

Research Article.

Rough Estimates of Solar System Gravitomagnetic Effects in Post-Newtonian Gravity

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In order to describe properly the gravity interactions including the mass currents, in the gravitomagnetism we construct four Maxwell type gravitational equations which are shown to be analogs of the Maxwell equations in the electromagnetism. Next, exploiting the Maxwell type gravitational equations, we explicitly predict the mass magnetic fields for both the isolated system of the spinning Moon orbiting the spinning Earth and that of the Sun and solar system planets orbiting the spinning Sun, whose phenomenological values have not been evaluated in the preceded Newtonian gravity formalisms. In the gravitomagnetism we also phenomenologically investigate the mass magnetic general relativity (GR) forces associated with the mass magnetic fields, to find that they are extremely small but non-vanishing compared to the corresponding mass electric Newtonian forces. Moreover, the directions of the mass magnetic GR forces for the solar system planets except Venus and Uranus are shown to be anti-parallel to those of their mass electric Newtonian forces. Next we investigate the mass magnetic dipole moment related with the B-ring of Saturn, to evaluate $\vec{m}_M(\text{Ring}) = -1.141 \times 10^4 \text{ m}^3 \text{ s}^{-1} \hat{\omega}$ with $\hat{\omega}$ being the unit vector along the axis direction of the spinning B-ring. The predicted value of $\vec{m}_M(\text{Ring})$ is shown to be directly related with the Cassini data on the total mass of the rings of Saturn.

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I. Introduction

In the weak field approximation, the Einstein field equations in general relativity are similar to the Maxwell equations of electromagnetism (EM). For a brief formalism of the EM, see Appendix A. Next in

the post-Newtonian corrections to the gravitational law there have been discussions on the frame dragging effects^{[1][2]} and the Lense-Thirring effects^{[3][4]}. For more details on these effects, see Refs.^{[1][2][3][4]} and references therein. In investigating phenomenology related with spinning compact objects such as the Sun, the solar system planets and the Moon for example, one can effectively exploit the gravitomagnetism possessing the rotational degrees of freedom (DOF). For more details of the gravitomagnetism, see Refs.^{[5][6][7][8][9]} and references therein.

Recently, we have proposed^[10] a new formalism of the modified linear general relativity (MLGR) which uses the vectorial scheme for the mass scalar and mass vector potentials of the gravitational wave (GW). To do this, in the MLGR we have employed the Wald approximation^{[9][10][11]}. To be specific, we have found the GW radiation intensity profile possessing a prolate ellipsoid geometry due to the merging binary compact objects source. Here the radiation lobe is prolate with long axis perpendicular to the orbital plane^{[10][12]}. To be more specific, at a given radial distance from the binary compact objects, the GW radiation intensity on the revolution axis of the binary compact objects has been shown to be twice that on the equatorial plane^[10].

On the other hand, the obliquity of Saturn has been known to be too large to have risen during Saturn's formation from a protoplanetary disk or from a large impact^[13]. The rings of Saturn have been known to appear about 100 million years ago, based on the estimated strength of satellite ring torques^[14] and the estimated rate of darkening of the ice-rich materials^{[15][16]}. However, it is uncertain how the young rings of Saturn could have formed so recently. The Cassini data recently have been exploited to refine estimates of Saturn's moment of inertia. Moreover, it has been proposed^[17] that Saturn previously possessed an additional satellite, called Chrysalis, which caused the obliquity of Saturn to increase through the Neptune resonance. Next destabilization of Chrysalis's orbit about 100 million years ago can then explain the proximity of the system to the resonance and the formation of the young rings through a grazing encounter of Chrysalis with Saturn. Moreover the simulated Chrysalis has been shown to have multiple encounters with Titan which is the largest satellite of Saturn^[17]. The other Cassini data also have explicitly shown the total mass of the rings of Saturn^[16].

In this paper, we will theoretically investigate the precision astrophysics phenomenology having the gravitomagnetism related with mass M which is the analogue of the EM^{[10][11][18][19]} associated with charge Q . Next we will study the mass magnetic fields for the spinning compact objects in the gravitomagnetism. To be more specific, we will evaluate the mass magnetic fields for both the isolated

system of the Moon orbiting the spinning Earth and that of the solar system planets orbiting the spinning Sun. Making use of the mass magnetic fields, we will proceed to predict the ensuing Lorentz type mass magnetic general relativity (GR) forces for these systems. Next we will construct the mass magnetic dipole moment in the gravitomagnetism to investigate the phenomenology of the B-ring of Saturn.

In Sec. II, we will first construct the formalism of the gravitomagnetism by exploiting the MLGR. Next, in this gravitomagnetism we will formulate the mass electric and mass magnetic fields, and the corresponding mass electric Newtonian force and mass magnetic one. In Sec. III, we will investigate the phenomenology associated with the mass magnetic fields of both the spinning Moon orbiting the spinning Earth, and the spinning solar system planets orbiting the spinning Sun. Moreover we will evaluate the mass electric Newtonian forces and mass magnetic ones for these systems. Next we will evaluate the mass magnetic dipole moment associated with the B-ring of Saturn. Sec. IV includes conclusions.

II. Formalism of astrophysical phenomenology

A. Gravitomagnetism originated from MLGR

In this subsection, we construct the formalism of the gravitomagnetism by using the MLGR^[10]. To do this, we will briefly recapitulate the MLGR originated from the Einstein GR. Now we start with the Einstein-Hilbert action

$$S_{EH} = \int d^4x \sqrt{-g} R \quad (2.1)$$

from which we construct the Einstein GR field equation^{[11][5][20]}

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \frac{8\pi G}{c^4}T_{\alpha\beta}, \quad (2.2)$$

where $R_{\alpha\beta}$ and R are the Riemann tensor and scalar curvature, respectively, and $T_{\alpha\beta}$ is the energy stress tensor. Next we assume that the deviation $h_{\alpha\beta}$ of the spacetime metric $g_{\alpha\beta}$ from a flat metric $\eta_{\alpha\beta}$ is small

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}. \quad (2.3)$$

Exploiting the Einstein GR field equation in (2.2) and $g_{\alpha\beta}$ in (2.3), we construct the linearized Einstein equation^{[11][10][5][20]}

$$-\frac{1}{2}\square\bar{h}_{\alpha\beta} + \partial^\gamma\partial_{(\alpha}\bar{h}_{\beta)\gamma} - \frac{1}{2}\eta_{\alpha\beta}\partial^\gamma\partial^\delta\bar{h}_{\gamma\delta} = \frac{8\pi G}{c^4}T_{\alpha\beta}, \quad (2.4)$$

where $\square = \partial^\beta\partial_\beta$ with the metric $\eta_{\alpha\beta} = \text{diag}(-1, +1, +1, +1)$. Here the trace reversed perturbation $\bar{h}_{\alpha\beta}$ is given by $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}h$ with $h = \eta^{\alpha\beta}h_{\alpha\beta}$. Next exploiting (2.4) and the Lorentz type gauge condition in gravity

$$\partial^\beta\bar{h}_{\alpha\beta} = 0, \quad (2.5)$$

we arrive at the Maxwell type equation in the MLGR

$$\square\bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4}T_{\alpha\beta}. \quad (2.6)$$

Now we consider the non-vanishing $\bar{h}_{00}$ and $\bar{h}_{0i}$ ($i = 1, 2, 3$), together with $\bar{h}_{ij} = 0$ [9][10][11]. We define $\bar{h}_{00}$ and T_{00} as

$$\bar{h}_{00} = -\frac{4}{c^2}\phi_M, \quad T_{00} = c^2\rho_M, \quad (2.7)$$

where $\phi_M \equiv cA_M^0$ is the linearized physical quantity in the MLGR. From now on, the subscripts M and Q denote the mass and charge physical quantities respectively. Inserting (2.7) into the Maxwell type equation in (2.6) yields

$$\square\phi_M = 4\pi G\rho_M. \quad (2.8)$$

Next we define $\bar{h}_{0i}$ and T_{0i} as

$$\bar{h}_{0i} = \frac{4}{c}A_M^i, \quad T_{0i} = -c\rho_M v^i = -cJ_M^i, \quad (2.9)$$

where the energy stress tensor is approximated to linear order in velocity in the Wald Approximation [9][10][11]. In the MLGR, inserting (2.9) into the Maxwell type equation in (2.6) produces

$$\square A_M^i = \frac{4\pi G}{c^2}J_M^i. \quad (2.10)$$

Note that in the Wald approximation associated with T_{00} in (2.7) and T_{0i} in (2.9), we construct $T_{\alpha\beta}$ in terms of $J_M^i = \rho_M v^i$ [9][10][11]

$$T_{\alpha\beta} = \begin{pmatrix} c^2\rho_M & -cJ_M^j \\ -cJ_M^i & 0 \end{pmatrix}, \quad (2.11)$$

where the space-space components T_{ij} vanish.

In vacuum, inserting $T_{\alpha\beta} = 0$ into (2.6) and exploiting the definitions of $\bar{h}_{00}$ and $\bar{h}_{0i}$ in (2.7) and (2.9), we obtain the wave equations for the non-vanishing fields $\bar{h}_{00}$ and $\bar{h}_{0i}$

$$\square \bar{h}_{00} = \square A_M^0 = 0, \quad \square \bar{h}_{0i} = \square A_M^i = 0, \quad (2.12)$$

from which we arrive at the wave equations in the MLGR

$$\square A_M^\alpha = 0, \quad \alpha = 0, 1, 2, 3 \quad (2.13)$$

which describes a massless spin-one graviton propagating in the flat spacetime. This phenomenology in the gravitomagnetism is analogous to that in the EM. Note also that we have the Lorentz type gauge condition for A_M^α

$$\partial_\alpha A_M^\alpha = 0. \quad (2.14)$$

Next we investigate the formalism of the gravitomagnetism. To this end, we define the *mass electric* and *mass magnetic* fields E_M^i and B_M^i in terms of $\bar{h}_{00}$ in (2.7) and $\bar{h}_{0i}$ in (2.9) constructed in the relativistic MLGR

$$E_M^i = -\partial_i \phi_M - \frac{\partial A_M^i}{\partial t} = \frac{c^2}{4} (\partial_i \bar{h}_{00} - \partial_0 \bar{h}_{0i}), \quad B_M^i = \epsilon^{ijk} \partial_j A_M^k = \frac{c}{4} \epsilon^{ijk} \partial_j \bar{h}_{0k}. \quad (2.15)$$

Note that E_M^i contains the DOF related with the term $\frac{\partial A_M^i}{\partial t}$. After some algebra exploiting $\vec{E}_M$ and $\vec{B}_M$ in (2.15) in the MLGR, we construct the Maxwell type equations in the gravitomagnetism^[6]

$$\begin{aligned} \nabla \cdot \vec{E}_M &= -4\pi G \rho_M, & \nabla \times \vec{E}_M + \frac{\partial \vec{B}_M}{\partial t} &= 0, \\ \nabla \cdot \vec{B}_M &= 0, & \nabla \times \vec{B}_M - \frac{1}{c^2} \frac{\partial \vec{E}_M}{\partial t} &= -\frac{4\pi G}{c^2} \rho_M \vec{v}. \end{aligned} \quad (2.16)$$

Note that the Maxwell type gravitational equations in (2.16) in the gravitomagnetism are the analogs of the Maxwell equations in the EM^{[18][19]}. Note also that in the gravitomagnetism the equations in (2.16) include the time derivative terms related with the dynamics of the gravitomagnetism interactions.

Now it seems appropriate to address comments on the gravitomagnetism in Refs.^{[5][7][8][9]}, where the gravitational Maxwell equations are given by

$$\begin{aligned} \nabla \cdot \vec{E}_g &= -4\pi G \rho_M, & \nabla \times \vec{E}_g &= 0, \\ \nabla \cdot \vec{B}_g &= 0, & \nabla \times \vec{B}_g &= -\frac{16\pi G}{c^2} \rho_M \vec{v}. \end{aligned} \quad (2.17)$$

First, in the gravitomagnetism in (2.17), we find the curl equations which do not possess the dynamic DOF associated with the time derivative terms in (2.16). Note that the equations for $\vec{E}_g$ denote the gravitational field produced by a static mass configuration. Next the equations for $\vec{B}_g$ yield a notationally means of determining the extra gravitational field produced by moving masses associated with $\rho_M \vec{v}$ ^[5].

Second, even in the static case, the last equation in (2.17) has a *coefficient different* from that in the gravitomagnetism in (2.16). This feature in the gravitomagnetism in Refs.^{[5][7][8][9]} originates from the identification^[5]:

$$\epsilon_0 \leftrightarrow -\frac{1}{4\pi G}, \mu_0 \leftrightarrow -\frac{16\pi G}{c^2}. \quad (2.18)$$

In contrast, in the gravitomagnetism associated with (2.16), we construct the last equation possessing $-\frac{4\pi G}{c^2}$, by using the treatment of the Einstein GR field equation in (2.2) and $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ in (2.3) used in the MLGR. Especially in order to predict the astrophysical quantities, which will be *numerically* evaluated in Section 3 for instance, we need to exploit the equations in (2.16). This feature in the MLGR and the ensuing gravitomagnetism associated with (2.16) is one of the main points of this paper.

Next we have comments on the gauge invariance of the gravitomagnetism and that of the EM described in Appendix A. First, in the MLGR^[10], we do not exploit the traceless transverse (TT) gauge^{[11][5][20]}^[10] which is a second-rank tensorial formalism in the linearized general relativity (LGR). In the TT gauge we choose $\bar{h}_{TT}^{0i} \equiv 0$. Next, for the MLGR, exploiting the Wald approximation^{[9][10][11]}, we have constructed a vectorial formalism which is similar to that of the EM^[10]. Note that in the MLGR we have the vanishing $\bar{h}^{ij} = 0$ associated with the Wald approximation, while in the LGR we possess the nonzero $\bar{h}_{TT}^{ij} \neq 0$ in the TT gauge^{[11][5][20][10]}.

Second, the equations of motion (EOM) for the fields ϕ_M and $\vec{A}_M$ in (2.8) and (2.10) produce the covariant EOM of $A_M^\alpha \equiv (\frac{\phi_M}{c}, A_M^i)$

$$\square A_M^\alpha = \frac{4\pi G}{c^2} J_M^\alpha, \quad (2.19)$$

where $J_M^\alpha \equiv (c\rho_M, J_M^i)$. Next, (2.19) is the same form of the EOM for the fields $A_Q^\alpha \equiv (\frac{\phi_Q}{c}, A_Q^i)$ obtainable from (A.3) and $J_Q^\alpha \equiv (c\rho_Q, J_Q^i)$:

$$\square A_Q^\alpha = -\frac{1}{c^2 \epsilon_0} J_Q^\alpha, \quad (2.20)$$

which are the covariant EOM. Note that, in the MLGR, the GW radiation intensity profile possess a prolate ellipsoid geometry possessing the angle dependence of the merging binary compact objects source^[10].

Third, motivated by the mathematical similarity in the vectorial forms of the EOM for A_M^α ($= -c/4 \bar{h}^{0\alpha}$) ($\alpha = 0, 1, 2, 3$) in (2.19) and A_Q^α in (2.20), we construct the vectorial formalism for the spin-one graviton in the MLGR^[10]. Now we investigate the gauge invariance and U(1) transformation in the gravitomagnetism.

To do this, we start with the Dirac equation for the electron wave function ψ and the mass scalar and mass vector potentials A_M^μ in the unit of $\hbar = c = 1$

$$i\gamma^\mu \partial_\mu \psi - m_e \psi - e\gamma^\mu A_{M,\mu} \psi = 0, \quad (2.21)$$

with m_e and e being the electron mass and charge, respectively. Next, keeping in mind that in the MLGR we have the spin-one graviton, we study the quantum gravitomagnetic dynamics (QGD) which describes the interactions between electrons and gravitons. Now we introduce the QGD Lagrangian which is analogous to that of the quantum electrodynamics (QED) in (A.7)^[21]

$$\mathcal{L}_{QGD} = -\frac{1}{4}F_{M,\mu\nu}F_M^{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu - m_e)\psi - e\bar{\psi}\gamma^\mu A_{M,\mu}\psi, \quad (2.22)$$

where $F_M^{\mu\nu} = \partial^\mu A_M^\nu - \partial^\nu A_M^\mu$ and the third term denotes the interaction between the electron and the gravitomagnetic wave (or spin-one graviton). Now we find that the QGD Lagrangian in (2.22) is invariant under the gauge transformations

$$\begin{aligned} \psi(x) &\rightarrow \psi'(x) = e^{-ie\Theta(x)}\psi(x), \\ A_M^\mu(x) &\rightarrow A_M'^\mu(x) = A_M^\mu(x) + \partial^\mu \Theta(x), \end{aligned} \quad (2.23)$$

where the first relation is the U(1) transformation.

B. Mass magnetic GR force

Now we construct the mass magnetic GR force by exploiting the gravitomagnetism which is formulated in the previous subsection. For the cases of *mass electrostatics* and *mass magnetostatics* which are practically applicable to the phenomenological predictions in the next section, the mass electric and mass magnetic fields are given by

$$\vec{E}_M = -G \int \frac{\rho_M \hat{R}}{R^2} d^3 x', \quad \vec{B}_M = -\frac{G}{c^2} \int \frac{\vec{J}_M \times \hat{R}}{R^2} d^3 x', \quad (2.24)$$

where $\vec{R} = \vec{x} - \vec{x}'$ is the vector from $d^3 x'$ to the field point $\vec{x}$ and $\hat{R} = \vec{R}/R$ with $R = |\vec{x} - \vec{x}'|$. Using (2.24) we obtain the corresponding force acting on a test mass M_0 moving with velocity v as follows¹

$$\vec{F}_M = M_0(\vec{E}_M + \vec{v} \times \vec{B}_M) \equiv \vec{F}_M^E + \vec{F}_M^B. \quad (2.25)$$

Now, in the mass magnetostatics, we construct the mass vector potential $\vec{A}_M$ as

$$\vec{A}_M = -\frac{G}{c^2} \int \frac{\vec{J}_M}{R} d^3 x'. \quad (2.26)$$

Note that, exploiting $\vec{B}_M (= \nabla \times \vec{A}_M)$ we find that $\vec{A}_M$ in (2.26) reproduces $\vec{B}_M$ in (2.24).

Now it seems appropriate to address some comments on the applications of the formulas (2.24) and (2.26). First, exploiting (2.24), we construct the mass magnetic field $\vec{B}_M$ at the center of the loop of orbital radius R which consists of total mass M and orbits its center with orbital period T

$$\vec{B}_M = -\frac{2\pi GM}{c^2 RT} \hat{\omega}. \quad (2.27)$$

Here $\hat{\omega}$ is the unit vector along the axis of the loop orbiting its center. Note that, inside (outside) the loop, the direction of $\vec{B}_M$ is anti-parallel (parallel) to $\hat{\omega}$.

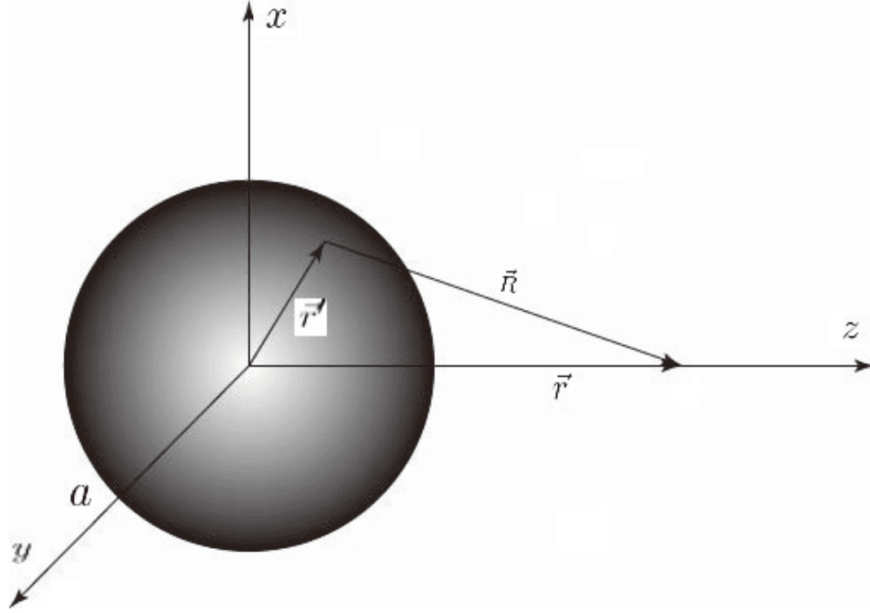


Figure 1. The geometry for a solid sphere of mass M and radius a spinning with angular velocity $\vec{\omega} = \omega \hat{\omega}$ where $\hat{\omega}$ is the unit vector along $\hat{x}$. Here

$$\vec{r}' = r'(\hat{x} \sin \theta' \cos \phi' + \hat{y} \sin \theta' \sin \phi' + \hat{z} \cos \theta') \text{ and}$$

$$R = (r^2 + r'^2 - 2rr' \cos \theta')^{1/2} \text{ with } \theta' \text{ being the polar angle between } \vec{r} \text{ and } \vec{r}'.$$

Second, we consider a solid sphere of mass M and radius a spinning with angular velocity $\vec{\omega} = \omega \hat{\omega}$ as shown in Figure 1. Note that $\vec{J}_M = \rho_M \vec{v} = \rho_M \vec{\omega} \times \vec{r}'$ produces

$$\vec{J}_M = \frac{3M}{4\pi a^3} (-\hat{y} \omega r' \cos \theta' + \hat{z} \omega r' \sin \theta' \sin \phi'). \quad (2.28)$$

Inserting (2.28) into (2.26), we find the mass vector potential $\vec{A}_M$ as follows

$$\vec{A}_M = -\frac{3GM\omega}{2c^2 a^3} \int_0^a dr' r'^3 \int_0^\pi \frac{d\theta' \sin \theta' \cos \theta'}{R} \hat{\omega} \times \hat{r}, \quad (2.29)$$

where we have used the relation $\hat{y} = -\hat{\omega} \times \hat{z}$. Next, on the equatorial plane of the solid sphere, we arrive at $\vec{A}_M$ at $\vec{r} = r\hat{r}$ with r ($r \geq a$)²

$$\vec{A}_M = -\frac{GMa^2\omega}{5c^2r^2}\hat{\omega} \times \hat{r}. \quad (2.30)$$

Exploiting (2.30), we finally formulate the mass magnetic field $\vec{B}_M (= \nabla \times \vec{A}_M)$ of the form

$$\vec{B}_M = \frac{4\pi GMa^2}{5c^2r^3t}\hat{\omega}, \quad (2.31)$$

where t is the rotational period of the spinning solid sphere. Note that, outside the spinning solid sphere, the direction of $\vec{B}_M$ is parallel to $\hat{\omega}$ on the equatorial plane. Note also that in the EM the direction of $\vec{B}_Q$ circles around the charge current following the right hand thumb rule since the charges interact repulsively^{[18][19]}. In contrast, in the gravitomagnetism the direction of $\vec{B}_M$ circles around the mass current following the left hand thumb rule due to the fact that the masses interact attractively. Explicitly these features can be explained by the comparison of (2.26) and (A.11).

III. Phenomenology of astrophysical systems in gravitomagnetism

A. Earth and Moon

Now in the gravitomagnetism we consider the *isolated two body system* of the Earth of mass $M_\oplus$ and the spinning Moon of mass M_m orbiting the spinning Earth to find the phenomenological prediction for the mass magnetic field $\vec{B}_M(M_\oplus) = B_M(M_\oplus)\hat{\omega}$ at the surface of the Earth where $\hat{\omega}$ is the unit vector along the axis direction of the spinning Earth. Note that $\hat{\omega}$ is also the unit vector along both the axis direction of the spinning Moon and the orbital axis direction of the Moon orbiting the Earth. The mass magnetic field at the surface of the Moon will be considered later. Exploiting (2.27) and (2.31), we construct the expression of mass magnetic field $B_M(M_\oplus)$ for the isolated system of the Earth and Moon, both of which possess the rotational DOF, for an observer located on the surface of the Earth³

$$\begin{aligned} B_M(M_\oplus) &= \frac{4\pi GM_\oplus}{5c^2r_\oplus t_\oplus} - \frac{2\pi GM_m}{c^2R_m T_m} + \frac{4\pi GM_m r_m^2}{5c^2 R_m^3 t_m} \\ &= (2.029 \times 10^{-14} - 3.777 \times 10^{-19} + 3.085 \times 10^{-24}) \text{ sec}^{-1}, \end{aligned} \quad (3.1)$$

where the first, second and third terms originate from the contributions of the spinning effect of the Earth and orbital effect of the Moon and spinning effect of the Moon, respectively. Here the subscript m denotes the physical quantities of the Moon. Next $r_\oplus$ and $t_\oplus$ are the radius and rotational period of the

spinning Earth, while r_m , R_m , t_m and T_m are the radius, average orbital distance, rotational period and average orbital period of the spinning Moon orbiting the Earth, respectively.⁴

Note that the contributions from the orbital effect of Moon and spinning effect of Moon are negligible. The contributions of the other solar system planets to $B_M(M_\oplus)$ will be discussed in (3.8).

Next we evaluate the mass magnetic field $\vec{B}_M(M_m) = B_M(M_m)\hat{\omega}$ for an observer located on the surface of the Moon by using (2.31). To do this, we obtain the expression of the mass magnetic field $B_M(M_m)$ for the isolated system of the Earth and Moon

$$B_M(M_m) = \frac{4\pi G M_m}{5c^2 r_m t_m} + \frac{4\pi G M_\oplus r_\oplus^2}{5c^2 R_m^3 t_\oplus} = (3.343 \times 10^{-17} + 9.267 \times 10^{-20}) \text{ sec}^{-1}, \quad (3.2)$$

where the first and second terms originate from the contributions of the spinning Moon and the spinning Earth, respectively.

Now we investigate the force acting on the spinning Moon of mass M_m due to the spinning Earth of mass $M_\oplus$. Exploiting $\vec{F}_M^B$ in (2.25), we are left with $\vec{F}_M^B(M_m - M_\oplus) = F_M^B(M_m - M_\oplus)\hat{r}$ with $\hat{r}$ being the radial direction from the Earth⁵

$$F_M^B(M_m - M_\oplus) = (2.513 \times 10^9 + 6.966 \times 10^6) N. \quad (3.3)$$

Next, making use of $\vec{E}_M$ in (2.24) and $\vec{F}_M^E$ in (2.25), one can readily construct the mass electric Newtonian force acting on the Moon $\vec{F}_M^E(M_m - M_\oplus) = F_M^E(M_m - M_\oplus)\hat{r}$ for the isolated system of the Earth and Moon, for an observer located on the surface of the Moon⁶

$$F_M^E(M_m - M_\oplus) = -1.983 \times 10^{20} N, \quad (3.4)$$

from which we arrive at the characteristic ratio $\chi(M_m - M_\oplus)$ of the form

$$\chi(M_m - M_\oplus) = |\vec{F}_M^B(M_m - M_\oplus)| / |\vec{F}_M^E(M_m - M_\oplus)| = 1.271 \times 10^{-11}. \quad (3.5)$$

The ratio $\chi(M_m - M_\oplus)$ in (3.5) implies that the gravitomagnetic correction, namely $|\vec{F}_M^B(M_m - M_\oplus)|$ in the gravitomagnetism, to the Newtonian classical prediction $|\vec{F}_M^E(M_m - M_\oplus)|$ is extremely small but it is non-vanishing.

B. Sun and solar system planets

In the gravitomagnetism we now investigate the isolated system of the Sun and the solar system planets orbiting the spinning Sun to evaluate the mass magnetic field $\vec{B}_M(M_\odot) = B_M(M_\odot)\hat{\omega}$ for an observer located on the surface of the Sun where $\hat{\omega}$ is the unit vector along the axis direction of the spinning Sun.

Here, for simplicity, we assume that the rotation axes of the solar system planets, except Venus and Uranus, are approximately parallel to that of the Sun to produce the vanishing obliquities and that these axes reside on the equatorial plane of the Sun. We also assume that on the equatorial plane of the Sun, the rotation axes of Venus and Uranus are anti-parallel to and perpendicular to that of the Sun, respectively.

Similar to (3.1), exploiting (2.27) and (2.31) we find the formula of mass magnetic field $B_M(M_\odot)$ for the isolated system of the Sun and solar system planets, for an observer located on the surface of the Sun,

$$B_M(M_\odot) = \frac{4\pi G M_\odot}{5c^2 r_\odot t_\odot} - \sum_{i=1}^8 \frac{2\pi G M_i}{c^2 R_i T_i} + \sum_{i=1}^8 \frac{4\pi G M_i r_i^2}{5c^2 R_i^3 t_i} \\ = (2.468 \times 10^{-12} - 5.266 \times 10^{-20} + 3.769 \times 10^{-25}) \text{ sec}^{-1}, \quad (3.6)$$

where the first, second and third terms come from the contributions from the spinning effect of the Sun, orbital effects of the solar system planets and spinning effects of the solar system planets, respectively. Here $M_\odot$, $r_\odot$ and $t_\odot$ are the mass, radius and rotational period of the spinning Sun, while M_i , r_i , R_i , t_i and T_i ($i = 1, 2, 3, \dots, 8$) are the mass, radius, average orbital distance, rotational period and average orbital period of the i -th solar system planet orbiting the Sun, respectively. In treating $B_M(M_\odot)$ in (3.6), we have ignored the effect of the sun radius $r_\odot$ which is negligible comparing to $R_{Mercury}$ due to the factor $r_\odot/R_{Mercury} = 1.202 \times 10^{-2}$ [20], for instance, in calculating the second term in (3.6). From now on we will ignore the contributions from the spinning effects of the solar system planets in the third term, since these effects are extremely small, as shown in (3.1) for the case of the Earth and Moon system.

Now we investigate the mass magnetic fields $\vec{B}_M(M_i) = B_M(M_i)\hat{\omega}$ ($i = 1, 2, 3, \dots, 8$) for the solar system planets with M_i for an observer located on the surface of the planets where $\hat{\omega}$ is the unit vector along the axis direction of the spinning Sun. To do this, using (2.31) we formulate $B_M(M_i)$ as follows

$$B_M(M_i) = \frac{4\pi G M_i}{5c^2 r_i t_i} + \frac{4\pi G M_\odot r_\odot^2}{5c^2 R_i^3 t_\odot} + \sum_{j \neq i} \frac{4\pi G M_j r_j^2}{5c^2 |R_j - R_i|^3 t_j}, \quad (3.7)$$

where the first, second and third terms originate from the contributions from the spinning effect of the i -th planet, spinning effect of the Sun and spinning effects of the other j -th planets, respectively. Here we have made approximation that all the solar system planets are aligned on the straight line in planetary order for simplicity, and approximation that for the case of Uranus the direction of $\vec{B}_M(M_7)$ is along the centripetal direction $-\hat{r}$.

Making use of (3.7), for the case of the Earth we arrive at

$$B_M(M_\oplus) = (2.029 \times 10^{-14} + 2.486 \times 10^{-19} + 2.078 \times 10^{-24}) \text{ sec}^{-1}, \quad (3.8)$$

where for simplicity we have not included the contribution of $\vec{B}_M(M_7)$ originated from Uranus, since $\vec{B}_M(M_7)$ is perpendicular to the directions of $\vec{B}_M(M_i)$ for the other solar system planets. Note that in (3.8) we can ignore the negligible contributions of the spinning effect of the Sun and spinning effects of the other solar system planets. Next in (3.8) we also have excluded the negligible contributions of the orbital effect of Moon and spinning effect of Moon which are listed in $B_M(M_\oplus)$ in (3.1). Moreover, since the contributions of the spinning effects of the other solar system planets with M_j ($j = 1, 2, 4, \dots, 8$) are negligible as shown in (3.8), in evaluating $B_M(M_i)$ in (3.7) we effectively do not need the constraint that all the solar system planets are aligned on the straight line in planetary order.

Next, similar to (3.3) for the case of the Moon, we evaluate the mass magnetic GR force acting on the Earth $\vec{F}_M^B(M_\oplus - M_\odot) = F_M^B(M_\oplus - M_\odot)\hat{r}$ with $\hat{r}$ being the unit radial vector from the Sun. Exploiting (2.25) one can readily evaluate $F_M^B(M_\oplus - M_\odot)$

$$\begin{aligned} F_M^B(M_\oplus - M_\odot) &= (3.612 \times 10^{15} + 4.426 \times 10^{10} + 3.529 \times 10^5)N \\ &\cong 1.019 \times 10^{-7} \times |F_M^E(M_\oplus - M_\odot)|, \end{aligned} \quad (3.9)$$

where $|F_M^E(M_\oplus - M_\odot)|$ is given below. Now we readily formulate the mass electric Newtonian force acting on the Earth $\vec{F}_M^E(M_\oplus - M_\odot) = F_M^E(M_\oplus - M_\odot)\hat{r}$ where $\hat{r}$ is again the unit radial vector from the Sun and $F_M^E(M_\oplus - M_\odot)$ is given by

$$\begin{aligned} F_M^E(M_\oplus - M_\odot) &= -\frac{GM_\oplus M_\odot}{R_\oplus^2} - \sum_{j=1,2} \frac{GM_\oplus M_j}{|R_j - R_\oplus|^2} + \sum_{j>3} \frac{GM_\oplus M_j}{|R_j - R_\oplus|^2}, \\ &= (-3.543 \times 10^{22} - 1.149 \times 10^{18} + 2.103 \times 10^{18})N, \end{aligned} \quad (3.10)$$

where the first, second and third terms come from the contributions from the Sun, the solar system planets ($i = 1, 2$) inside and the solar system planets ($i = 4, 5, \dots, 8$) outside the Earth orbital radius, respectively. Here we again have made approximation that all the solar system planets are aligned on the straight line in planetary order, for simplicity, and approximation that in evaluating $F_M^E(M_\oplus - M_\odot)$ the Sun and the solar system planets do not have the rotational DOE, for simplicity. Note that the leading order prediction in $F_M^E(M_\oplus - M_\odot)$ in (3.10) is dominant comparing to the corresponding subleading order ones. Since the contributions of the spinning effects of the other solar system planets with M_j ($j = 1, 2, 4, \dots, 8$) are negligible as shown in (3.10), as in the case of the evaluation of $B_M(M_\oplus)$ in (3.8), for the leading order prediction of $F_M^E(M_\oplus - M_\odot)$ in (3.10) we can again remove effectively the constraint that all the solar system planets are aligned on the straight line in planetary order.

i -th Planet	$B_M(M_i)$ (sec^{-1})	$F_M^B(M_i - M_\odot)$	$F_M^E(M_i - M_\odot)$	$\chi(M_i - M_\odot)$
Mercury	5.423×10^{-17}	2.422×10^{-11}	-3.691×10^{-1}	6.563×10^{-11}
Venus	$-7.081 \times 10^{-17*}$	$-3.409 \times 10^{-10*}$	-1.558	2.188×10^{-10}
Earth	2.029×10^{-14}	1.019×10^{-7}	-1.000	1.019×10^{-7}
Mars	3.973×10^{-15}	1.737×10^{-9}	-4.622×10^{-2}	3.758×10^{-8}
Jupiter	1.388×10^{-12}	9.723×10^{-4}	-1.174×10	8.280×10^{-5}
Saturn	4.583×10^{-13}	7.097×10^{-5}	-1.046	6.784×10^{-5}
Uranus	$1.021 \times 10^{-13*}$	$1.703 \times 10^{-6*}$	-3.948×10^{-2}	4.314×10^{-5}
Neptune	1.355×10^{-13}	2.126×10^{-6}	-1.895×10^{-2}	1.122×10^{-4}

Table 1. $\vec{B}_M(M_i) = B_M(M_i)\hat{\omega}$, $\vec{F}_M^B(M_i - M_\odot) = F_M^B(M_i - M_\odot)\hat{r}$, $\vec{F}_M^E(M_i - M_\odot) = F_M^E(M_i - M_\odot)\hat{r}$ and $\chi(M_i - M_\odot) = |\vec{F}_M^B(M_i - M_\odot)|/|\vec{F}_M^E(M_i - M_\odot)|$ are listed for the i -th planet ($i = 1, 2, 3, \dots, 8$). Here $\hat{\omega}$ and $\hat{r}$ are the unit vector along the axis direction of the spinning Sun, and that along the radial vector from the Sun, respectively. The magnitudes of $F_M^B(M_i - M_\odot)$ and $F_M^E(M_i - M_\odot)$ are given in the unit of $F_M^E(M_\oplus - M_\odot)$. For the case of Venus, the direction of $\vec{B}_M(M_2)$ is along $-\hat{\omega}$ to produce the centripetal force $\vec{F}_M^B(M_2 - M_\odot)$, while for the case of Uranus, the direction of $\vec{B}_M(M_7)$ is along $-\hat{r}$ to yield the force $\vec{F}_M^B(M_7 - M_\odot)$ along $\hat{\omega}$. The predictions for the exceptional cases of Venus and Uranus are indicated by *.

$\vec{B}_M(M_i) = B_M(M_i)\hat{\omega}$, $\vec{F}_M^B(M_i - M_\odot) = F_M^B(M_i - M_\odot)\hat{r}$, $\vec{F}_M^E(M_i - M_\odot) = F_M^E(M_i - M_\odot)\hat{r}$ and $\chi(M_i - M_\odot) = |\vec{F}_M^B(M_i - M_\odot)|/|\vec{F}_M^E(M_i - M_\odot)|$ are listed for the i -th planet ($i = 1, 2, 3, \dots, 8$). Here, $\hat{\omega}$ and $\hat{r}$ are the unit vector along the axis direction of the spinning Sun and that along the radial vector from the Sun, respectively. The magnitudes of $F_M^B(M_i - M_\odot)$ and $F_M^E(M_i - M_\odot)$ are given in units of $|F_M^E(M_\oplus - M_\odot)|$. For the case of Venus, the direction of $\vec{B}_M(M_2)$ is along $-\hat{\omega}$ to produce the centripetal force ($\vec{F}_M^B(M_2 - M_\odot)$), while for the case of Uranus, the direction of $\vec{B}_M(M_7)$ is along $-\hat{r}$ to

yield the force ($\vec{F}_M^B(M_7 - M_\odot)$) along $\hat{\omega}$. The predictions for the exceptional cases of Venus and Uranus are indicated by *.

Next even the subleading order contributions of about order of $10^{18} N$ in $F_M^E(M_\oplus - M_\odot)$ in (3.10) are greater than the total contribution in $F_M^B(M_\oplus - M_\odot)$ in (3.9), implying that $F_M^B(M_\oplus - M_\odot)$ has been hidden in the Newtonian gravity. From (3.9) and (3.10) we arrive at the characteristic ratio $\chi(M_\oplus - M_\odot)$ of the form

$$\chi(M_\oplus - M_\odot) = |\vec{F}_M^B(M_\oplus - M_\odot)| / |\vec{F}_M^E(M_\oplus - M_\odot)| = 1.019 \times 10^{-7}, \quad (3.11)$$

implying that the correction $|\vec{F}_M^B(M_\oplus - M_\odot)|$ to the Newtonian classical prediction $|\vec{F}_M^E(M_\oplus - M_\odot)|$ is extremely small but it is non-vanishing as in the case of (3.5) for the isolated system of the Earth and the spinning Moon orbiting the spinning Earth.

Now, we investigate the general cases of the solar system planets with masses M_i ($i = 1, 2, 3, \dots, 8$) including the Earth with mass $M_3 = M_\oplus$. The predictions of $B_M(M_i)$ in (3.7) for the solar system planets with masses M_i are given in Table 1. Exploiting (2.25), we can readily construct $\vec{F}_M^B(M_i - M_\odot) = F_M^B(M_i - M_\odot)\hat{r}$ where $F_M^B(M_i - M_\odot)$ are listed in Table 1.⁷ Next, in order to find the characteristic ratios, we construct the mass electric Newtonian forces acting on the solar system planets $\vec{F}_M^E(M_i - M_\odot) = F_M^E(M_i - M_\odot)\hat{r}$ where $F_M^E(M_i - M_\odot)$ is given by

$$F_M^E(M_i - M_\odot) = -\frac{GM_i M_\odot}{R_i^2} - \sum_{j < i} \frac{GM_i M_j}{|R_j - R_i|^2} + \sum_{j > i} \frac{GM_i M_j}{|R_j - R_i|^2}. \quad (3.12)$$

Furthermore the predictions for the mass magnetic GR forces $F_M^B(M_i - M_\odot)$ associated with $B_M(M_i)$ in (3.7), the mass electric Newtonian forces $F_M^E(M_i - M_\odot)$ in (3.12), and the corresponding characteristic ratios

$$\chi(M_i - M_\odot) = |\vec{F}_M^B(M_i - M_\odot)| / |\vec{F}_M^E(M_i - M_\odot)| \sim 10^{-11} - 10^{-4}, \quad (3.13)$$

are also listed in Table 1. Here we observe that $F_M^B(M_i - M_\odot)$ and $F_M^E(M_i - M_\odot)$ acting on the solar system planets with M_i show the magnitudes of about order of $(10^{11} - 10^{19})N$ and $(10^{20} - 10^{23})N$, respectively. Note that for the case of Jupiter $B_M(M_5)$, $F_M^B(M_5 - M_\odot)$ and $F_M^E(M_5 - M_\odot)$ have the largest predicted values among those of the solar system planets because of its heaviest mass.

Note that the smallness of these characteristic ratios $\chi(M_m - M_\oplus)$ and $\chi(M_i - M_\odot)$ originate from the factor $\frac{G}{c^2}$ involved in $B_M(M_m)$ in (3.2) and $B_M(M_i)$ in (3.7), with which $F_M^B(M_m - M_\oplus)$ in (3.3) and $F_M^B(M_i - M_\odot)$ are constructed. In contrast, $F_M^E(M_m - M_\oplus)$ and $F_M^E(M_i - M_\odot)$ possess the factor

G only as shown in (3.4) and (3.12). Note also that, for the case of Venus, the direction of $\vec{B}_M(M_2)$ is along $-\hat{\omega}$ to produce the centripetal force $\vec{F}_M^B(M_2 - M_\odot)$. Next, for the case of Uranus, the direction of $\vec{B}_M(M_7)$ is along $-\hat{r}$ to yield the force $\vec{F}_M^B(M_7 - M_\odot)$ along $\hat{\omega}$.

C. B-ring of Saturn

Now it seems appropriate to address some comments on the rings of Saturn, to investigate the corresponding mass magnetic dipole moments. It is well known that the B-ring of Saturn is the brightest, most opaque and the most likely massive of Saturn's rings^[22]. Motivated by this, we study the isolated system of the B-ring of Saturn for simplicity. Making use of a multipole expansion valid at distant observation point, we find $\vec{A}_M$ in terms of the dominant mass magnetic dipole moment contribution⁸

$$\vec{A}_M = \frac{\vec{m}_M \times \vec{x}}{r^3}, \quad (3.14)$$

where $r = |\vec{x}|$ and the mass magnetic dipole moment $\vec{m}_M$ is given by

$$\vec{m}_M = -\frac{G}{2c^2} \int \vec{x}' \times \vec{J}_M d^3x'. \quad (3.15)$$

Note that the mass magnetic dipole moment $\vec{m}_M$ in (3.15) is the analogue of the charge magnetic dipole moment $\vec{m}_Q$ ^{[18][19]}, and the direction of $\vec{m}_M$ is opposite to that of $\vec{m}_Q$ because of the difference of the interaction patterns between repulsive EM and attractive mass gravitational forces.

Now we construct the mass magnetic dipole moment of the B-ring orbiting Saturn. To do this, we first find the period t_B of the B-ring orbiting Saturn. In the absence of non-gravitational fields, the B-ring of Saturn moves in circular orbit according to the Kepler's third law to a high degree of approximation^[23], where the period t_B is given by^[24]

$$t_B^2 = \frac{4\pi^2 R_B^3}{GM_6}. \quad (3.16)$$

Here R_B is the orbital radius of the B-ring. Note that in (3.16) we have neglected the effect of the B-ring mass M_B which is extremely small compared to the mass M_6 . Combining the formulas $\vec{m}_M$ in (3.15) and t_B in (3.16), for the B-ring of Saturn we finally arrive at the mass magnetic dipole moment $\vec{m}_M(Ring) = m_M(Ring)\hat{\omega}$ where⁹

$$m_M(Ring) = -\frac{G^{3/2} M_6^{1/2} M_B R_B^{1/2}}{2c^2} = -1.141 \times 10^4 \text{ m}^3 \text{ sec}^{-1}. \quad (3.17)$$

Note that the direction of this mass magnetic dipole moment is opposite to that of the charge magnetic dipole moment in the electromagnetic interaction, because of the difference of the interaction patterns between charge repulsive electromagnetic and mass attractive gravitational forces. Now the predicted value for $m_M(Ring)$ in (3.17) is intrinsically associated with the Cassini data^[16] concerning the total mass of the rings of Saturn via the explicit formula possessing M_B in (3.17), which is the approximated total mass of Saturn's rings^[22].

Next we investigate the recent Cassini data^[17] on the formation of the young rings through a grazing encounter of an additional satellite, which is named Chrysalis and is assumed to be predominantly composed of water ice, with Saturn. This encounter of Chrysalis with Saturn would have caused Chrysalis to break apart to yield debris which could have developed into Saturn's young rings^{[25][17]}. The loss of the hypothetical satellite Chrysalis can then explain the obliquity of Saturn and the young age of its rings. Moreover the value of $m_M(Ring)$ in (3.17), which could come from the debris mass of Chrysalis via (3.17), thus could be related with the the formation mechanism^[17] of Saturn's rings.

IV. Conclusions

In summary, we have constructed the formalism of the gravitomagnetism by making use of the MLGR. Exploiting this gravitomagnetism we have evaluated *numerically* the astrophysical quantities such as mass magnetic GR force and the mass magnetic dipole moment of the Saturn's B-ring, for instance. To be more specific, in order to identify the effectiveness of the gravitomagnetism, we have treated the astrophysical observables such as the mass magnetic fields for both the isolated system of the Moon orbiting the spinning Earth and that of the solar system planets orbiting the spinning Sun. Exploiting the mass magnetic fields, we have predicted the ensuing Lorentz type mass magnetic GR forces for these systems. Note that, in investigating the gravitomagnetism, we have used the MLGR associated with the Wald Approximation, different from the LGR possessing the TT gauge. Note also that, in the MLGR related with the gravitomagnetism, the angle dependence of the GW radiation can be explained in the vectorial formalism^[10]. In contrast, the LGR is defined in the second rank tensorial formalism mathematically different from the vectorial one in the MLGR, and the LGR can not present the angle dependence of the GW radiation^[10]. The gravitomagnetism in this paper has predicted the existence and ensuing numerical estimates of the physical observable such as the mass magnetic GR force which is hidden in the Newtonian gravity. Note that the LGR can not predict the astrophysical physical quantities such as the mass magnetic fields and the mass magnetic GR forces for instance.

Next, exploiting the Maxwell type gravitational equations in the gravitomagnetism we have obtained the mass electric and mass magnetic fields $\vec{E}_M$ and $\vec{B}_M$ with which we have constructed the corresponding mass electric Newtonian and Lorentz type mass magnetic GR forces $\vec{F}_M^E$ and $\vec{F}_M^B$. One of the advantages of the gravitomagnetism is to evaluate the mass magnetic field $\vec{B}_M$, whose phenomenological value is hidden in the Newtonian gravity and is newly predicted in this paper. For instance, in the isolated Earth of mass $M_\oplus$ and the Moon of mass M_m we have found $|\vec{B}_M(M_\oplus)|$ acting on the Earth, while in the isolated system of the Sun of mass $M_\odot$ and solar system planets of masses M_i ($i = 1, 2, 3, \dots, 8$) we have obtained $|\vec{B}_M(M_\odot)|$ acting on the Sun, which is about $10^2 \times |\vec{B}_M(M_\oplus)|$. Moreover we have noticed that the mass magnetic GR force $\vec{F}_M^B$ is relatively small, comparing with $\vec{F}_M^E$. To be specific, considering the isolated system of the Moon and Earth, in the gravitomagnetism we have estimated $|\vec{F}_M^B(M_m - M_\oplus)|$ acting on the Moon which is approximately 10^{-11} of the mass electric Newtonian force $|\vec{F}_M^E(M_m - M_\oplus)|$. For the case of the Earth and the Sun we have found that $|\vec{F}_M^B(M_\oplus - M_\odot)|$ is about 10^{-7} of the corresponding mass electric Newtonian force $|\vec{F}_M^E(M_\oplus - M_\odot)|$. Next for the isolated Sun and solar system planets, in the gravitomagnetism we have evaluated $|\vec{F}_M^B(M_i - M_\odot)|$ acting on the solar system planets, which are about order of $10^{-11} - 10^{-4}$ of the corresponding mass electric Newtonian forces $|\vec{F}_M^E(M_i - M_\odot)|$ as shown in Table 1. This feature implies that the gravitomagnetic correction, namely $|\vec{F}_M^B(M_i - M_\odot)|$, to the Newtonian classical prediction $|\vec{F}_M^E(M_i - M_\odot)|$ is extremely small but it is non-vanishing. To be specific, this gravitomagnetic correction has been hidden in the astrophysical phenomenology until now. The theoretical prediction of the mass magnetic GR force in the gravitomagnetism is one of the main points of this paper. Note that $|\vec{F}_M^B(M_i - M_\odot)|$ possesses the leading order correction, which originates from the spinning effects of the solar system planet of M_i , and this correction is dominant comparing to the other next order ones. Note also that the directions of $\vec{F}_M^B(M_i - M_\odot)$ for the solar system planets except Venus of mass M_2 and Uranus of mass M_7 are anti-parallel to those of $\vec{F}_M^E(M_i - M_\odot)$, while the directions of $\vec{F}_M^B(M_2 - M_\odot)$ and $\vec{F}_M^B(M_7 - M_\odot)$ for Venus and Uranus are parallel to and perpendicular to those of $\vec{F}_M^E(M_i - M_\odot)$ ($i = 2, 7$), respectively.

Now, as a typical example of the mass magnetic dipole moment $\vec{m}_B$, in the gravitomagnetism we have investigated Saturn's B-ring that orbits Saturn, to predict $\vec{m}_M(\text{Ring})$. Note that the prediction of $\vec{m}_M(\text{Ring})$ has been shown to be closely related with the Cassini data^[16] on the total mass of the rings of Saturn. Next $\vec{m}_M(\text{Ring})$, which could originate from the debris mass of the hypothetical satellite Chrysalis^[17], also could be associated with the formation of the young rings of Saturn. Note that the mass

magnetic GR force $\vec{F}_M^B(\text{Ring})$ acting on the B-ring of Saturn contributes in small amounts to the formation of the width of the B-ring.

Next, we have *predicted* phenomenologically in this paper the physical quantities such as the mass magnetic fields, their ensuing mass magnetic GR forces which exist in the Universe as far as we propose the gravitomagnetism (or MLGR). It will be interesting to search for observational evidences for these physical quantities in the Universe. Once this is done, the theoretical formalism developed in this paper could give some progressive impacts on the *precision astrophysics phenomenology*. Note that there have been searches on the GR effects such as the relativistic precession and frame dragging conducted saliently by the Gravity Probe B satellite that is well documented in Refs.^{[2][26][3]} and references therein.

Appendix A. Sketch of EM

In order to construct the gravitomagnetism formalism discussed in Section 2, we will digress to pedagogically recapitulate the EM associated with charge Q ^{[18][19]}. To do this, we start with the charge electric and charge magnetic fields in the EM where we find the Maxwell equations in the EM

$$\begin{aligned}\nabla \cdot \vec{E}_Q &= \frac{1}{\epsilon_0} \rho_Q, & \nabla \times \vec{E}_Q + \frac{\partial \vec{B}_Q}{\partial t} &= 0, \\ \nabla \cdot \vec{B}_Q &= 0, & \nabla \times \vec{B}_Q - \frac{1}{c^2} \frac{\partial \vec{E}_Q}{\partial t} &= \frac{1}{\epsilon_0 c^2} \vec{J}_Q,\end{aligned}\tag{A.1}$$

where ρ_Q and $\vec{J}_Q$ are the volume charge density and volume charge current, respectively. Here we have used $\epsilon_0 \mu_0 = c^{-2}$ where ϵ_0 , μ_0 and c are the permittivity, permeability and speed of light of free space, respectively. Note that the charge electric and mass magnetic fields E_Q^i and B_Q^i are given in terms of the four charge vector potentials $A_Q^\alpha \equiv \left(\frac{\phi_Q}{c}, \vec{A}_Q \right)$

$$\vec{E}_Q = -\nabla \phi_Q - \frac{\partial \vec{A}_Q}{\partial t}, \quad \vec{B}_Q = \nabla \times \vec{A}_Q.\tag{A.2}$$

We also find that the charge vector potentials A_Q^α fulfill

$$\square \phi_Q = -\frac{1}{\epsilon_0} \rho_Q, \quad \square \vec{A}_Q = -\frac{1}{\epsilon_0 c^2} \vec{J}_Q,\tag{A.3}$$

where we have exploited the Lorentz gauge condition

$$\partial_\alpha A_Q^\alpha = 0.\tag{A.4}$$

In vacuum, inserting $\rho_Q = \vec{J}_Q = 0$ into (A.3), we find the wave equation for A_Q^α

$$\square A_Q^\alpha = 0, \quad \alpha = 0, 1, 2, 3\tag{A.5}$$

which implies a massless spin-one photon propagating in the flat spacetime.

Next we investigate the gauge invariance and U(1) transformation in the EM. To do this, we start with the Dirac equation for the electron wave function ψ and the charge vector potentials A_Q^μ in the unit of $\hbar = c = 1$

$$i\gamma^\mu \partial_\mu \psi - m_e \psi - e\gamma^\mu A_{Q,\mu} \psi = 0, \quad (\text{A.6})$$

where m_e and e are the electron mass and charge, respectively. Next the QED describes the interactions between electrons and photons. Now we introduce the QED Lagrangian^[21]

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{Q,\mu\nu}F_Q^{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu - m_e)\psi - e\bar{\psi}\gamma^\mu A_{Q,\mu}\psi, \quad (\text{A.7})$$

where $F_Q^{\mu\nu} = \partial^\mu A_Q^\nu - \partial^\nu A_Q^\mu$ and the third term denotes the interaction between the electron and the electromagnetic wave (or spin-one photon). Note that the QED Lagrangian is invariant under the gauge transformations

$$\begin{aligned} \psi(x) &\rightarrow \psi'(x) = e^{-ie\Gamma(x)}\psi(x), \\ A_Q^\mu(x) &\rightarrow A_Q'^\mu(x) = A_Q^\mu(x) + \partial^\mu \Gamma(x), \end{aligned} \quad (\text{A.8})$$

where the first relation is the U(1) transformation.

Next, for the cases of charge electrostatics and charge magnetostatics, corresponding to ρ_Q and $\vec{J}_Q$ inside a given volume we obtain the charge electric and charge magnetic fields

$$\begin{aligned} \vec{E}_Q &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho_Q \hat{R}}{R^2} d^3x', \\ \vec{B}_Q &= \frac{1}{4\pi\epsilon_0 c^2} \int \frac{\vec{J}_Q \times \hat{R}}{R^2} d^3x', \end{aligned} \quad (\text{A.9})$$

where $\vec{R} = \vec{x} - \vec{x}'$ is the vector from d^3x' to the field point $\vec{x}$ and $\hat{R} = \vec{R}/R$ with $R = |\vec{x} - \vec{x}'|$. Exploiting the charge electric and magnetic fields $\vec{E}_Q$ and $\vec{B}_Q$ in (A.9) we end up with the corresponding force acting on a test charge Q_0 moving with velocity v

$$\vec{F}_Q = Q_0(\vec{E}_Q + \vec{v} \times \vec{B}_Q). \quad (\text{A.10})$$

Moreover, in the charge magnetostatics, we construct the charge vector potential $\vec{A}_Q$ as follows

$$\vec{A}_Q = \frac{1}{4\pi\epsilon_0 c^2} \int \frac{\vec{J}_Q}{R} d^3x'. \quad (\text{A.11})$$

Making use of a multipole expansion valid at distant observation point, we reformulate $\vec{A}_Q$ in terms of the dominant charge magnetic dipole moment contribution

$$\vec{A}_Q = \frac{\vec{m}_Q \times \vec{x}}{r^3}, \quad (\text{A.12})$$

where $r = |\vec{x}|$ and the charge magnetic dipole moment $\vec{m}_Q$ is given by

$$\vec{m}_Q = \frac{1}{8\pi\epsilon_0 c^2} \int \vec{x}' \times \vec{J}_Q d^3 x'. \quad (\text{A.13})$$

Statements and Declarations

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Data Availability

No data has been used in the work.

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Conflicts of Interest

The author declares no conflict of interest.

Footnotes

¹ The superscripts E and B stand for the mass *electric* Newtonian force and the mass *magnetic* force, respectively.

² Here we have exploited the identity $\int_0^\pi \frac{d\theta' \sin \theta' \cos \theta'}{R} = \frac{2r'}{3r^2}$.

³ Here we have assumed that this observer resides on a space fixed frame and does not co-rotate together with the Earth so that we can include the spinning effect of the Earth. The same statement will be applied to the observers who are located on the surface of the Moon, the Sun and the solar system planets.

⁴ For the observational data for M_I , r_I , R_I , t_I and T_I ($I = m, \oplus$) we exploit [20][27][28]. Similarly, we will also use [20][27][28] for the observational data for M_i , r_i , R_i , t_i and T_i ($i = 1, 2, \dots, 8$) for the i -th planets in the solar system which will be discussed in the next subsections.

⁵ From now on $\vec{F}_M^B(M_m - M_\oplus)$ denotes that the force $\vec{F}_M^B$ acts on the position at M_m under the field at $M_\oplus$ for instance. Here the argument $M_m - M_\oplus$ indicates the vector of the direction from $M_\oplus$ to M_m with magnitude $|M_m - M_\oplus|$.

⁶ From now on the minus sign in $F_M^E(M_m - M_\oplus)$ indicates the attractive Newtonian force between M_m and $M_\oplus$ for example.

⁷ In Table 1, the magnitudes of $F_M^B(M_i - M_\odot)$ and $F_M^E(M_i - M_\odot)$ are given in the unit of $|F_M^E(M_\oplus - M_\odot)|$ and these values are dimensionless. The minus signs in $F_M^E(M_i - M_\odot)$ again denote the attractive Newtonian forces between M_i and $M_\odot$.

⁸ Here we have exploited the approximation at distant observation point $\vec{A}_M = -\frac{G}{c^2} \left(\frac{1}{r} \oint \vec{J}_M d^3x' + \frac{x_i}{r^3} \int \vec{J}_M x'_i d^3x' + \dots \right)$ where the ellipsis stands for the higher order terms which can be ignored for the mass vector potential of a localized mass current distribution. Note that the first term denotes the mass magnetic monopole term which vanishes for the closed loop integral, as expected. In this paper, we thus exclude the possibility of mass magnetic monopole in addition to that of charge magnetic monopole.

⁹ The recent Cassini data show that the total mass of the rings of Saturn is $1.54 \times 10^{19} \text{ kg}$ [16]. In evaluating (3.17), we have exploited this value for the B-ring mass M_B for simplicity, since the B-ring is known to be probably the most massive of Saturn's rings [22]. Next the orbital radius of the B-ring is given by $R_B = (9.232 \times 10^7 - 1.177 \times 10^8) \text{ m}$ [20], and thus we have used the average value of $R_B = 1.050 \times 10^8 \text{ m}$. Moreover we have ignored the effect of the width of the B-ring of Saturn in the evaluation in (3.17), for simplicity.

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Declarations

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