigma)$ and the chosen factorization $b = b_-^{-1} b_+$ is such, that for some mapping $\omega : K \rightarrow \mathbb{C}$ there hold the embeddings $b_+(z)\omega(z) \in L_{\bar{q}}(K, d\sigma)$, where $1/\bar{q} < \min(1/p - 2/r, 1 - 1/p)$, $r > 2p$, and $b_-(z)^{-1}\omega(z) \in L_\infty(\mathbb{C}, d\sigma)$, as well as the condition $|\omega b_-^{-1}| \geq \delta$ holds for some $\delta > 0$ almost everywhere.*

First of all, take into account that for $q > p \geq 1$ the image $\text{Range } A \subset L_p(K, d\sigma)$ can not be closed, as $\dim \text{Range } A = \infty$. Moreover, the image $\text{Range } A^* \subset L_q(K, d\sigma)'$ of the adjoint singular transformation $A^* = A - \{T, b\} : L_p(K, d\sigma)' \rightarrow L_q(K, d\sigma)'$, where $\{T, b\} := T \circ b + b \circ T$, is not also closed, as owing to the Banach closed image theorem one obtains both the equality $\text{codim } \text{Range } A^* = \dim \text{Ker } A < \infty$ and $\dim \text{Ker } A^* = \infty$, contradicting the condition $\text{codim } \text{Range } A < \infty$. Thus, to state that $\dim \text{Ker } A < \infty$, we will construct a new suitably reduced operator on the kernel subspace $\text{Ker } A \subset L_q(K, d\sigma)$, next extend it on the whole space $L_q(K, d\sigma)$ and state its compactness by means of both the interpolating properties of the singular Bergman-Beltrami operator (1.4) and the classical N. Kolmogorov Theorem 1.3, formulated in the case of the Banach space $L_q(K, d\sigma)$. As this compact operator on $L_q(K, d\sigma)$, if reduced on its closed subspace $\text{Ker } A \subset L_q(K, d\sigma)$, persists to be compact and its image is closed, we right away derive, as a consequence from the F. Riesz Theorem 1.4, that $\dim \text{Ker } A < \infty$.

Namely, consider the closed kernel subspace $\text{Ker } A \subset L_q(K, d\sigma)$ of the operator (1.5), that is the condition $A \text{Ker } A = 0$, which is equivalently rewritten as the equation

$$-\omega(z)b_+^{-1}Taf = \omega(z)b_-^{-1}f \quad (3.2)$$

for some nonzero mapping $\omega : K \rightarrow \mathbb{C}$, defining a singular operator $A_T : L_q(K, d\sigma) \rightarrow L_q(K, d\sigma)$ as

$$A_T f := \omega(z)b_+^{-1}Taf \quad (3.3)$$

for any $f \in L_q(K, d\sigma)$. The operator (3.3) is characterized by the following lemma.

Lemma 3.2. *Let the factorization $b(z) := b_-(z)^{-1}b_+(z) \in L_r(K, d\sigma)$, $1/r = 1/p - 1/q$, satisfy for almost all $z \in K$ the following conditions: there exists a mapping $\omega : K \rightarrow \mathbb{C}$, such that $b_+(z)^{-1}\omega(z) \in L_{\tilde{q}}(K, d\sigma)$, where $1/\tilde{q} < \min(1/p - 2/r, 1 - 1/p)$ and $r > 2p$. Then the linear operator (3.3) is bounded and compact.*

Proof. To state the above lemma, we will take into account that the boundedness and compactness of the operator $A_T : L_q(K, d\sigma) \rightarrow L_q(K, d\sigma)$ is equivalent to the compactness of the adjoint operator $A_T^* : L_{\tilde{q}}(K, d\sigma) \rightarrow L_{\tilde{q}}(K, d\sigma)$, where $1/q + 1/\tilde{q} = 1$. To demonstrate the latter, consider for any $h \in L_{\tilde{q}}(K, d\sigma)$ and $f \in L_q(K, d\sigma) \cap C_0^1(K)$ the following expression:

$$\begin{aligned} |(A_T^* h | f)| &= \lim_{\varepsilon \rightarrow 0} |(h | b_+^{-1} \omega T_\varepsilon (af))| = \\ &= \lim_{\varepsilon \rightarrow 0} |(h \omega b_+^{-1} | T_\varepsilon (af))| = \lim_{\varepsilon \rightarrow 0} |(a T_\varepsilon^* h \omega b_+^{-1} | f)|, \\ &\leq \alpha_{\tilde{p}, \tilde{s}} \|h \omega b_+^{-1}\|_{\tilde{s}} \|a\|_r \|f\|_q \leq \alpha_{\tilde{p}, \tilde{s}} \|h\|_{\tilde{q}} \|a\|_r \|f\|_q, \end{aligned} \quad (3.4)$$

yielding the norm estimation

$$\|A_T^* h\|_{\tilde{q}} \leq \alpha_{p, s} \|a\|_r \|h\|_{\tilde{q}} \|\omega b_+^{-1}\|_{\tilde{q}} \quad (3.5)$$

for arbitrary $h \in L_{\tilde{q}}(K, d\sigma)$, if $\omega b_+^{-1} \in L_{\tilde{q}}(K, d\sigma)$, $1/\tilde{q} < \min(1/p - 2/r, 1 - 1/p)$, $1/\tilde{q} + 1/s = 1/q$, where we took into account that the closure $\overline{C_0(K)} = L_q(K, d\sigma)$ and $1/\tilde{s} + 1/s = 1$, as well as the interpolated singular Bergman-Beltrami operator

$$T_\varepsilon^* : L_{\tilde{s}}(K, d\sigma) \rightarrow L_{\tilde{p}}(K, d\sigma) \quad (3.6)$$

is for any $\varepsilon > 0$ bounded, if $s > q > 1$, that is $\|T_\varepsilon^*\| = \alpha_{\tilde{p}, \tilde{s}} < \infty$. Moreover, owing to the Krasnoselskii interpolation theorem [\[18\]\[20\]\[10\]](#), the above operator (3.6) is compact, thus giving rise to the compactness of the operator $A_T^* : L_{\tilde{q}}(K, d\sigma) \rightarrow L_{\tilde{q}}(K, d\sigma)$ and, respectively, to that of the operator $A_T : L_q(K, d\sigma) \rightarrow L_q(K, d\sigma)$. Simultaneously, one can observe that the estimation (3.5) makes it possible to state the additional integral inequality:

$$\int_{K \cap \{|z| > 1/\varepsilon\}} |A_T^* h(z)| d\sigma(z, \bar{z}) \leq \alpha_{\tilde{p}, \tilde{s}} \|a\|_r \|h\|_{\tilde{q}} \left(\int_{K \cap \{|z| > 1/\varepsilon\}} |\omega b_+^{-1}|^{\tilde{q}} \right)^{1/\tilde{q}}, \quad (3.7)$$

from which one obtains for $h \in L_{\tilde{q}}(K, d\sigma)$ the uniform limit

$$\lim_{\varepsilon \rightarrow 0} \int_{K \cap \{|z| > 1/\varepsilon\}} |A_T^* h(z)| d\sigma(z, \bar{z}) = 0. \quad (3.8)$$

The latter, owing to the classical [\[12\]\[15\]\[16\]\[17\]](#) N. Kolmogorov Theorem 1.3, confirms the compactness of the adjoint operator $A_T^* : L_{\bar{q}}(K, d\sigma) \rightarrow L_{\bar{q}}(K, d\sigma)$, and respectively. $\square$

Proof. (Proof of the main Theorem) We first observe that the reduced operator $A_T : \text{Ker } A \rightarrow L_q(K, d\sigma)$ is compact too, and whose image $\text{Range } A_T|_{\text{Ker } A} = \omega b_-^{-1} \text{Ker } A \subset L_q(K, d\sigma)$ is a priori closed, if $\omega b_-^{-1} \in L_\infty(\mathbb{C}, d\sigma)$ and $|\omega b_-^{-1}| \geq \delta$ for some $\delta > 0$ almost everywhere. Then, based on the compactness F. Riesz Theorem 1.4, we derive the boundedness $\dim \text{Ker } A < \infty$. The latter means that the operator (1.1) possesses the weakly semi-Fredholm structure, thus proving main theorem. $\square$

Remark 3.3. Taking into account the properties (2.2) of the P - operator (2.1), the above equation (3.2) can be reformulated as the following partial differential equation

$$b_+ \frac{\partial P f(z)}{\partial \bar{z}} + b_- \frac{\partial P (af)(z)}{\partial z} = 0, \quad (3.9)$$

generating quasi-conformal mappings on the complex plane $\mathbb{C}$, and whose solutions naturally generate the kernel $\text{Ker } A \subset L_q(K, d\sigma)$ of the operator (1.1). This observation provides an alternative viewpoint on equation (3.2) and a more detailed analysis of the subspace $\text{Ker } A \subset L_q(\mathbb{C}, d\sigma)$ makes it possible to establish Theorem 1.2 in a somewhat more general form. It is also worth to mention that similar to (3.9) equations were extensively studied in [\[21\]](#), and analytical techniques there devised can be useful for studying problems, treated in the present work.

4. Conclusion

Based on the functional analytic properties of the singular Cauchy type integral operator

$$Pf(z) := \frac{1}{2\pi i} \int_K f(\xi) \left(\frac{1}{\xi - z} - \frac{1}{\xi} \right) d\sigma(\xi, \bar{\xi})$$

on the Banach space $L_q(K, d\sigma)$ of functions with compact support $K \subset \mathbb{C}$, we have studied the kernel $\text{Ker } A \subset L_q(K, d\sigma)$ of the linear singular operator expression

$$Af(z) := T(a(z)f(z)) + b(z)f(z) \quad (4.1)$$

with coefficients $a, b \in L_r(K, d\sigma)$, $b = b_+^{-1}b_-$, where for some mapping $\omega : K \rightarrow \mathbb{C}$ there hold the embeddings $b_+(z)^{-1}\omega(z) \in L_{\bar{q}}(\mathbb{C}, d\sigma)$ for $1/\bar{q} < \min(1/p - 2/r, 1 - 1/p)$, $b_-(z)^{-1}\omega(z) \in L_\infty(K, d\sigma)$, $|b_-(z)^{-1}\omega(z)| \geq \delta$ for some $\delta > 0$, and the operator $T : L_p(K, d\sigma) \rightarrow L(K, d\sigma)$ is defined by the differential expression

$$Tf(z) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{K \cap \{|z-\xi| > \varepsilon\}} \frac{f(z) - f(\xi)}{(z - \xi)^2} d\sigma(\xi, \bar{\xi}),$$

on the compact set $K \subset \mathbb{C}$. Making use of the reduction operator approach and classical functional analysis results Theorem 1.4 and Lemma 3.2, we succeeded in stating the weakly semi-Fredholm property of the above operator expression (4.1).

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Potential Competing Interests

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Data Availability

All available data are contained in the manuscript.

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