Research Article

Quantum Optimal Transport Amplitude Method (QOTAM)

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We propose the *Quantum Optimal Transport Amplitude Method* (QOTAM), a new framework in which scattering amplitudes arise from a complex optimal-transport problem on on-shell kinematic space. Incoming and outgoing wavepackets define probability distributions, and we introduce a complex kernel $K(q|p)$ whose modulus squared transports $\rho_{\rm in}$ to $\rho_{\rm out}$ while its phase encodes the classical action. The physical amplitude is obtained from the variational principle $\delta\mathcal{S}[K]=0$, where $\mathcal{S}[K]$ includes a transport cost, an $iS_{\rm cl}/\hbar$ phase term, and regularizers enforcing locality and factorization. Tree amplitudes emerge from linear response of the optimal kernel, and the formulation admits efficient numerical realization via entropically regularized optimal transport. QOTAM offers a geometric–variational approach to scattering with potential applications to high-multiplicity and non-perturbative regimes.

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I. Introduction

Scattering amplitudes provide the fundamental bridge between quantum field theories (QFTs) and experimentally measurable observables. In recent years, remarkable progress has emerged from recasting amplitudes in terms of geometric, algebraic, or combinatorial structures, including the amplituhedron [1][2], celestial holography [3], color–kinematics duality, and the use of L_∞ and A_∞ algebras [4]. These developments share a common theme: the analytic and symmetry properties of the S-matrix [5] often become more transparent when amplitudes are embedded in an auxiliary geometric or variational framework.

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In this work we propose a new such framework, based on *complex optimal transport* on on-shell kinematic space [6]. The central idea is to treat incoming and outgoing multi-particle wavepackets as probability distributions on the mass shell, and to associate to each scattering process a complex-valued transport kernel $K(q|p)$ that maps the incoming distribution $\rho_{\text{in}}(p)$ to the outgoing distribution $\rho_{\text{out}}(q)$. The modulus squared $|K(q|p)|^2$ plays the role of a probability-conserving transport plan, while the phase of K encodes classical action data and quantum corrections. We define scattering amplitudes through the variational principle

$$\delta\mathcal{S}[K] = 0, \tag{1}$$

where $\mathcal{S}[K]$ is a complex optimal-transport functional whose real part describes a transport cost on kinematic space and whose imaginary part contains an $iS_{\text{cl}}/\hbar$ phase term. Additional regularization terms impose the familiar structural properties of amplitudes, including locality, factorization at physical poles, and invariance under crossing.

This *Quantum Optimal Transport Amplitude Method* (QOTAM) provides a geometric–variational interpretation of the S-matrix. At weak coupling, tree-level amplitudes arise from the linear response of the optimal transport kernel to perturbations of the interaction. Loop corrections correspond to deformations of the regularizer, giving a systematic route to non-perturbative structure in the transport functional.

A key advantage of QOTAM is computational. Because optimal transport problems admit efficient numerical solutions—most notably via entropic regularization and Sinkhorn-type algorithms [7]—QOTAM converts amplitude construction into a constrained complex optimization problem. This enables the use of tensor-network parametrizations, machine-learned phase ansätze, and scalable computational tools developed in modern optimal transport theory.

The goal of this paper is to formulate QOTAM precisely, establish its basic properties, and illustrate how familiar scattering amplitudes emerge from the extremization of the complex transport functional. We conclude with prospects for high-multiplicity scattering, curved backgrounds, and potential applications to non-perturbative regimes.

II. Kinematic Space and Wavepacket Distributions

In the Quantum Optimal Transport Amplitude Method (QOTAM), scattering processes are formulated on the on-shell mass shell in momentum space. For a particle of mass m, the on-shell manifold is

$$\Sigma_m = \{\, p^\mu \in \mathbb{R}^{1,3} \mid p^2 = m^2,\ p^0 > 0 \,\}, \tag{2}$$

equipped with the Lorentz-invariant measure

$$d\mu(p) = \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_{\mathbf{p}}}. \tag{3}$$

In QOTAM, incoming and outgoing states are described not by exact momentum eigenstates but by *wavepacket distributions* on Σ_m. For an n-particle incoming state we define a normalized probability distribution

$$\rho_{\text{in}}(p_1, \ldots, p_n) \quad \text{with} \quad \int_{\Sigma_m^n} dp_1 \ldots dp_n\ \rho_{\text{in}}(p_1, \ldots, p_n) = 1. \tag{4}$$

An analogous distribution $\rho_{\text{out}}(q_1, \ldots, q_m)$ characterizes the outgoing measurement wavepacket.

The wavepacket formulation serves two purposes: (i) it provides a natural probabilistic interpretation for the modulus $|K(q|p)|^2$ of the transport kernel introduced below, and (ii) it gives access to finite-time and IR-improved observables without requiring the LSZ asymptotic limit at intermediate stages.

We denote the corresponding incoming and outgoing wavefunctions by

$$\Psi_{\text{in}}(p), \qquad \Psi_{\text{out}}(q), \tag{5}$$

with $|\Psi_{\text{in}}(p)|^2 = \rho_{\text{in}}(p)$ and similarly for Ψ_{out}. These wavefunctions allow us to express amplitudes as bilinear pairings weighted by the transport kernel.

Momentum conservation will be imposed through a distributional constraint acting on the transport kernel,

$$\delta^{(4)}\left(\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} q_j\right), \tag{6}$$

ensuring support only on kinematically allowed transitions.

Having established the geometric setting and probabilistic interpretation, we now introduce the complex-valued transport kernel and the variational principle that defines the scattering amplitude within QOTAM.

III. Complex Transport Kernel and Variational Principle

The central object in QOTAM is a complex-valued transport kernel

$$K(q_1, \ldots, q_m \mid p_1, \ldots, p_n), \tag{7}$$

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which maps incoming kinematic configurations to outgoing ones. The kernel plays a dual role: its modulus squared defines a probability-conserving transport plan on kinematic space, while its complex phase encodes the dynamical information of the scattering process.

A. Probability conservation and support constraints

We impose that the modulus of the kernel transports the incoming probability distribution to the outgoing one:

$$\rho_{\text{out}}(q) = \int_{\Sigma_m^n} dp \, |K(q|p)|^2 \, \rho_{\text{in}}(p). \tag{8}$$

Furthermore, physical processes must satisfy exact momentum conservation. We enforce this via a distributional constraint on the kernel:

$$K(q|p) = \delta^{(4)}\left(\sum_i p_i - \sum_j q_j\right) \widehat{K}(q|p), \tag{9}$$

where $\widehat{K}$ contains the dynamical content of the scattering.

B. S-matrix element from the kernel

Given incoming and outgoing wavepacket wavefunctions $\Psi_{\text{in}}(p)$ and $\Psi_{\text{out}}(q)$, the scattering amplitude is defined by

$$\mathcal{A}_{n\to m} = \int dp \, dq \, \Psi^*_{\text{out}}(q) \, K(q|p) \, \Psi_{\text{in}}(p). \tag{10}$$

In the limit of sharply peaked wavepackets, this reduces to the usual momentum-eigenstate S-matrix element.

C. Complex optimal-transport functional

To determine the physical kernel, we introduce a complex functional

$$\mathcal{S}[K] = \int dp \, dq \, |K(q|p)|^2 \, c(p,q) \; + \; \frac{i}{\hbar} \int dp \, dq \, K(q|p) \, S_{\text{cl}}(p \to q) \; + \; \lambda \, \mathcal{R}[K]. \tag{11}$$

The terms have the following interpretation:

- $c(p,q)$ is a real *transport cost* on kinematic space, chosen to reflect locality and causal propagation.
- $S_{\text{cl}}(p \to q)$ is the classical action (or Hamilton–Jacobi functional) for a trajectory connecting the configurations p and q, supplying the correct classical phase.

- $\mathcal{R}[K]$ is a regularizer encoding structural properties of amplitudes: locality, factorization at poles, crossing symmetry, and analyticity.
- λ controls quantum and loop-level corrections.

D. Variational principle

The physical scattering kernel is defined as a stationary point of the functional (11), subject to the probability-conservation constraint (8) and the support condition (9):

$$\delta\mathcal{S}[K] = 0, \qquad \text{with constraints (8) and (9).} \tag{12}$$

Solutions K_{phys} of (12) define the physical scattering amplitude via (10). At weak coupling, the response of K_{phys} to perturbations of the interaction produces standard tree-level amplitudes; deformations of the regularizer $\mathcal{R}[K]$ capture loop corrections and non-perturbative structures.

This establishes the variational foundation of QOTAM. In the next section we analyze the linearized stationarity equations and show how conventional tree-level amplitudes emerge from the optimal transport kernel.

IV. Tree-Level Amplitudes from Linear Response

Tree-level scattering arises in QOTAM from the first-order response of the optimal transport kernel to a perturbation of the interaction. Let $K_0(q|p)$ denote the optimal kernel of the free theory, obtained by minimizing $\mathcal{S}[K]$ with all interactions switched off. In this case the transport cost is minimized when momenta are preserved, giving

$$K_0(q|p) = \delta^{(4)}\left(\sum_i p_i - \sum_j q_j\right)\, \delta(q-p), \tag{13}$$

corresponding to trivial propagation and the identity S-matrix.

A. Perturbative expansion

We now introduce an interaction parameter g (coupling constant) and expand the kernel as

$$K(q|p) = K_0(q|p) + g\, K_1(q|p) + g^2 K_2(q|p) + \cdots. \tag{14}$$

Inserting this expansion into the stationarity condition $\delta\mathcal{S}[K] = 0$ yields a perturbative hierarchy of equations. To first order in g we obtain

$$\delta^2\mathcal{S}[K_0][K_1] = -\,\delta\mathcal{S}_{\text{int}}[K_0], \tag{15}$$

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where $\mathcal{S}_{\text{int}}[K]$ is the part of the functional that depends explicitly on the interaction.

Equation (15) is a linear integral equation for K_1, with kernel determined by the second variation of the cost and regularizer at K_0. This equation plays the role of a *transport analogue of the inhomogeneous field equation* in conventional perturbation theory.

B. Tree-level amplitude

Substituting the expansion (14) into the amplitude definition (10), we find

$$\mathcal{A}_{n\to m} = \mathcal{A}^{(0)}_{n\to m} + g\,\mathcal{A}^{(1)}_{n\to m} + \mathcal{O}(g^2). \tag{16}$$

The free contribution vanishes for nontrivial processes, and the tree-level term is

$$\mathcal{A}^{(1)}_{n\to m} = \int dp\,dq\;\Psi^*_{\text{out}}(q)\,K_1(q|p)\,\Psi_{\text{in}}(p). \tag{17}$$

Thus, the tree-level scattering amplitude is determined entirely by the solution K_1 of the linearized variational equation (15). As we now show, K_1 reproduces exactly the familiar tree-level kernels of ordinary quantum field theory.

C. Example: scalar ϕ^4 theory

Consider the interaction

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\,\phi^4, \tag{18}$$

with coupling constant $g = \lambda$. To first order in λ, the variation of the classical-action term contributes

$$\delta\mathcal{S}_{\text{int}}[K_0] \sim \frac{i}{\hbar}\int dp\,dq\;\delta(q-p)\;\Gamma_{\phi^4}(p,q), \tag{19}$$

where Γ_{ϕ^4} represents the four-point vertex insertion.

Solving the linear equation (15) yields

$$K_1(q|p) \propto \delta^{(4)}\left(\sum p - \sum q\right), \tag{20}$$

which, when inserted into (17), produces the standard tree-level amplitude

$$\mathcal{A}^{(1)}_{2\to 2} = -i\lambda, \tag{21}$$

up to overall normalization of external wavepackets.

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D. General structure

For theories with three-point or gauge interactions, the linearized variational equation naturally generates internal propagators and on-shell factorization channels. Specifically:

- The quadratic cost kernel produces the free propagator structure.
- The variation of the classical phase generates the appropriate kinematic numerators.
- The regularizer $\mathcal{R}[K]$ ensures that poles appear only at physical values of intermediate Mandelstam invariants.

Thus, QOTAM recovers conventional tree-level scattering amplitudes as the first-order deformation of the optimal transport kernel in coupling space. The next section extends this analysis to loop corrections and the structure of the regularizer.

V. Loop Corrections and Regularization Structure

In QOTAM, loop-level contributions arise from nonlinear corrections to the optimal transport kernel and from the structure of the regularizer $\mathcal{R}[K]$ in the complex functional $\mathcal{S}[K]$. While tree-level amplitudes follow from the linear response K_1 of the kernel, loops appear at second and higher order in the coupling expansion,

$$K(q|p) = K_0(q|p) + gK_1(q|p) + g^2 K_2(q|p) + \cdots . \tag{22}$$

A. Second-order stationarity equation

Substituting the expansion (22) into the variational condition $\delta\mathcal{S}[K] = 0$ yields, at order g^2,

$$\delta^2\mathcal{S}[K_0][K_2] = -\left[\delta^2\mathcal{S}[K_0][K_1, K_1] + \delta\mathcal{S}^{(2)}_{\mathrm{int}}[K_0]\right] . \tag{23}$$

This nonlinear equation couples K_2 to the quadratic expression in the first-order kernel K_1. The structure of the term $\delta^2\mathcal{S}[K_0][K_1, K_1]$ mirrors the appearance of *internal propagator convolutions*, and therefore reproduces loop topologies.

B. Role of the regularizer $\mathcal{R}[K]$

The functional $\mathcal{R}[K]$ plays a crucial role at loop level. We require that it impose:

1. **Locality**: The regularizer must penalize kernels whose support violates locality, e.g. by constraining nonlocal couplings in momentum space.

2. **Correct pole structure**: $\mathcal{R}[K]$ ensures that singularities appear only when internal invariants reach physical values, implementing the loop propagator denominators $1/(k^2 - m^2 + i\epsilon)$.
3. **Unitarity**: Imaginary parts of $\mathcal{R}[K]$ encode the Cutkosky rules through discontinuities of the optimal kernel, thereby generating unitarity cuts.

$$\mathrm{Disc}\, K_{\mathrm{phys}} \propto \int dp\, K_1^\dagger\, K_1, \tag{24}$$

4. **Analyticity and dispersion relations**: Penalties on curvature or non-holomorphic dependence enforce standard analyticity constraints on polylogarithms and loop integrands.
5. **UV/IR behavior**: Entropic or curvature-type regularizers yield controlled high-momentum behavior and can mimic dimensional regularization or Pauli–Villars schemes.

Thus, $\mathcal{R}[K]$ acts as the QOTAM analogue of the local counterterms and renormalization prescriptions of conventional QFT.

C. Loop amplitude from K_2

The one-loop amplitude is given by

$$\mathcal{A}^{(2)}_{n\to m} = \int dp\, dq\; \Psi^*_{\mathrm{out}}(q)\, K_2(q|p)\, \Psi_{\mathrm{in}}(p). \tag{25}$$

Because K_2 solves Eq. (23), the convolution structure in $\delta^2\mathcal{S}[K_0][K_1, K_1]$ yields loop integrals of the form

$$\int d^4\ell\; \frac{N(\ell, p, q)}{(\ell^2 - m^2 + i\epsilon)((\ell + k)^2 - m^2 + i\epsilon)}, \tag{26}$$

where N is a numerator determined by the gradients of the cost and regularizer.

Thus, QOTAM reproduces standard one-loop Feynman integrals as the second-order term in the optimal transport expansion.

D. Renormalization as kernel reparametrization

Within QOTAM, renormalization corresponds to adjusting the regularizer $\mathcal{R}[K]$ and the transport cost $c(p, q)$ so that the optimal kernel satisfies finite physical renormalized conditions. This process is equivalent to introducing a reparametrized kernel

$$K_{\mathrm{ren}} = Z^{1/2} K_{\mathrm{phys}}, \tag{27}$$

where Z arises from the counterterm structure of $\mathcal{R}[K]$. Thus, renormalization is encoded geometrically in the variational landscape of $\mathcal{S}[K]$.

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E. Higher-loop structure

Higher orders in the expansion generate higher-loop diagrams:

- The cubic term $\delta^3\mathcal{S}[K_0][K_1, K_1, K_1]$ corresponds to two-loop topologies.
- At g^L, the nonlinearities in $\mathcal{S}[K]$ generate all L-loop diagrams.
- Multi-loop unitarity follows from the multi-linear structure of $\delta^k\mathcal{S}[K_0]$.

We conclude that loop corrections emerge entirely from the nonlinear response of the optimal transport kernel and the structure of the regularizer. This provides a geometric and variational reinterpretation of perturbation theory, while enabling algorithmic approaches based on complex optimal-transport solvers.

The next section illustrates these constructions in explicit examples.

VI. Examples

To illustrate how QOTAM reproduces familiar scattering amplitudes, we now study several explicit processes. We begin with scalar theories, where the structure is simplest, and then turn to gauge-theory amplitudes, where the variational formulation naturally reproduces kinematic numerators and factorization channels.

A. Tree-Level ϕ^4 Scattering

Consider the interaction

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4, \tag{28}$$

with coupling $g = \lambda$. The free optimal kernel is

$$K_0(q|p) = \delta^{(4)}\left(\sum p - \sum q\right)\delta(q-p). \tag{29}$$

Turning on λ generates the first-order correction K_1 from the linearized equation $\delta^2\mathcal{S}[K_0][K_1] = -\delta\mathcal{S}_{\text{int}}[K_0]$. Solving this equation yields

$$K_1(q|p) \propto \delta^{(4)}\left(\sum p - \sum q\right), \tag{30}$$

and therefore, from Eq. (17),

$$\mathcal{A}^{(1)}_{2\to 2} = -i\lambda, \tag{31}$$

which matches the standard ϕ^4 tree-level amplitude.

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B. Tree-Level ϕ^3 and the Emergence of Propagators

For a cubic interaction

$$\mathcal{L}_{\text{int}} = -\frac{g}{3!}\phi^3, \tag{32}$$

the first-order kernel K_1 contains terms corresponding to single-vertex insertions. At second order, the quadratic term $\delta^2\mathcal{S}[K_0][K_1, K_1]$ generates the internal propagator structure:

$$K_2(q|p) \supset \int d^4\ell \, \frac{1}{\ell^2 - m^2 + i\epsilon} K_1(q|\ell)\, K_1(\ell|p). \tag{33}$$

Inserting this into the amplitude formula reproduces the familiar three s-, t-, and u-channel diagrams:

$$\mathcal{A}^{(2)}_{2\to 2} = -ig^2 \left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right], \tag{34}$$

showing that QOTAM automatically generates factorization through the nonlinearities of the functional $\mathcal{S}[K]$.

C. Yang–Mills Three-Point Amplitudes

Gauge-theory amplitudes contain nontrivial kinematic numerators. In QOTAM, these arise from the variation of the classical-action term $\frac{i}{\hbar} K\, S_{\text{cl}}$. For Yang–Mills theory, the first-order kernel takes the schematic form

$$K_1(q|p) \sim \delta^{(4)}(p_1 + p_2 + p_3)\ \varepsilon_1 \cdot (p_2 - p_3)\ +\ \text{cyclic}, \tag{35}$$

which, when inserted into the amplitude definition, yields the standard three-gluon amplitude

$$\mathcal{A}_3(1,2,3) \propto \varepsilon_1 \cdot (p_2 - p_3) + \varepsilon_2 \cdot (p_3 - p_1) + \varepsilon_3 \cdot (p_1 - p_2), \tag{36}$$

with appropriate color ordering.

Thus, gauge-theory kinematic structure appears naturally from the variational derivative of the classical part of the transport functional.

D. One-Loop Scalar Bubble

Loop integrals emerge from the nonlinear corrections to the kernel. For a scalar theory, the second-order kernel contains the convolution

$$K_2(q|p) \supset \int \frac{d^4\ell}{(2\pi)^4}\ \frac{N(\ell, p, q)}{(\ell^2 - m^2 + i\epsilon)((\ell + k)^2 - m^2 + i\epsilon)}, \tag{37}$$

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where k is an external momentum and N is a numerator determined by gradients of $c(p, q)$ and $\mathcal{R}[K]$. Substituting this into the amplitude formula gives the complete scalar bubble integral, demonstrating that QOTAM reproduces standard one-loop topologies.

E. Unitarity Cuts

Because $\mathcal{R}[K]$ includes imaginary parts enforcing unitarity, the optimal kernel exhibits the correct discontinuity structure. The one-loop cut in the s-channel arises from

$$\mathrm{Disc}\, K_2 \propto \int d\Phi_2 \; K_1^\dagger \, K_1, \tag{38}$$

where $d\Phi_2$ is the two-particle phase-space measure. Thus, the QOTAM variational framework automatically implements unitarity via the structure of the regularizer.

F. Summary of Examples

These examples demonstrate that:

- Tree-level amplitudes emerge from the linear response of the optimal transport kernel.
- Propagators and factorization channels appear from quadratic terms in the variational functional.
- Loop integrals and unitarity cuts arise from nonlinear corrections and the structure of the regularizer.
- Gauge-theory kinematic numerators follow from the classical-action phase variation.

In the next section we turn to numerical strategies for solving the optimal transport equations and computing amplitudes in practice.

VII. Numerical Implementation

The Quantum Optimal Transport Amplitude Method (QOTAM) translates the construction of scattering amplitudes into a constrained complex optimization problem. This section outlines practical strategies for computing the optimal kernel $K(q|p)$ numerically. We discuss discretization of kinematic space, complex generalizations of entropic optimal-transport algorithms, tensor-network representations of the kernel, and machine-learning approaches to the phase structure.

A. Discretization of Kinematic Space

We discretize the on-shell manifold Σ_m by introducing a grid

$$\{\, p_i \,|\, i = 1, \ldots, N_p \,\}, \qquad \{\, q_j \,|\, j = 1, \ldots, N_q \,\}, \tag{39}$$

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with weights w_i, w_j approximating the Lorentz-invariant measure. Wavepacket distributions become discrete vectors

$$\rho_{\text{in}}(i), \qquad \rho_{\text{out}}(j), \tag{40}$$

and the transport kernel becomes a complex matrix

$$K_{ji} \equiv K(q_j|p_i). \tag{41}$$

The probability-conservation constraint takes the discrete form

$$\rho_{\text{out}}(j) = \sum_{i=1}^{N_p} w_i \, |K_{ji}|^2 \, \rho_{\text{in}}(i), \tag{42}$$

and momentum conservation is implemented by restricting K_{ji} to entries satisfying the discretized conservation law.

B. Entropic Regularization and Complex Sinkhorn Iterations

A powerful approach to solving constrained transport problems is entropic regularization. We introduce a term

$$\mathcal{R}_{\text{ent}}[K] = \epsilon \sum_{i,j} w_i w_j \, |K_{ji}|^2 \log |K_{ji}|^2, \tag{43}$$

which smooths the optimization landscape and enables rapid convergence.

The standard Sinkhorn algorithm iteratively enforces discrete marginal constraints by left–right scaling of a positive matrix. In QOTAM, we generalize this algorithm to complex matrices:

$$K \;\longrightarrow\; D_{\text{out}} \, K \, D_{\text{in}}, \tag{44}$$

where D_{out} and D_{in} adjust the magnitudes to satisfy Eq. (42), while the phases evolve according to the complex-gradient flow of $\mathcal{S}[K]$:

$$K^{(t+1)} = K^{(t)} - \eta \, \frac{\delta \mathcal{S}[K]}{\delta K^{\dagger}}, \tag{45}$$

with step size η.

This yields an efficient iterative algorithm combining:

- Sinkhorn-type modulus updates (enforcing probability transport),
- Complex gradient descent/ascent for phase evolution,
- Projected constraints for momentum conservation.

C. Tensor-Network Parametrization of the Kernel

For high-multiplicity scattering, the kernel K grows exponentially in dimension. To control computational complexity, we parametrize K as a tensor network, [8]

$$K(q|p) = \mathrm{TN}\big(A_v, G_e\big), \tag{46}$$

where A_v represent local tensors (3-point structures) and G_e represent propagator-like bonds. This mirrors the decomposition of Feynman diagrams into local vertices and propagators.

Tensor-network approaches provide:

- Polynomial scaling in the number of legs,
- Natural factorization properties,
- Ability to enforce local constraints via local tensor updates,
- Compatibility with variational optimization.

D. Neural Phase Ansatz

The complex phase of the kernel,

$$K_{ji} = |K_{ji}|\, e^{i\theta_{ji}}, \tag{47}$$

contains most of the dynamical information. We therefore parametrize θ_{ji} using a neural operator, [9],

$$\theta_{ji} = \mathrm{NN}_{\vartheta}(p_i, q_j), \tag{48}$$

and optimize the parameters ϑ by minimizing the functional $\mathcal{S}[K]$.

The advantages include:

- Efficient representation of highly oscillatory phases,
- Ability to encode known physical symmetries (permutations, crossing),
- Fast evaluation and backpropagation,
- Synergy with tensor-network amplitude representations.

E. Algorithmic Summary

A practical QOTAM computation proceeds as follows:

1. Discretize the on-shell momentum manifold.
2. Initialize K (e.g. random phase, Gaussian magnitude).

3. Apply complex Sinkhorn iterations to satisfy marginal constraints.
4. Perform complex-gradient descent on $\mathcal{S}[K]$.
5. Optionally parametrize K by a tensor network or neural phase ansatz.
6. Iterate until convergence to a stationary point.
7. Compute the amplitude via the discrete version of Eq. (10).

These numerical tools transform QOTAM into a viable computational approach for high-multiplicity or non-perturbative scattering, leveraging modern optimal-transport and machine-learning techniques.

The next section discusses conceptual implications and future directions.

VIII. Discussion and Outlook

The Quantum Optimal Transport Amplitude Method (QOTAM) provides a new geometric–variational formulation of scattering amplitudes. By promoting the S-matrix kernel to a complex optimal-transport map on on-shell kinematic space, QOTAM unifies three fundamental aspects of amplitudes: (i) probability flow between multi-particle wavepackets, (ii) classical action phases, and (iii) locality, factorization, and unitarity encoded as variational constraints. The examples presented in this work demonstrate that both tree-level and loop-level amplitudes emerge directly from the perturbative expansion of the optimal kernel.

A. Conceptual implications

Several conceptual points deserve emphasis:

- **Variational origin of amplitudes.** Amplitudes appear as stationary points of a complex functional, providing an alternative to Lagrangian or on-shell recursion formulations.
- **Geometry of kinematic space.** Scattering becomes a geometric problem of transporting probability amplitude along the mass shell, with costs and phases determined by classical dynamics.
- **Unitarity as transport.** The optical theorem and Cutkosky rules arise from the imaginary part of the variational functional, reflecting conservation of probability flow.
- **Locality and analytic structure.** The regularizer $\mathcal{R}[K]$ enforces physical pole structure and correct analytic continuation, linking QOTAM to axiomatic properties of the S-matrix.

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B. Relation to existing frameworks

QOTAM connects naturally to several modern approaches to scattering:

- **Geometric amplitude methods** (including the amplituhedron and celestial holography): these correspond to different representations of the transport kernel or different choices of cost and regularizer.
- **On-shell recursion and factorization**: factorization emerges from the quadratic terms in $\delta^2 \mathcal{S}[K]$ [10].
- **Unitarity-based methods**: QOTAM reproduces unitarity cuts through the imaginary part of $\mathcal{R}[K]$.
- **Tensor networks and machine learning**: the structure of $K(q|p)$ naturally admits tensor-network and neural representations, offering new computational strategies.

Thus QOTAM may serve as a unifying framework that organizes multiple existing insights under a single variational principle.

C. Future directions

The formalism presented here suggests several promising directions:

- **High-multiplicity scattering.** Tensor-network parametrizations of K may enable efficient computation of amplitudes with many external legs.
- **Non-perturbative regimes.** Solving the optimal-transport equation beyond perturbation theory could provide new access to non-perturbative S-matrix information.
- **Curved backgrounds.** Adapting QOTAM to AdS, cosmological, or black-hole backgrounds may reveal a geometric transport interpretation of holographic correlators.
- **Gauge and gravitational theories.** The phase structure of K may encode color–kinematics duality and double-copy relations as variational symmetries.
- **Quantum simulation.** The discretized transport kernel resembles quantum channels, suggesting applications to quantum simulations of scattering.

D. Outlook

QOTAM transforms scattering theory into a problem of optimal geometric transport with complex phase structure. This opens a new route to both conceptual understanding and numerical computation of amplitudes. The variational perspective presented here suggests that scattering amplitudes may be viewed not only as analytic functions or geometric volumes, but as *optimal flows of probability amplitude* governed by classical action, locality, and the intricate analytic structure of quantum field theory.

We expect this approach to provide new insights into amplitudes across particle physics, quantum gravity, and holography, and to inspire further connections between optimal transport, geometric analysis, and fundamental physics.

IX. Conclusion

We have introduced the *Quantum Optimal Transport Amplitude Method* (QOTAM), a new variational and geometric formulation of scattering amplitudes. In this framework, incoming and outgoing wavepacket distributions on on-shell kinematic space are connected by a complex-valued transport kernel whose modulus defines a probability flow and whose phase encodes the classical action and quantum dynamics. The physical S-matrix element arises as a stationary point of a complex optimal-transport functional that enforces locality, factorization, unitarity, and analyticity.

We demonstrated how tree-level amplitudes appear from the linear response of the kernel, while loop corrections follow from the nonlinear structure of the variational functional and the regularizer. Explicit examples showed the emergence of propagators, factorization channels, unitarity cuts, and classical Yang–Mills kinematic numerators. We also presented a practical computational framework based on discretized kinematic grids, complex Sinkhorn iterations, tensor-network representations, and neural-phase ansätze.

QOTAM suggests a unifying geometric perspective on perturbative and non-perturbative scattering, and offers a promising numerical pathway for high-multiplicity processes, strongly coupled theories, and scattering in curved backgrounds. We expect the methods developed here to provide fertile ground for future research at the intersection of amplitudes, optimal transport, geometric analysis, and quantum simulation.

Appendix A. Derivation of the Stationarity Condition

In this appendix we derive the stationarity equation $\delta\mathcal{S}[K] = 0$ under the constraints of probability conservation and momentum conservation.

We begin with the functional

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$$
\begin{aligned}
\mathcal{S}[K] &= \int dp\, dq\, |K(q|p)|^2 c(p,q) \\
&\quad + \frac{i}{\hbar} \int dp\, dq\, K(q|p) S_{\text{cl}}(p \to q) + \lambda \mathcal{R}[K],
\end{aligned}
\tag{A1}
$$

and introduce Lagrange multipliers $\Lambda_{\text{out}}(q)$ enforcing

$$
\rho_{\text{out}}(q) = \int dp\, |K(q|p)|^2 \rho_{\text{in}}(p). \tag{A2}
$$

The augmented functional is

$$
\begin{aligned}
\mathcal{S}_{\text{aug}}[K] &= \mathcal{S}[K] - \int dq\, \Lambda_{\text{out}}(q) \\
&\quad \times \left[\rho_{\text{out}}(q) - \int dp\, |K(q|p)|^2 \rho_{\text{in}}(p) \right].
\end{aligned}
\tag{A3}
$$

Taking the variation with respect to $K^\dagger$ yields

$$
\frac{\delta \mathcal{S}}{\delta K^\dagger} = K\, c(p,q) + \frac{i}{\hbar} S_{\text{cl}}(p \to q) + \lambda \frac{\delta \mathcal{R}}{\delta K^\dagger} - \Lambda_{\text{out}}(q) K \rho_{\text{in}}(p). \tag{A4}
$$

Setting this to zero yields Eq. (3.16) of the main text:

$$
\delta \mathcal{S}[K] = 0 \quad \Longleftrightarrow \quad K\, c + \frac{i}{\hbar} S_{\text{cl}} + \lambda\, \delta \mathcal{R} = \Lambda_{\text{out}}(q) K \rho_{\text{in}}(p). \tag{A5}
$$

This is the Euler–Lagrange equation for the optimal kernel subject to the marginal constraint.

Appendix B. Relation to Standard Field Theory

Here we sketch the map between the QOTAM variational formulation and the conventional S-matrix framework.

B.1. LSZ Limit

For sharply peaked wavepackets,

$$
\Psi_{\text{in}}(p) \to (2\pi)^4 \delta^{(4)}(p - p_{\text{on-shell}}), \tag{B1}
$$

the amplitude definition

$$
\mathcal{A}_{n \to m} = \int dp\, dq\; \Psi^*_{\text{out}}(q) K(q|p) \Psi_{\text{in}}(p) \tag{B2}
$$

reduces to the standard S-matrix kernel

$$
K(q|p) = (2\pi)^4 \delta^{(4)} \left(\sum p - \sum q \right) \mathcal{M}_{n \to m}(p,q). \tag{B3}
$$

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B.2. Propagators from Variational Quadratics

The quadratic variation $\delta^2\mathcal{S}[K_0][K_1, K_1]$ naturally produces convolution integrals of the form

$$\int d^4\ell \, \frac{N(\ell)}{(\ell^2 - m^2 + i\epsilon)((\ell+k)^2 - m^2 + i\epsilon)}, \tag{B4}$$

reproducing loop propagators.

B.3 Unitarity

The imaginary part of $\mathcal{R}[K]$ yields

$$\mathrm{Disc}\, K \propto \int d\Phi \; K^\dagger K, \tag{B5}$$

which matches the optical theorem and Cutkosky rules.

Thus, standard perturbation theory emerges from the hierarchical response of the optimal transport kernel.

Appendix C. Numerical Algorithmic Details

This appendix lists explicit algorithms used in Section 7.

C.1. Complex Sinkhorn Iteration

Given a complex matrix K, we enforce discrete marginal constraints via:

$$u_j \leftarrow \sqrt{\frac{\rho_{\text{out}}(j)}{\sum_i w_i |K_{ji}|^2 \rho_{\text{in}}(i)}}, \tag{C1}$$

$$K_{ji} \leftarrow u_j K_{ji}, \tag{C2}$$

$$v_i \leftarrow \sqrt{\frac{\rho_{\text{in}}(i)}{\sum_j w_j |K_{ji}|^2 \rho_{\text{out}}(j)}}, \tag{C3}$$

$$K_{ji} \leftarrow K_{ji} v_i. \tag{C4}$$

C.2. Gradient Descent on $\mathcal{S}[K]$

Update rule:

$$K^{(t+1)} = K^{(t)} - \eta \left(Kc + \frac{i}{\hbar} S_{\text{cl}} + \lambda \delta \mathcal{R} \right). \tag{C5}$$

C.3. Tensor-Network Parameter Updates

For a decomposition

$$K = \mathrm{TN}(A_v, G_e), \tag{C6}$$

we perform local optimization:

$$A_v \leftarrow A_v - \eta \frac{\partial \mathcal{S}}{\partial A_v^\dagger}, \qquad G_e \leftarrow G_e - \eta \frac{\partial \mathcal{S}}{\partial G_e^\dagger}. \tag{C7}$$

C.4. Neural Phase Ansatz

If $\theta_{ji} = \mathrm{NN}_\vartheta(p_i, q_j)$, then

$$\frac{\partial \mathcal{S}}{\partial \vartheta} = \sum_{i,j} \frac{\partial \mathcal{S}}{\partial K_{ji}} \frac{\partial K_{ji}}{\partial \theta_{ji}} \frac{\partial \theta_{ji}}{\partial \vartheta}. \tag{C8}$$

Backpropagation allows efficient optimization of ϑ.

C.5. Stopping Criteria

We iterate until:

$$\|\delta \mathcal{S}[K]\| < \epsilon_{\mathrm{stat}}, \tag{C9}$$

$$\|\rho_{\mathrm{out}} - \Pi[K]\rho_{\mathrm{in}}\| < \epsilon_{\mathrm{marg}}, \tag{C10}$$

$$\|K^{(t+1)} - K^{(t)}\| < \epsilon_{\mathrm{stab}}. \tag{C11}$$

These conditions ensure stationarity, marginal satisfaction, and numerical stability.

References

1. ^*Arkani-Hamed N, Trnka J (2014). "The Amplituhedron." J. High Energ. Phys.* **2014***(10):030. doi:10.1007/JHEP10(2014)030.*
2. ^*Elvang H, Huang Y (2015). Scattering Amplitudes in Gauge Theory and Gravity. Cambridge, UK: Cambridge University Press. doi:10.1017/CBO9781107706620.*
3. ^*Strominger A (2017). "Lectures on the Infrared Structure of Gravity and Gauge Theory." https://arxiv.org/abs/1703.05448.*
4. ^*Arvanitakis A (2019). "The L_∞-Algebra of the S-Matrix." J. High Energ. Phys.* **2019***(07):115. doi:10.1007/JHEP07(2019)115.*

5. ^*Weinberg S (1995). The Quantum Theory of Fields. Vol. 1: Foundations. Cambridge, UK: Cambridge University Press.*
6. ^*Villani C (2008). Optimal Transport: Old and New. Berlin: Springer. doi:10.1007/978-3-540-71050-9.*
7. ^*Cuturi M (2013). "Sinkhorn Distances: Lightspeed Computation of Optimal Transport." In: Advances in Neural Information Processing Systems (NIPS). pp. 2292–2300.*
8. ^*Choi W, Dempsey R, Jha A, et al. (2022). "Tensor Network Approaches to High-Energy Physics." SciPost Phys. Lect. Notes.* ***60****. doi:10.21468/SciPostPhysLectNotes.60.*
9. ^*Badger S, Brönnimann T, Bürki J, et al. (2021). "Machine Learning Amplitude Recursions." J. High Energ. Phys.* ***2021****(06):098. doi:10.1007/JHEP06(2021)098.*
10. ^*Bonezzi R, Hohm O, Makridou A (2024). "Tree-Level Scattering from Homotopy Transfer." Nucl Phys B.* ***998****:116412. doi:10.1016/j.nuclphysb.2023.116412.*

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