Research Article

Classification of the Travelling Wave Solutions of the Sharma–Tasso–Olver Equation

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The Sharma–Tasso–Olver (STO) equation is an integrable nonlinear partial differential equation that arises in various physical contexts, including fluid dynamics, plasma physics, and nonlinear optics. In this work, a complete analytical treatment of the travelling wave solutions of the STO equation is provided. The analysis reveals a family of exact solutions broader than those found in the literature.

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1. Introduction

The Sharma–Tasso–Olver (STO) equation is a nonlinear partial differential equation (PDE) used in mathematical physics and engineering. It is typically written in the form:

$$u_t + \alpha(u^3)_x + \frac{3}{2}\alpha(u^2)_{xx} + \alpha u_{xxx} = 0, \tag{1}$$

where $u(x,t)$ represents the wave profile depending on the spatial coordinate x and temporal coordinate t, while α is a non-zero real parameter.

This equation has attracted considerable attention due to its mathematical richness and physical relevance. Its double-dispersive nature makes it significant for studying complex nonlinear wave phenomena and soliton interactions in diverse physical systems, including fluid dynamics, plasma physics, optical fibers, and ocean wave dynamics[1][2]. On the other hand, the STO equation belongs to the Burgers hierarchy of equations and possesses integrability properties, including infinitely many symmetries and the existence of Lax pairs and Bäcklund transformations[3].

Moreover, several generalizations of the STO equation have been proposed, including the conformable fractional version[4], the external force version[5], and the generalized version with variable coefficients[6].

A standard technique for finding exact solutions to PDEs like the STO equation is the travelling wave reduction. This involves the search of solutions that maintain their shape while propagating at a constant velocity. The reduction is given by the ansatz

$$u(x,t) = y(\xi),$$

where $\xi = kx - \omega t$. The parameter k is the wave number and ω represents the wave speed. We will restrict ourselves to the case where k and ω are both positive, for simplicity of the exposition.

Applying this transformation to equation (1) we obtain the third-order ordinary differential equation (ODE):

$$-\omega y' + 3\alpha k y^2 y' + 3\alpha k^2 (y y'' + (y')^2) + \alpha k^3 y''' = 0.$$

This ODE can be integrated once with respect to ξ, giving rise to the second-order ODE

$$-\omega y + \alpha k y^3 + 3\alpha k^2 y y' + \alpha k^3 y'' = C, \qquad (2)$$

where $C \in \mathbb{R}$ is an arbitrary constant.

A critical examination of the existing literature reveals a pattern: the vast majority of studies investigating travelling wave solutions of the STO equation proceed by setting particular values for the integration constant C, and frequently assuming implicitly $C = 0$. This simplification is common across numerous papers, where different analytical techniques are applied to solve the STO equation, including the (G'/G)-expansion method[7], sub-equation methods[2], tanh-coth methods[8], extended hyperbolic function method[9], and various other approaches[1][10][11][12]. The prevalence of this assumption appears to be driven primarily by mathematical convenience, as many standard solution techniques are explicitly designed for, or significantly simplified by, homogeneous equations.

However, from both mathematical and physical perspectives, there is no inherent reason why the integration constant C must vanish. The general form of the ODE with $C \neq 0$ represents a broader class of solutions. The systematic neglect of this non-zero constant case constitutes a significant gap in the understanding of the full solution space of the STO equation.

In this paper, we present a comprehensive analysis of the travelling wave solutions of the STO equation with explicit consideration of the non-zero integration constant case, which significantly extends the insight into the dynamics of the system. Our investigation employs basic techniques of mathematical analysis, but represents, to the best of our knowledge, the first complete study of the STO travelling wave ODE.

2. Travelling wave solutions of the STO equation

In order to classify the travelling wave solutions of the STO equation (1), we apply to the second-order ODE (2) the transformation

$$y = k\frac{w'}{w}, \tag{3}$$

with $w = w(\xi)$ a smooth function to be determined. We obtain the following third-order ODE:

$$w''' - \frac{\omega}{\alpha k^3}w' - \frac{C}{\alpha k^4}w = 0. \tag{4}$$

The characteristic polynomial of this homogeneous linear equation with constant coefficients is

$$\lambda^3 - \frac{\omega}{\alpha k^3}\lambda - \frac{C}{\alpha k^4} = 0. \tag{5}$$

The nature of the roots of this depressed cubic depends on the sign of the discriminant, which is given by

$$\Delta = \frac{4\omega^3 - 27C^2\alpha k}{\alpha^3 k^9}. \tag{6}$$

Taking into account that $\omega, k > 0$, the sign of Δ is determined solely by the parameters α and C:

Case 1: $\Delta > 0$. This case occurs when $\alpha > 0$ and $C^2 < \frac{4\omega^3}{27\alpha k}$. The characteristic polynomial has three *distinct real* roots, given by the expression

$$\lambda_j = 2\sqrt{\frac{\omega}{3\alpha k^3}}\cos\left(\frac{1}{3}\arccos\left(\frac{3C}{2}\sqrt{\frac{3\alpha k}{\omega^3}}\right) - \frac{2\pi j}{3}\right), \tag{7}$$

for $j = 1, 2, 3$.

This way, the general solution to equation (4) is given by

$$w(\xi) = A_1 e^{\lambda_1\xi} + A_2 e^{\lambda_2\xi} + A_3 e^{\lambda_3\xi}, \tag{8}$$

where A_1, A_2, and A_3 are arbitrary constants. The corresponding solution to equation (2) is given by

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$$y(\xi) = k\frac{A_1\lambda_1 e^{\lambda_1\xi} + A_2\lambda_2 e^{\lambda_2\xi} + A_3\lambda_3 e^{\lambda_3\xi}}{A_1 e^{\lambda_1\xi} + A_2 e^{\lambda_2\xi} + A_3 e^{\lambda_3\xi}}. \tag{9}$$

Assuming, without loss of generality, that $A_3 \neq 0$, we can rename $C_1 = A_1/A_3, C_2 = A_2/A_3$, and then the general solution to the STO equation (1) is given by

$$u(x,t) = k\frac{C_1\lambda_1 e^{\lambda_1(kx-\omega t)} + C_2\lambda_2 e^{\lambda_2(kx-\omega t)} + \lambda_3 e^{\lambda_3(kx-\omega t)}}{C_1 e^{\lambda_1(kx-\omega t)} + C_2 e^{\lambda_2(kx-\omega t)} + e^{\lambda_3(kx-\omega t)}}, \tag{10}$$

where C_1, C_2 and C are the parameters.

In Figure 1, we can find a graphic representation of solution (10) for particular values of the parameters.

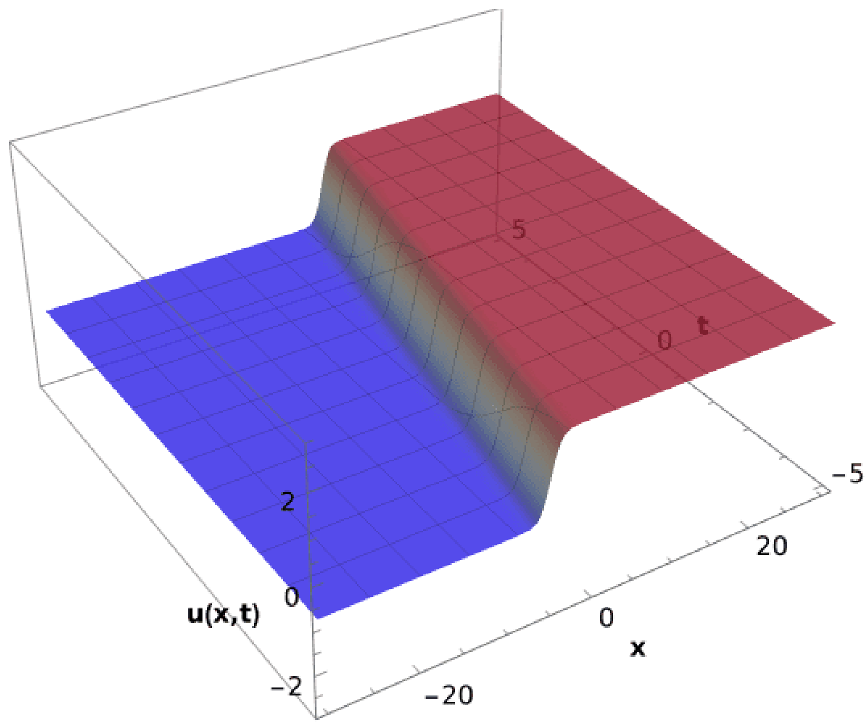

Figure 1. Graphic representation of solution (10) for $C = 0.2, C_1 = 0, C_2 = 2, \alpha = 0.5, \omega = 1, k = 1$.

Case 2: $\Delta < 0$. This happens when $\alpha > 0$ and $C^2 > \frac{4\omega^3}{27\alpha k}$; or when $\alpha < 0$ and C takes any value. In this case, the characteristic polynomial has one real root and a pair of complex conjugate roots. The real root is given by the expression

$$\lambda = \sqrt[3]{\frac{C}{2\alpha k^4} + \sqrt{\frac{27C^2\alpha k - 4\omega^3}{108\alpha^3 k^9}}} + \sqrt[3]{\frac{C}{2\alpha k^4} - \sqrt{\frac{27C^2\alpha k - 4\omega^3}{108\alpha^3 k^9}}}. \tag{11}$$

On the other hand, the complex conjugate roots are given by $a \pm ib$, where

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$$
\begin{aligned}
a &= -\frac{1}{2}\left(\sqrt[3]{\frac{C}{2\alpha k^4}+\sqrt{\frac{27C^2\alpha k-4\omega^3}{108\alpha^3k^9}}}+\sqrt[3]{\frac{C}{2\alpha k^4}-\sqrt{\frac{27C^2\alpha k-4\omega^3}{108\alpha^3k^9}}}\right),\\
b &= \frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{C}{2\alpha k^4}+\sqrt{\frac{27C^2\alpha k-4\omega^3}{108\alpha^3k^9}}}-\sqrt[3]{\frac{C}{2\alpha k^4}-\sqrt{\frac{27C^2\alpha k-4\omega^3}{108\alpha^3k^9}}}\right).
\end{aligned}
\tag{12}
$$

Observe that $\lambda = -2a$, so the general solution to equation (4) can be written as

$$
w(\xi) = A_1 e^{a\xi}\cos(b\xi) + A_2 e^{a\xi}\sin(b\xi) + A_3 e^{-2a\xi}, \tag{13}
$$

where A_1, A_2, and A_3 are arbitrary constants. The corresponding solution to equation (2) is then

$$
y(\xi) = k\,\frac{(aA_1 + bA_2)\cos(b\xi) + (aA_2 - bA_1)\sin(b\xi) - 2a\,A_3 e^{-3a\xi}}{A_1\cos(b\xi) + A_2\sin(b\xi) + A_3 e^{-3a\xi}}. \tag{14}
$$

By assuming $A_3 \neq 0$, we can define $C_1 = A_1/A_3, C_2 = A_2/A_3$, in such a way that we obtain the three-parameter family of solutions to the STO equation (1):

$$
u(x,t) = k\,\frac{(aC_1 + bC_2)\cos(b(kx-\omega t)) + (aC_2 - bC_1)\sin(b(kx-\omega t)) - 2ae^{-3a(kx-\omega t)}}{C_1\cos(b(kx-\omega t)) + C_2\sin(b(kx-\omega t)) + e^{-3a(kx-\omega t)}}, \tag{15}
$$

where C_1, C_2 and C are arbitrary constants. This solution is visualized in Figure 2 for particular values of the parameters.

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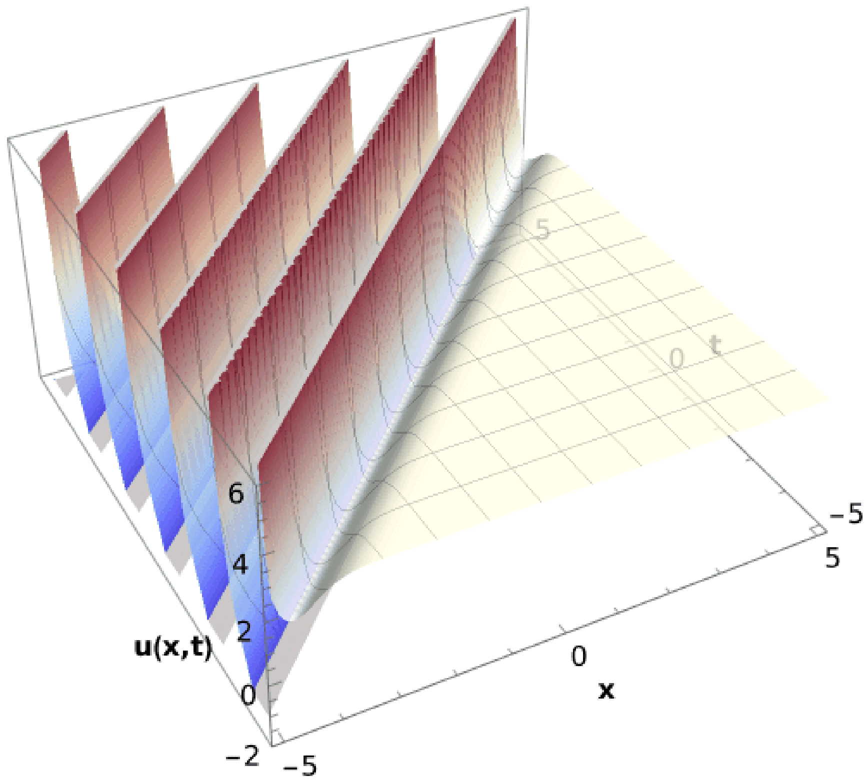

Figure 2. Graphic representation of solution (15) for

$C = 10, C_1 = 1, C_2 = 2, \alpha = 1, \omega = 1, k = 1.$

For $A_3 = 0$, and assuming $A_2 \neq 0$, we can define $C_1 = A_1/A_2$, and we have the corresponding two-parameter family of solutions to the STO equation (1) given by

$$u(x,t) = k\frac{(aC_1 + b)\cos(b(kx - \omega t)) + (a - bC_1)\sin(b(kx - \omega t))}{C_1\cos(b(kx - \omega t)) + \sin(b(kx - \omega t))}. \tag{16}$$

This solution is visualized in Figure 3 for particular values of the parameters.

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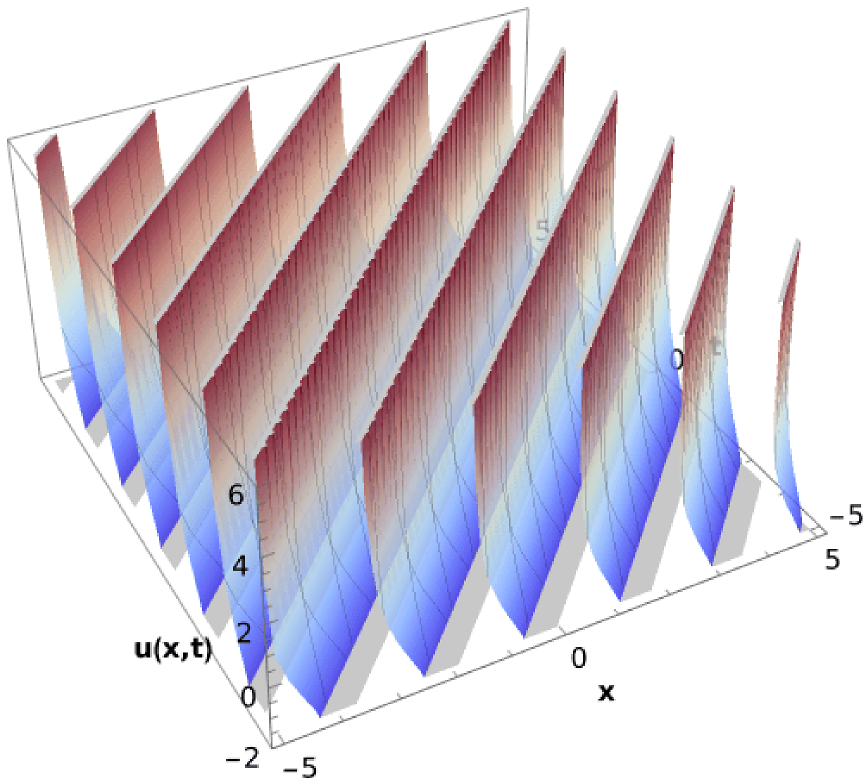

Figure 3. Graphic representation of solution (16) for

$C=10, C_1=1, \alpha=1, \omega=1, k=1.$

Finally, if $A_3=A_2=0$, we can define $C_1=A_1$, and we have the one-parameter family of solutions to the STO equation (1) given by

$$u(x,t)=ka-kb\tan(b(kx-\omega t)), \tag{17}$$

with C a parameter implicit in a and b.

Case 3: $\Delta=0$. This case occurs when $\alpha>0$ and we fix the value of the parameter C to

$$C=\pm\frac{2}{3}\sqrt{\frac{\omega^3}{3\alpha k}}.$$

The cubic (5) has a double root and a simple root. A straightforward factorization shows that the simple root is

$$\lambda_1=\pm 2\sqrt{\frac{\omega}{3\alpha k^3}},$$

and the double root is

 doi.org/10.32388/21RDLB

$$\lambda_2 = \mp\sqrt{\frac{\omega}{3\alpha k^3}}.$$

Therefore, the general solution to equation (4) is given by

$$w(\xi) = A_1 e^{\pm 2\sqrt{\frac{\omega}{3\alpha k^3}}\xi} + A_2 \xi e^{\mp\sqrt{\frac{\omega}{3\alpha k^3}}\xi} + A_3 e^{\mp\sqrt{\frac{\omega}{3\alpha k^3}}\xi}, \tag{18}$$

where A_1, A_2, and A_3 are arbitrary constants. The associated solution to equation (2) is given by

$$y(\xi) = \pm\sqrt{\frac{\omega}{3\alpha k}}\frac{2A_1 e^{\pm\sqrt{\frac{3\omega}{\alpha k^3}}\xi} - A_2\left(\xi \mp \sqrt{\frac{3\alpha k^3}{\omega}}\right) - A_3}{A_1 e^{\pm\sqrt{\frac{3\omega}{\alpha k^3}}\xi} + A_2\xi + A_3}. \tag{19}$$

Assuming $A_3 \neq 0$, we can rename $C_1 = A_1/A_3, C_2 = A_2/A_3$, and we obtain the two-parameter family of travelling wave solutions to the STO equation (1)

$$u(x,t) = \pm\sqrt{\frac{\omega}{3\alpha k}}\frac{2C_1 e^{\pm\sqrt{\frac{3\omega}{\alpha k^3}}(kx-\omega t)} - C_2\left(kx - \omega t \mp \sqrt{\frac{3\alpha k^3}{\omega}}\right) - 1}{C_1 e^{\pm\sqrt{\frac{3\omega}{\alpha k^3}}(kx-\omega t)} + C_2(kx-\omega t) + 1}, \tag{20}$$

with parameters given by C_1, C_2. A graphic representation of this solution is shown in Figure 4 for particular values of the parameters.

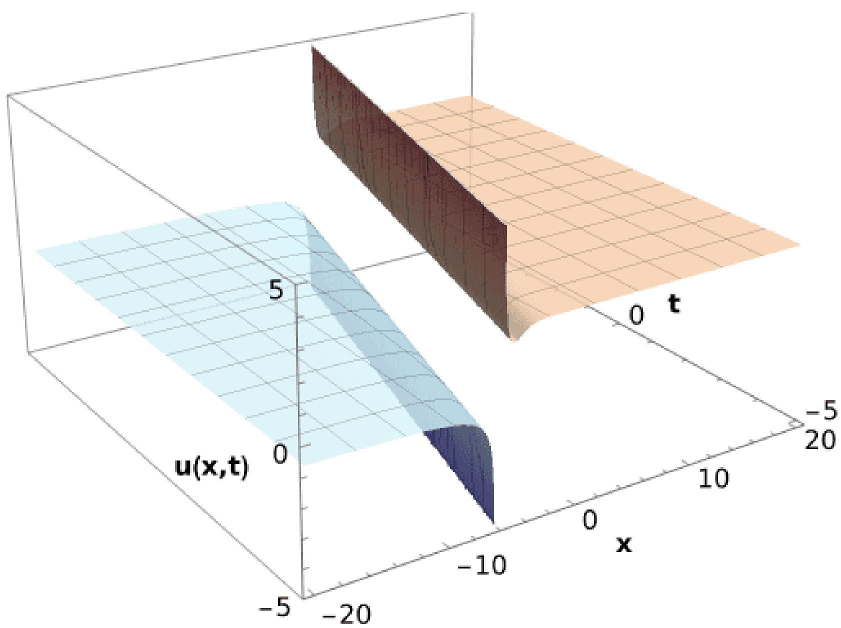

Figure 4. Graphic representation of solution (20) for $C_1 = 1, C_2 = 2, \alpha = 0.5, \omega = 1, k = 1.$

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For $A_3 = 0$, and assuming $A_2 \neq 0$, we can rename $C_1 = A_1/A_2$, and we obtain the one-parameter family of travelling wave solutions to the STO equation (1) given by:

$$u(x,t) = \pm\sqrt{\frac{\omega}{3\alpha k}}\frac{2C_1 e^{\pm\sqrt{\frac{3\omega}{\alpha k^3}}(kx-\omega t)} - kx + \omega t \pm \sqrt{\frac{3\alpha k^3}{\omega}}}{C_1 e^{\pm\sqrt{\frac{3\omega}{\alpha k^3}}(kx-\omega t)} + kx - \omega t}, \tag{21}$$

with C_1 a parameter.

3. The particular case $C = 0$

In[10], N. A. Kudryashov considers equation (2) for the case $C = 0$ (equation (3.13) of[10]), and provides the general solution in equation (3.17) of[10]. Translated into our notation, this family is expressed as

$$y(\xi) = \sqrt{\frac{\omega}{\alpha k}}\frac{A_2 e^{\sqrt{\frac{\omega}{\alpha k^3}}\xi} - A_3 e^{-\sqrt{\frac{\omega}{\alpha k^3}}\xi}}{A_1 + A_2 e^{\sqrt{\frac{\omega}{\alpha k^3}}\xi} + A_3 e^{-\sqrt{\frac{\omega}{\alpha k^3}}\xi}}. \tag{22}$$

The condition $\alpha > 0$ is implicitly assumed in[10], and since

$$C = 0 < \frac{4\omega^3}{27\alpha k},$$

equation (22) corresponds to Case 1 in our classification: three disinct real roots. According to equation (7), the roots are given by

$$\lambda_1 = 0, \quad \lambda_2 = \sqrt{\frac{\omega}{\alpha k^3}}, \quad \lambda_3 = -\sqrt{\frac{\omega}{\alpha k^3}}. \tag{23}$$

One can verify that substituting the roots (23) into equation (10) yields the same expression as equation (22). Therefore, all the travelling wave solutions of the STO equation obtained from the assumption $C = 0$ are included in our classification.

4. Conclusions

In this paper, we have presented a full classification of the travelling wave solutions of the STO equation, explicitly considering the general case with a non-zero integration constant C in equation (2). To the best of our knowledge, this work represents the first complete study of the STO travelling wave ODE.

The classification is based on the analysis of the characteristic polynomial (equation (5)) of the transformed ODE (equation (4)), and in particular, on the discriminant Δ:

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- When $\Delta > 0$, the characteristic polynomial has three distinct real roots, leading to solutions expressed as combinations of exponential terms (equation (10)).
- When $\Delta < 0$, there is one real root and a pair of complex conjugate roots, resulting in solutions involving combinations of trigonometric and exponential terms (equations (15), (16), and (17)).
- When $\Delta = 0$, the cubic has a double root and a simple root, yielding solutions that combine exponential and polynomial terms (equations (20) and (21)).

These results significantly extend the existing literature by introducing a broader family of exact solutions and providing a more complete view of the solution space of the STO equation. This could have implications for the study of complex nonlinear wave phenomena in various physical systems where the STO equation is applied, such as fluid dynamics, plasma physics, and optical fibers.

Notes

2020 MSC: 35C07, 37K10, 34C20.

Statements and Declarations

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work, the author used ChatGPT and Grok exclusively for grammar and language refinement. After using this service, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.

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Declarations

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