REPRESENTATIONS OF LIE GROUPOIDS ON BUNDLES

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ABSTRACT. Groupoids and Lie groupoids are interesting mathematical structures which have vital applications in mathematical physics. Bundles are certain structures which are avilable in several instants. In this paper we discuss $\mathcal{VB}$-groupoids, provide several examples of such $\mathcal{VB}$-groupoids. The representation of Lie groupoids on vector bundles are also discussed and it is shown that provide $\mathcal{VB}$-groupoids.

Groups arise primarily in connection with symmetry; that is as set of automorphisms of geometric or other mathematical structures. From this viewpoint, groupoid are the natural formulation of a symmetry system for objects which have a bundle structure. A bundle is just a map viewed as an object in a particular category and it is considered as the basic underlying structure for complicated systems such as fibre bundle, vector bundle and the like. In the following we discuss bundles and bundle categories in general and more specifically Lie groupoids and their bundle representations.

1. Bundles and bundle categories

A *bundle* is a triple $\eta = (E, p, B)$ is where $p : E \to B$ is a map. The space B is called the base space and the space E is called total space and the map p is called the projection of the bundle . For each $b \in B$, the space $p^{-1}(b)$ is called the *fibre* of the bundle over $b \in B$.

Intuitively a bundle is regarded as a union of fibres $p^{-1}(b)$ for $b \in B$ parametrized by B and 'glued together' by the topology of the space E. A cross section of a bundle (E, p, B) is a map $s : B \to E$ such that $ps = 1_B$.

Given any space B, a product bundle over B with fibre F is the bundle $(B \times F, p, B)$ where p is the projection on the first factor.

Example 1. *A covering space is a(continuous, surjective) map $p : X \to Y$ such that for every $y \in Y$ there exist an open neighborhood U containing y such that $p^{-1}(U)$ is homeomorphic to a disjoint union of open sets in X, each being mapped homeomorphically onto U by p . Hence it is a fibre bundle such that the bundle projection is a local homeomorphism. Here each fibre is a discrete space.*

Definition 1. *A product bundle over B with fibre F is $(E \times F, p, B)$ where p is the projection on to the first factor.*

Definition 2. *A bundle $\eta' = (E', p', B')$ is a subbundle of $\eta = (E, p, B)$ provided E' is a subspace of E, B' is a subspsce of B and $p' = p|\, E' : E' \to B'$.*

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Two bundles η and ζ over B are locally isomorphic if for each $b \in B$ there exists an open neighbourhood U of b such that $\eta|U$ and $\zeta|U$ are U-isomorphic.

Definition 3. *A bundle η over B is locally trivial with fibre F if η is locally isomerphic with the product bundle $(B \times F, p, B)$*

Definition 4. *A space F is a fibre of a bundle (E, p, B) provided every fibre $p^{-1}(b)$ for $b \in B$ is homoeomorphic to F. A bundle (E, p, B) is trivial with fibre F provided (E, p, B) is B-isomorphic to the product bundle $(E \times F, p, B)$.*

Example 2. *The tangent bundle over S^n denoted as (T, p, S^n) and the normal bundle (N, q, S^n) are two subbundles of the product bundle $(S^n \times R^{n+1}, p, S^n)$ whose total space are defined by the relation $(b, x) \in T$ if and only if the inner product $\langle b, x\rangle = 0$ and by $(b, x) \in N$ if and only if $x = kb$ for some $k \in R$.*

1.1. **Category of bundles, vector bundles and principal G-bundles.** A bundle morphism is a pair of fibre preserving map between two bundles $\eta = (E, p, B)$ and $\eta' = (E', p', B')$ ie.,

$$(u, f) = \eta = (E, p, B) \to \eta' = (E', p', B')$$

where $u : E \to E'$ and $f : B \to B'$ such that $p'u = fp$, ie., the diagram of arrows commutes. In particular when $\eta = (E, p, B)$ and $\eta' = (E', p', B)$ bundles over B, then the bundle morphism $u : (E, p, B) \to (E', p', B)$ is a map $u : E \to E'$ such that $p = p'u$.

Definition 5. *The category of bundles, denoted* **Bun***, has as its objects all bundles (E, p, B) and as morphisms (E, p, B) to (E', p', B') the set of all bundle morphisms. For each spsce B, the sub category of bundles over B, denoted as $\mathbf{Bun_B}$ has objects bundles with base B and B-morphisms as its morphisms.*

A vector bundle is a bundle with an additional vector space structure on each fibre.

Definition 6. *A k- dimensional vector bundle over F is a bundle (E, p, B) together with the structure of a k- dimensional vector space over F on each fibre $p^{-1}(b)$ such that the following local triviality condition is satisfied. Each point of B has an open neighborhood U and a U- isomorphism h: $U \times F^k \to p^{-1}(U)$ such that the restriction $b \times F^k \to p^{-1}(b)$ is a vectorspace isomorphism for each $b \in U$.*

Example 3. *The tangent bundle of a differentiable manifold M is a manifold TM which assembles all the tangent vectors in M . As a set it is given by disjoint union of tangent spaces of M . Elements of TM can be thought of as a pair (x,v) where x is a point in M and v is a tangent vector to M at x. There is a natural projection $\pi : TM \to M$ denoted $\pi(x, v) = x$. It forms a vector bundle where each fibre is the tangent space at some point of M.*

Example 4. *The tangent bundle over S^n, denoted (T, p, S^n), and the normal bundle over S^n, denoted (N, q, S^n) are two subbundles of the product bundle $(S^n \times R^{n+1}, p, S^n)$ whose total spaces are defined by the relation $(b, x) \in T$ if and only if the inner product $\langle b, x\rangle = 0$ and by $(b, x) \in N$ if and only if $x = kb$ for some $k \in R$.*

A principal bundle is a mathematical object that formalizes some of the essential features of the cartesian product $X \times G$ of a space X with a group G.

Definition 7. *Let G be a topological group. A principal G bundle is a fibre bundle $\pi : P \to X$ together with a continuous right action $P \times G \to G$ such that G preserves fibers and acts freely and transitively on them in such a way that for each $x \in X$ and $y \in P_x$, the map $G \to P_x$ sending $g \mapsto gg$ is a homoeomorphism.*

2. Groupoids and Lie groupoids

Definition 8. *A groupoid consists of two sets G and M, called respectively the groupoid and the base, together with two maps α and β from G to M, called respectively the source projection and target projection, a map $1 : x \mapsto 1_x$ called the object inclusion map, and a partial multiplication $(h, g) \mapsto hg$ in G defined on the set $G * G = \{(h, g) \in G \times G \,|\, \alpha(h) = \beta(g)\}$, subject to the following conditions:*

(1) *$\alpha(hg) = \alpha(g)$ and $\beta(hg) = \beta(h)$ for all $(h, g) \in G * G$:*
(2) *$j(hg) = (jh)g$ for all $j, h, g \in G$ such that $\alpha(j) = \beta(h)$ and $\alpha(h) = \beta(g)$;*
(3) *$\alpha(1_x) = \beta(1_x) = x$ for all $x \in M$;*
(4) *$g1_{\alpha(g)} = g$ and $(1_{\beta(g)}g = g$ for all $g \in G$;*
(5) *each $g \in G$ has a two sided inverse g^{-1} such that $\alpha(g^{-1}) = \beta(g), \beta(g^{-1}) = \alpha(g)$ and $g^{-1}g = 1_{\alpha(g)}, gg^{-1} = 1_{\beta(g)}$.*

Element of M are called objects of the groupoid G and elements of G are called arrows. The arrow 1_x corresponds to the object $x \in M$ may also be called the unity or identity corresponding to x.

Definition 9. *A Lie groupoid $G \rightrightarrows M$ is a groupoid G on base M together with smooth structures on G and M such that the maps $\alpha, \beta : G \to M$ are surjective submersions, the object inclusion map $x \mapsto 1_x$, $M \to G$ is smooth and the partial multiplication $G * G \to G$ is smooth.*

Here $G * G = (\alpha \times \beta)^{-1}(\Delta_M)$ is a closed embedded submanifold of $G \times G$, since α and β are submersions.

Example 5. *Let $\mathbf{G} \times \mathbf{M} \to \mathbf{M}$ be a smooth action of a Lie group $\mathbf{G}$ on a manifold $\mathbf{M}$. Give the product manifold $\mathbf{G} \times \mathbf{M}$ the structure of a Lie groupoid on $\mathbf{M}$ in the following way.*

(1) *α be the projection into the second factor of $\mathbf{G} \times \mathbf{M}$, β be the group action itself.*
(2) *The object inclusion map is $x \mapsto 1_x = (1, x)$.*
(3) *partial multiplication is $(g_2, y)\ (g_1, x) = (g_2g_1, x)$ which is defines iff $y = g_1x$.*
(4) *The inverse of (g, x) is (g^{-1}, gx).*

with this structure we denote $\mathbf{G} \times \mathbf{M}$ by $\mathbf{G} \lhd \mathbf{M}$ and call it the action groupoid of $\mathbf{G} \times \mathbf{M} \to \mathbf{M}$.

Example 6. *Any manifold $\mathbf{M}$ may be regarded as a Lie groupoid on itself with $\alpha = \beta = id_{\mathbf{M}}$ and every element a unity . A groupoid in which every element is a unity will be called a base groupoid*

Example 7. *Let $\mathbf{M}$ be a manifold and $\mathbf{G}$ a Lie group. We give $\mathbf{M} \times \mathbf{G} \times \mathbf{M}$ the structure of a Lie groupoid on $\mathbf{M}$ in the following way:*

- *α is the projection into the third factor of $\mathbf{M} \times \mathbf{G} \times \mathbf{M}$ and β is the projection into the first factor.*
- *The object inclusion map is $x \mapsto 1_x = (x, 1, x)$.*

- *partial multiplication is* $(z, h, y')\ (y, g, x) = (z, hg, x)$ *which is defines iff* $y = y'$.
- *The inverse of* (y, g, x) *is* (x, g^{-1}, y).

this is usually called the trivial groupoid on $\mathbf{M}$ *with group* $\mathbf{G}$. *In particular any Lie group may considered to be a Lie groupoid on a singleton manifold and any cartesian square* $\mathbf{M} \times \mathbf{M}$ *is a groupoid on* $\mathbf{M}$.

2.1. **$\mathcal{VB}$-groupoids and representations of Lie groupoids.** Consider a commutative diagram of Lie groupoids and vector bundles as follows:

$$\begin{array}{ccc} \Gamma & \rightrightarrows & E \\ \downarrow & & \downarrow \\ G & \rightrightarrows & M \end{array}$$

ie., we mean that $\Gamma \rightrightarrows E$ is a Lie groupoid (with source, target, multiplication, identity and inverse maps $\tilde{s}, \tilde{t}, \tilde{m}, \tilde{1}$ and $\tilde{i}$), and $G \rightrightarrows M$ is a Lie groupoid (with source, target, multiplication, identity and inverse maps $s, t, m, 1$ and i), $\Gamma \to G$ is a vector (with projection map and zero section $\tilde{q}$ and $\tilde{0}$), $E \to M$ is a vector bundle (with projection map and zero section q and 0) and such that $q\tilde{s} = s\tilde{q}$ and $q\tilde{t} = t\tilde{q}$. For the rest of this subsection we will always start with this data.

Definition 10. *A $\mathcal{VB}$-groupoid is a commutative diagram of Lie groupoids and vector bundles like the one above such that the following conditions hold:*

(1) $(\tilde{s}, s)$ *is a morphism of vector bundles.*
(2) $(\tilde{t}, t)$ *is a morphism of vector bundles.*
(3) $(\tilde{q}, q)$ *is a morphism of Lie groupoids.*
(4) Interchange law:

$$(\gamma_1 + \gamma_3)(\gamma_2 + \gamma_4) = \gamma_1\gamma_2 + \gamma_3\gamma_4$$

for any $\gamma_i \in \Gamma$ *for which the equation makes sense; specifically, for any* $(\gamma_1, \gamma_2) \in \Gamma^{(2)}, (\gamma_3, \gamma_4) \in \Gamma^{(2)}$ *such that* $\tilde{q}(\gamma_1) = \tilde{q}(\gamma_3)$ *and* $\tilde{q}(\gamma_2) = \tilde{q}(\gamma_4)$.

Example 8. *Consider Lie groupoids* $G \rightrightarrows M$ *and* $TG \rightrightarrows TM$ *Then the following diagarm of Lie groupoids and vector bundles is a $\mathcal{VB}$-groupoid.*

$$\begin{array}{ccc} TG & \rightrightarrows & TM \\ \downarrow & & \downarrow \\ G & \rightrightarrows & M \end{array}$$

Note that the existence of the commutative diagram of Lie groupoids and vector bundles as below

$$\begin{array}{ccc} \Gamma & \rightrightarrows & E \\ \downarrow & & \downarrow \\ G & \rightrightarrows & M \end{array}$$

is equivalent to say that $\Gamma \to G$ is a Lie groupoid object in the category of vector bundles or $\Gamma \rightrightarrows E$ is a vector bundle object in the category of Lie groupoids.

Definition 11. *Let* $E \to M$ *be a vector bundle. The frame groupoid* $\mathcal{G}(E)$ *is the groupoid whose set of objects is* M *and whose morphisms are set of all isomorphisms* $E_x \to E_y$ *for* $x, y \in M$. *It is easy to see that the frame groupoid is a Lie groupoid.*

Definition 12. *A representation of a Lie groupoid $G \rightrightarrows M$ on a is a vector bundle $E \to M$ is a smooth functor (homomorphism) of Lie groupoids over M viz.,*

$$\rho : G \to \mathcal{G}(E) \quad \textit{where } \mathcal{G}(E) \textit{ is the frame groupoid.}$$

In the light of all foregoing discussions it can be seen that every representation of a Lie groupoid on a vector bundle yeilds a commutative diagram of Lie groupoids and bundles, ie., $\mathcal{VB}$-groupoid. However the converse question - whether all $\mathcal{VB}$-groupoids yeilds representations of Lie groupoids needs more investigations.

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