Review of: "Representation of physical quantities: From scalars, vectors, tensors and spinors to multivectors"

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Potential competing interests: No potential competing interests to declare.

I will send a .pdf file. The text is hereunder

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Review YOQ9KZ

Representation of physical quantities:

From scalars, vectors, tensors and spinors to multivectors

General assessment

The paper is relatively original, as it proposes a wide approach on fundamental concepts.**This could even lead to a change in the title of the paper (see conclusion)**.

The study is well-founded, clear, and suitably formatted. It is, in fact, a synthetic (educational) paper, which could ask for additional precisions (but it would be too long). Thus, it may be published **subject to the numerous slight modifications detailed below**. They are **explicitly indicated in red** (33 notes in total).

The general introduction precisely proposes a pertinent description of the concerned conceptual model and historical context. This is a pertinent point.

Page 2

Tensors, then spinors by cyclic transforms, are addressed, mentioning the special Penrose's twistor (6D representation, stopped). The relative 3-D limitations/specificities of Vectorial Algebra (VA) are evocated, as well as Clifford's Geometric Products (GP) with scalars, vectors, bivectors, multivectors.

Note 1: **Hestenes prolongated Clifford's work in GA (Geometric Algebra) by including vectors, spinors, and complex numbers in a single system with geometrical significance. The advantage of GA vs GP is said to be similar to those of algebra over arithmetic… This point could be reinforced and precised at this level by one or two additional sentences.**

Page 3

Starting from the three products – dot, wedge & geometric – of two vectors, the process is described. Grassmann introduced the geometric product between multivectors of higher grades.

Note 2: **The perspective here is to embrace vector, complex, and spin in a single formalism, and make emerge the GA. A short addition could strengthen this point.**

§1. Products

The 3-D antisymmetric covariant tensor is used to present the cross-product.

Note 3: **The classical Einstein's index and Levi-Civita symbol are used, as it is mentioned that the e_i vectorial basis is orthogonal. It could be precised that the notation could be extended to a non-orthogonal basis, by explicit distinction between covariant and contravariant coordinates.**

Page 4

Note 4: **Among the examples mentioned on top of page 4 (polarizability tensor, inertia tensor), it could also be precised that mechanical (generally symmetrical) tensorial fields are relevant: stress, strain, piezoelectric, …. At the end of the same section, the Gaussian tensor of curvature on a surface can be mentioned, in the frame of the theory of shells, for instance.**

Note 5: **After equation (3) (sign and Jacobian), it is necessary to insist on the link between the product and the behavior of symmetrical/antisymmetrical or explicit/pseudo- entities vs geometrical transforms. In 3-D, the illustration is given by the invariance of a pseudo-vector (associated with a 3-D antisymmetrical tensor) vs the mirror-symmetry.**

Page 5

In the section concerning axial – polar vectors ("In the usual vector equation u = v …"), it is relevant to mention the V-AV Lagrangian (weak interaction, Sudarshan and Marshak).

Note 6: **It could be pertinent to indicate here that this distinction can be also applied to the gyroscopic component in classical solid rotating motion around a fixed point.**

In the following paragraph ("Finally, we note the difficulties…"), it is relevant to mention that "the product vector lacks an absolute direction."

Note 7: **It is possible to indicate here that Einstein himself, in the 1930 publication about gravitation (General Relativity), mentioned that he considers a translational non-galilean reference (and even with a given linear acceleration), but no rotational motion of the reference. This lead – in a non-relativistic approach – to the Coriolis term, which is not in the Ricci tensor.**

§1.1 Dyadic-tensor product of vectors

This section gives an excellent (educational) classical synthesis.

Page 6

§1.2 Properties of dyads

This section (page 6) is also an excellent presentation of the basic properties.

Note 8: **These properties could be explicitly linked with Lie-algebra structures whose general characteristics had been defined and classified by Bourbaki as Groups, Rings, Bodies, and finally vectorial spaces.**

Pages 7 and 8 (end §1.2)

Page 7 is a very interesting section, introducing the invariants, the Up-Lo decomposition.

Note 9: At this level (end of page 7, for instance), it could be interesting to add the mention of spherical – deviatoric contributions, whose role is fundamental in the physics of fields.

The end of §1.2 (page 8) precisely reflects the previous note.

Note 10: It could be possible to mention here (end of §1.2) that the Gaussian tensor of curvature on a surface (2-3-D) is equivalent to the tensor of Ricci (4-5-D) in Einstein's theory.

Page 8 §2

This first section (bottom of page 8) is fundamental regarding the exterior-Wedge product.

Note 11: **The mention of pseudo-scalars highlights the link – the identity – with the complex base. This could be said explicitly.**

Note 12: **It seems that §2 is too long. It could be structured in 2 or 3 sub-paragraphs. The fundamental definitions mentioned in the previous note could constitute §2.1.**

Page 9

This part is important as it is finally based on the permutation-rotation.

Note 13: **This could be explicitly mentioned.**

Page 10

Note 14: **Comment (ii) raises the link with associative but non-commutative (anti-commutative) algebraic laws.**

Note 15: **Comment (iii) mentions a " kind of closure property ". This is interesting and could be linked with the topological closure.**

Page 11

Note 16: **It could be interesting here to reinforce the 3 level particularities and their possible extension: 3-D space, orthogonal vectors, normed base.**

Pages 12 & 13

Note 17**: This section about k-blades is essential and could lead to an explicit sub-paragraph in §2.**

Note 18: **At the end of page 12, and on page 13, appears the numbering which links to the arithmetics… This could also lead to an explicit sub-paragraph.**

This part is particularly difficult, but the example with Coriolis (page 14) contributes to lightening it.

Page 14

Note 19: **The last section of paragraph 2 is a little confusing. In fact, it could be possible either to shorten it (recommended) or to detail it (but it could be too long).**

The beginning of §3 (end of page 14) is clear and convenient.

Pages 15 & 16 §3.1 Quaternion algebra

This paragraph is very clear and can be considered as a valid abstract with educational possible applications. It does not call for any additional comment.

Pages 16 & 17 §3.2 Euler-type formula

To pass from the arithmetical definition to the algebraic use, the Taylor expansion is mentioned in order to define the basic exponential function (which will be the base of any derivative-differential process).

Note 20: **In this paragraph – as well as in the previous one and in several parts of the paper… – it is necessary to refer – in the text – to the bibliographical references, cited with [].**

Note 21: **The mentioned properties could be illustrated – in this paragraph as well as in the previous one – with the success of Quaternionic Modeling for the classical Euler-Poinsot rotational motion, the Lagrange-Poisson potential solution, and also in the field of digital solid robotics.**

Pages 18 & 24 §3.4 Spinor

Note 22: **This paragraph is essential but much too long.**

Note 23: **It must be separated from the conclusion, which is not itself clearly indicated but seems to appear at the middle of page 24.**

Note 24: **Consequently, even if the sense of the text is interesting, the paper (the organization of the paper) must**

be restructured from page 18 to 25. One eventuality could be to suppress §3.4 and to replace it by a §4 dedicated to spinor and topology (§4 could be itself decomposed in 4.1, 4.2, …). §4.1 could be the topological sense of the spinor.

The part situated in the manuscript second part of page 18 and the first part of page 19 mentions the closure with inversion, and it is perfect to make an allusion to Möbius as that is actually a topological inversion. The link with the spin is immediate, and the mention of Pauli's matrix is clear.

The second part of page 19 ("Also, since a spinor represents a probability…") is less clear as there is a combination of a probabilistic approach and the Lorentz transform.

Note 25: **In order not to propose any definitive choice, it could be prudent to add a generic sentence on page 19 (just after "Further insightful developments by Dirac, Weil, and Majorana will be discussed in the next part of our study"). This generic sentence could be "This paper is a synthesis of several existing approaches."**

Note 26: **It could be pertinent to assemble the last section of page 19 and the first section of page 20 in a §4.2 (particle representation).**

Page 20 to the middle of page 21

Note 27: **The section from "The twistor theory, proposed by Penrose…" to "… important later work of Grassmann" could constitute a §4.3 (Fields and algebraic developments). But to be clear, it is very dense and relatively difficult to synthesize.**

Middle page 21 & to the middle of page 22

Note 28: **The section from "Clifford algebra incorporates…" to "…geometric product of the cliff and the pseudoscalar" could be §4.4 (orthogonality and multivectors in fields).**

Middle page 22 & to the middle of page 24

Note 29: **The section from "In geometrical calculus…" to "…and –1 for odd permutations" could close §4, as section §4.5, with the possible title: A variety of products for differential geometry.**

It is clear and relatively easy to visualize, as it is close to the conclusion.

Middle page 24 & to page 29

The section beginning with "Geometrical product equips the vector space with an algebraic structure…" could be considered as the beginning of the conclusion, but this is not feasible as this conclusion would be much too long (as this very complete section actually ends on page 29 with item 14).

Note 30: **A possibility could be to call all this section (without modification, and including the 14 items) §5 as "Synthesis of the presentation."**

But a real – and concise – conclusion is now missing.

Note 31: **Consequently, after item 14, we suggest to add a 10-15 line paragraph called "Conclusion" explaining the goal of this paper and how it is reached by a global approach and presentation.**

Note 32: **It is necessary to specify that the paper's topic is wide (!) and that its goal is descriptive. Thus, we suggest to slightly modify the title as: Representation of physical quantities, from scalars, vectors, tensors, and spinors to multivectors. A synoptic description.**

Bibliography

The bibliography is clear and convenient.

Note 33: **Nevertheless, even if the numbering of the references is correct, the information must be inventoried [1], [2], [3]… in the biblio and referenced like this (with this typography) in the text of the paper.**

The graphic quality of the paper is globally correct.