Research Article

Fixing the Measure: Deriving $|\Psi|^2$ From Symmetry in Deterministic Geometry

Zayn Blore[1]

1. Independent researcher

We present a symmetry-first account of Born weights for finite-dimensional quantum systems that does not take probabilistic postulates as primitive. The core observation is geometric: pure states are rays in $\mathbb{C}^n$, hence the operational state space is complex projective space CP^{n-1}, equipped with its natural $SU(n)$ action. We show that this symmetry fixes, up to normalisation, a unique $SU(n)$-invariant Borel probability measure on CP^{n-1}, namely the Fubini–Study measure μ_{FS}[1] (Ashtekar & Schilling 1999). Symmetry alone, however, does not determine how weights attach to specific outcome projectors. To close this gap without appealing to stochastic axioms, we introduce minimal operational consistency requirements for a probability assignment $p(P \mid \psi)$ over projectors: normalisation on orthonormal resolutions, additivity on orthogonal projectors, noncontextuality, and unitary covariance. For $n \geq 3$ these assumptions imply the quadratic form $p(P \mid \psi) = \langle\psi|P|\psi\rangle$ by a Gleason-class theorem[2], and the qubit case is handled by a standard strengthening such as extension to POVMs (Busch 2003). Finally, combining these Born weights with deterministic volume-typicality results from our companion work (Paper A) yields observed outcome frequencies as long-run volume ratios under measure-preserving dynamics. The result is a compact, finite-dimensional foundation in which Born statistics arise from symmetry plus operational consistency, with empirical content residing in the typicality and invariance assumptions.

Corresponding author: Zayn Blore, zayn.blore@ou.ac.uk

1. Introduction

In standard quantum mechanics, the Born rule is introduced as a postulate: for a measurement with orthonormal outcome rays $\{|\phi_i\rangle\}$, the probability of outcome i for a pure state $|\psi\rangle$ is $|\langle\phi_i|\psi\rangle|^2$. Although empirically successful, the Born rule is usually not derived from more primitive structural requirements.

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Why should long-run outcome frequencies match squared overlaps, and why does the same functional form arise across disparate quantum systems?

This paper gives a symmetry-first explanation in the finite-dimensional setting. We consider a complex Hilbert space $H = \mathbb{C}^n$ and focus on pure states as rays, so that the operational pure-state space is complex projective space CP^{n-1}. The projective space carries a natural $SU(n)$ action, reflecting the absence of any preferred basis in the operational description of preparations and measurements.

Our first result is geometric. We identify the epistemic measure on CP^{n-1} by imposing a single symmetry requirement: invariance under the $SU(n)$ action. Under mild regularity conditions, this fixes a unique $SU(n)$-invariant Borel probability measure on CP^{n-1}, namely the Fubini–Study measure μ_{FS}, up to normalisation[1] (Ashtekar & Schilling 1999). This step is purely structural and does not assume probabilistic postulates, stochastic dynamics, hidden variables, or interpretive additions. The role of complex rescaling $\psi \sim \lambda\psi$ is to define rays and hence the projective state space. It is not an additional symmetry acting nontrivially on CP^{n-1}.

Symmetry alone does not determine how weights attach to particular outcome projectors. To close this gap while remaining within an operational framework, we introduce standard consistency requirements for a probability assignment $p(P \mid \psi)$ on projectors P: normalisation on orthonormal resolutions of the identity, additivity on orthogonal projectors, noncontextuality, and unitary covariance. For $n \geq 3$, these assumptions imply the quadratic form $p(P \mid \psi) = \langle\psi| P |\psi\rangle$ by a Gleason-class theorem[2]. The qubit case $n = 2$ is handled by a standard strengthening such as extension from projective measurements to POVMs (Busch 2003). Taken together, the symmetry-fixed projective geometry and these operational constraints single out the Born weights for finite-dimensional systems.

Finally, we connect weights to empirical frequencies using deterministic volume-typicality results developed in the companion work (Paper A). In that framework, the system evolves deterministically under a measure-preserving flow on the underlying state space, and outcome frequencies arise as long-run volume ratios for typical initial microstates. Paper B provides the missing link that identifies the relevant operational weights with the Born form, thereby explaining why the volume-based frequencies match squared overlaps in standard quantum mechanics.

We work in finite n because every laboratory apparatus resolves only a finite set of distinguishable outcomes, yielding an effective finite-dimensional operational description. Appendix B sketches, at a

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heuristic level, how parts of the symmetry argument may extend under inductive limits; no infinite-dimensional uniqueness theorem is claimed in the main text.

The paper is structured as follows. Section 2 fixes the $SU(n)$-invariant measure on CP^{n-1} and records the relevant geometric consequences. Section 3 states the operational consistency assumptions and derives the Born weights via a Gleason-class argument, including the $n = 2$ case. Section 4 links these weights to observed frequencies using the deterministic typicality framework of Paper A. Section 5 discusses scope, limitations, and falsifiable assumptions, and outlines potential empirical tests.

1.1. Framework Overview

We work with a finite-dimensional complex Hilbert space $H = \mathbb{C}^n$. Pure states are identified with rays, so the operational pure-state space is the complex projective manifold CP^{n-1}. A measurement context is represented by a set of mutually orthogonal projectors $\{P_i\}$ with $\sum_i P_i = I$, corresponding to a finite set of distinguishable outcomes.

Our objective is to explain why outcome weights take the Born form without introducing probabilistic postulates as primitives. The strategy has two components.

(i) Symmetry fixes the epistemic measure on state space:

The unitary group acts transitively on CP^{n-1}, expressing the absence of any preferred basis in the operational description. We impose the requirement that the epistemic measure on CP^{n-1} is invariant under this action. Under mild regularity assumptions, this identifies a unique $SU(n)$-invariant Borel probability measure on CP^{n-1}, namely the Fubini–Study measure μ_{FS}, up to normalisation[1] (Ashtekar & Schilling 1999). This step is purely geometric. Complex rescaling $\psi \sim \lambda\psi$ plays only the role of defining rays and hence the projective space.

(ii) Operational consistency fixes the functional form of outcome weights:

Symmetry and projective geometry do not, by themselves, determine how weights attach to particular outcome projectors $\{P_i\}$. To obtain a unique weight rule, we introduce a probability assignment

$$p(P \mid \psi) \in [0,1], \tag{1}$$

defined for projectors P and rays $[\psi] \in CP^{n-1}$, satisfying standard operational consistency requirements: (a) normalisation on each orthogonal resolution of the identity, (b) additivity on orthogonal projectors, (c) noncontextuality, and (d) unitary covariance. For $n \geq 3$, a Gleason-class theorem implies that these conditions force the quadratic form

$$p(P \mid \psi) = \langle\psi| P |\psi\rangle \tag{2}$$

[2]. The qubit case $n = 2$ is treated by a standard strengthening such as extending the assignment from projective measurements to POVMs (Busch 2003).

The present paper determines the operational weights. To connect weights to observed frequencies in repeated trials, we rely on deterministic volume-typicality results established in the companion work (Paper A). There, the underlying microstate evolves under a measure-preserving flow, and long-run frequencies coincide with volume ratios for typical initial conditions. Combining that typicality principle with the uniquely fixed weights above yields the Born statistics for finite-dimensional quantum systems, conditional on the stated symmetry and operational assumptions.

In summary, the contribution of this paper is to separate the Born rule problem into a geometric part, which fixes the natural invariant measure on CP^{n-1}, and an operational part, which fixes the quadratic weight rule. Paper A then supplies the deterministic typicality link from weights to observed frequencies.

1.2. Postulates addressed

Standard presentations of finite-dimensional quantum mechanics organise the formalism into a small set of postulates. In this setting, these are commonly stated as: (i) pure states are represented by rays in a complex Hilbert space, (ii) measurement contexts are represented by orthogonal resolutions of the identity $\{P_i\}$, and (iii) outcome weights are given by the Born rule. The present work does not seek to replace the kinematic architecture of finite-dimensional quantum theory. Rather, it targets the specific question of why the Born weights take the quadratic form.

Accordingly, we treat the Hilbert-space and projective-state-space structure as background. We also assume that operational statistics are covariant under changes of basis, expressed by the natural $SU(n)$ action on CP^{n-1}. Within this setting, the postulate we aim to eliminate as primitive is the probability postulate.

More precisely, this paper addresses the following components.

Given a measurement context represented by a set of mutually orthogonal projectors $\{P_i\}$ with $\sum_i P_i = I$, the Born rule assigns

$$p_i = \langle\psi| P_i |\psi\rangle. \tag{3}$$

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Our aim is to show that this quadratic functional form is fixed by symmetry of the projective state space together with standard operational consistency requirements for probability assignments.

The map from weights to observed long-run frequencies in repeated trials is not assumed as an independent stochastic axiom. Instead, we adopt the deterministic typicality framework developed in the companion work (Paper A), in which a measure-preserving flow on the underlying microstate space yields empirical frequencies as typical volume ratios. The present paper supplies the missing identification of the operational weights with the Born form; Paper A supplies the frequency interpretation.

At no point do we assume irreducible randomness, collapse dynamics, or hidden-variable stochasticity. We also do not require interpretive commitments about wavefunction ontology. The argument is finite-dimensional and does not claim to resolve issues that arise uniquely in infinite-dimensional settings, relativistic quantum field theory, or thermodynamic limits.

In summary, the logical role of the present work is to replace the Born probability postulate with a derivation from symmetry and operational consistency, while leaving the remaining finite-dimensional kinematics intact, and to connect those derived weights to empirical frequencies via the deterministic typicality results of Paper A.

Standard postulate	Status	Location
1. Measurement outcomes are stochastic and replaced: Outcomes arise from deterministic branching structure	[3]	this work
2. Probability (interpretive rule)	Replaced: Frequencies emerge from volume ratios in deterministic geometry	[3]

2. Symmetry and the invariant measure

This section fixes the natural measure on the operational pure-state space. The key principle is basis-independence: if a preparation is described only up to a choice of coordinates in $H = \mathbb{C}^n$, then the statistical weight assigned to sets of rays should not depend on how we label the underlying degrees of freedom. Mathematically, this is expressed as invariance under the natural $SU(n)$ action on CP^{n-1}.

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We proceed in two steps. First, we state the symmetry requirement precisely as invariance of a Borel probability measure under the $SU(n)$ action. Second, we record the standard uniqueness result: there is a unique $SU(n)$-invariant Borel probability measure on CP^{n-1}, namely the Fubini–Study measure μ_{FS}. This provides the geometric input needed for the subsequent derivation of Born weights, where operational consistency requirements will fix the functional dependence of outcome probabilities on projectors and states.

2.1. State Space, Evolution, and Outcome Regions

Let $H = \mathbb{C}^n$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We restrict attention to pure states. A nonzero vector $\psi \in H \setminus \{0\}$ represents the same physical pure state as $\lambda\psi$ for any $\lambda \in \mathbb{C}^*$, so pure states are identified with rays. The resulting operational pure-state space is complex projective space

$$CP^{n-1} = (H \setminus \{0\}) / \mathbb{C}^*. \tag{4}$$

Equivalently, one may work on the unit sphere $S^{2n-1} = \{\psi \in H : \|\psi\| = 1\}$ and quotient by global phase $U(1)$, yielding

$$CP^{n-1} \cong S^{2n-1}/U(1). \tag{5}$$

We write $[\psi] \in CP^{n-1}$ for the ray containing ψ, and denote by

$$\pi : S^{2n-1} \to CP^{n-1}, \pi(\psi) = [\psi], \tag{6}$$

the canonical projection map.

A measurement context is represented by an orthogonal resolution of the identity $\{P_i\}_{i=1}^m$ on H, where $P_i P_j = \delta_{ij} P_i$ and $\sum_i P_i = I$. When convenient, we also refer to a corresponding orthonormal basis of outcome rays $\{|\phi_i\rangle\}$ in the nondegenerate case, with $P_i = |\phi_i\rangle \langle\phi_i|$.

Our central geometric object is a Borel probability measure μ on CP^{n-1}, interpreted as the epistemic measure on pure states. For a measurable set $A \subset CP^{n-1}$, $\mu(A)$ denotes its weight. In the next subsection we impose symmetry requirements that single out μ uniquely as the Fubini–Study measure.

2.2. SU(n) covariance and measure invariance

The special unitary group $SU(n)$ acts on $H = \mathbb{C}^n$ by $\psi \mapsto U\psi$, preserving the inner product. This induces a well-defined action on rays,

$$U \cdot [\psi] := [U\psi], U \in SU(n), [\psi] \in CP^{n-1}, \tag{7}$$

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since $U(\lambda\psi) = \lambda(U\psi)$ for all $\lambda \in \mathbb{C}^*$. We interpret this action as expressing basis-independence of the operational description: relabelling the coordinates of H by a unitary transformation should not change the statistical weights assigned to sets of pure states.

Let $\mathcal{B}(CP^{n-1})$ denote the Borel σ-algebra on CP^{n-1}. A Borel probability measure μ on CP^{n-1} is called $SU(n)$-invariant if, for every measurable set $A \in \mathcal{B}(CP^{n-1})$ and every $U \in SU(n)$,

$$\mu(U \cdot A) = \mu(A), \tag{8}$$

where $U \cdot A := \{U \cdot x : x \in A\}$. This invariance condition is the sole symmetry requirement used to fix the measure on the operational pure-state space.

Operational basis-independence is therefore implemented as $SU(n)$-invariance of μ. This reduces the measure-selection problem to the classification of $SU(n)$-invariant Borel probability measures on CP^{n-1}. In the next subsection we use the standard uniqueness result to conclude that μ must coincide with μ_{FS}.

2.3. Worked Example: qubit state space and the invariant measure

We illustrate the invariance requirement of Section 2.1 in the simplest nontrivial case, $n = 2$. For a qubit, the operational pure-state space is

$$CP^1 \cong S^2, \tag{9}$$

the Bloch sphere. Concretely, each ray $[\psi] \in CP^1$ corresponds to a unique Bloch vector $\mathbf{r} \in \mathbb{R}^3$ with $\|\mathbf{r}\| = 1$, defined by $\rho_\psi = |\psi\rangle\langle\psi| = \frac{1}{2}(I + \mathbf{r} \cdot \boldsymbol{\sigma})$, where $\boldsymbol{\sigma}$ are the Pauli matrices.

The action of $SU(2)$ on rays, $[\psi] \mapsto [U\psi]$, induces an action on Bloch vectors by spatial rotations. More precisely, the adjoint action $U\rho_\psi U^\dagger$ corresponds to $\mathbf{r} \mapsto R_U\mathbf{r}$ for some $R_U \in SO(3)$, and every rotation arises from two unitaries $\pm U$. Thus the invariance condition

$$\mu(U \cdot A) = \mu(A), A \in \mathcal{B}(CP^1), \tag{10}$$

is equivalent, under the identification $CP^1 \cong S^2$, to invariance of the corresponding measure under all rotations of the sphere.

Rotation invariance fixes the measure uniquely. Let $\widetilde{\mu}$ be the pushforward of μ under the identification $CP^1 \cong S^2$. If $\widetilde{\mu}$ is a Borel probability measure on S^2 invariant under $SO(3)$, then for any two spherical caps of equal solid angle the invariance implies they have equal $\widetilde{\mu}$-measure. By standard arguments, this forces $\widetilde{\mu}$ to coincide with the normalised area measure on S^2, hence

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$$d\widetilde{\mu}(\mathbf{r}) = \frac{1}{4\pi} d\Omega, \tag{11}$$

where $d\Omega$ is the solid-angle element. Under the identification $CP^1 \cong S^2$, this normalised area measure is precisely the Fubini–Study measure μ_{FS} on CP^1.

It is useful to visualise how measurement contexts transform under this symmetry. For the standard qubit measurement $\{P_0 = |0\rangle\langle 0|, P_1 = |1\rangle\langle 1|\}$, the outcome rays correspond to the north and south poles of the Bloch sphere. Applying $U \in SU(2)$ rotates this measurement axis and maps the pair of outcome rays accordingly. The invariance requirement asserts that the statistical weight assigned by μ to any set of rays is unchanged under such transformations. In higher dimensions, the same basis-independence principle is expressed by $SU(n)$-invariance on CP^{n-1} and leads to the unique invariant measure μ_{FS}.

2.4. General Properties of Volume-Based Outcome Weights

Section 2.3 illustrates the symmetry principle used in this paper: basis-independence is implemented as invariance of a Borel probability measure on CP^{n-1} under the natural $SU(n)$ action. This section introduces the volume-based weights that will be used to connect the symmetry-fixed state-space structure to empirical frequencies, following the deterministic typicality framework developed in the companion work (Paper A).

In that framework, a measurement context induces a measurable partition $\{\Omega_i\}_{i=1}^m$ of an outcome-relevant region Ω_0 in the underlying microstate space. Given a reference measure ν on that space, the corresponding outcome weights are defined by the normalised volumes

$$w_i = \frac{\nu(\Omega_i)}{\nu(\Omega_0)}, \Omega_i \subset \Omega_0, \bigsqcup_{i=1}^{m} \Omega_i = \Omega_0. \tag{12}$$

The role of the present paper is to determine the functional form of these operational weights for finite-dimensional quantum measurements from symmetry and operational consistency requirements. The role of Paper A is to justify the identification of such weights with observed long-run frequencies under repeated macroscopic preparations.

More precisely, volume-typicality results imply that if the underlying dynamics preserves ν and preparations sample typical initial conditions within Ω_0, then the relative volumes w_i coincide with outcome frequencies in the limit of many trials[3]. In this sense, empirical frequency data probe the assumptions of typicality and measure preservation, while the present paper fixes the operational weight rule to which those frequencies are compared.

2.5. Robustness properties of volume-based weights

We collect several basic invariance and robustness properties of the volume-based outcome weights defined in Section 2.4. Let $(\Sigma, \mathcal{F}, \nu)$ denote the underlying microstate space, equipped with a σ-algebra $\mathcal{F}$ and a reference measure ν. Let $\Omega_0 \in \mathcal{F}$ be an outcome-relevant region and let $\{\Omega_i\}_{i=1}^m$ be a measurable partition of Ω_0 with $\bigsqcup_i \Omega_i = \Omega_0$. The weights are

$$w_i = \frac{\nu(\Omega_i)}{\nu(\Omega_0)}. \tag{13}$$

Invariance under measure-preserving reparameterisation:

If $\Psi : \Sigma \to \Sigma$ is a measurable bijection whose pushforward preserves ν, in the sense that $\nu(\Psi(A)) = \nu(A)$ for all $A \in \mathcal{F}$, then

$$\nu(\Psi(\Omega_i)) = \nu(\Omega_i), \nu(\Psi(\Omega_0)) = \nu(\Omega_0), \tag{14}$$

and hence the weights are unchanged. This expresses coordinate-independence: the predictions depend only on the measure-class and the partition, not on an arbitrary choice of parametrisation of Σ.

Dependence on the observable partition:

The regions Ω_i depend on the measurement context, which can be represented abstractly by a measurable map $f : \Omega_0 \to \mathcal{O}$ to a finite outcome set $\mathcal{O} = \{o_i\}$, with $\Omega_i = f^{-1}(o_i)$. Different coarse-grainings generally yield different partitions and therefore different weights. However, if two measurement contexts are related by a symmetry $G : \Sigma \to \Sigma$ that preserves ν, and the preparation is transformed accordingly, then the induced partitions have equal weights:

$$w_i = \frac{\nu(\Omega_i)}{\nu(\Omega_0)} = \frac{\nu(G(\Omega_i))}{\nu(G(\Omega_0))}. \tag{15}$$

This formalises the requirement that physically equivalent measurements yield the same statistical predictions.

Stability under measure-preserving evolution:

Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a measurable flow on Σ that preserves ν, so that $\nu(\varphi_t(A)) = \nu(A)$ for all $A \in \mathcal{F}$ and all t. Then

$$\nu(\varphi_t(\Omega_i)) = \nu(\Omega_i), \tag{16}$$

and the weights are time-invariant. This expresses stationarity of the volume-based weights under measure-preserving deterministic dynamics.

Boundary insensitivity:

If Ω_i and Ω_i' differ only by a set of ν-measure zero, that is, $\nu(\Omega_i \triangle \Omega_i') = 0$, then they define the same weights. In particular, modifications of partition boundaries that remain contained within a ν-null set do not affect the predictions. This is the basic robustness requirement that microscopic boundary refinements should not change empirical weights.

2.6. Summary

We have fixed the geometric setting for the finite-dimensional problem. Pure states are rays, so the operational pure-state space is CP^{n-1} (Section 2.1). Basis-independence is implemented as invariance under the natural $SU(n)$ action on CP^{n-1} (Section 2.2). The qubit case $CP^1 \cong S^2$ illustrates this invariance concretely: $SU(2)$ covariance corresponds to rotational invariance on the Bloch sphere, which uniquely selects the uniform probability measure, identified with the normalised Fubini–Study measure μ_{FS} (Section 2.3).

Separately, following the deterministic typicality framework of the companion work (Paper A), we introduced volume-based outcome weights as normalised measures of outcome regions in an underlying microstate space $(\Sigma, \mathcal{F}, \nu)$

$$w_i = \frac{\nu(\Omega_i)}{\nu(\Omega_0)}. \tag{17}$$

These weights obey the formal properties of probability assignments and enjoy robustness under measure-preserving reparameterisations, symmetries, and measure-preserving evolution (Sections 2.4–2.5). No stochastic input is required at the level of their definition.

What remains is to connect the geometric and operational structures to the specific functional form of quantum weights. Section 3 therefore analyses the conditions under which a probability assignment

$p(P \mid \psi)$ for projectors P and rays $[\psi] \in CP^{n-1}$ is uniquely fixed. Symmetry fixes the natural invariant measure on state space, but to determine the dependence of weights on projectors one also requires standard operational consistency conditions such as normalisation on orthogonal resolutions, additivity on orthogonal projectors, noncontextuality, and unitary covariance. A Gleason-class theorem will then imply the Born form for $n \geq 3$, with the qubit case handled by a standard strengthening.

Symbol	Meaning
Σ	Underlying microstate space in the deterministic typicality framework (Paper A)
φ_t	Measure-preserving flow on Σ: $\varphi_t : \Sigma \to \Sigma$
ν	Reference measure on Σ, preserved by φ_t
$\mathcal{O}$	Finite outcome set $\mathcal{O} = \{o_1, \ldots, o_m\}$
f	Outcome map (measurement context) $f : \Omega_0 \to \mathcal{O}$
Ω_i	Outcome region $\Omega_i = f^{-1}(o_i) \subset \Omega_0 \subset \Sigma$
w_i	Volume-based weight $w_i = \nu(\Omega_i)/\nu(\Omega_0)$
H	Finite-dimensional Hilbert space $H = \mathbb{C}^n$
$SU(n)$	Special unitary group acting on H and inducing an action on CP^{n-1}
CP^{n-1}	Complex projective space of rays $[\psi]$, the operational pure-state space
μ_{FS}	Fubini–Study (normalised) $SU(n)$-invariant probability measure on CP^{n-1}

Table 1. Glossary Table

3. Symmetry, Operations, and the Probability Rule

This section constrains the form of the quantum probability rule from operational symmetry considerations. The object of analysis is the assignment of probabilities to measurement outcomes given a prepared quantum state. States are treated as equivalence classes under physically irrelevant transformations, represented by rays $[\psi]$ in a finite-dimensional Hilbert space, while measurement outcomes are represented by projectors or, more generally, by effects.

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Probabilities are viewed as functions $p(E \mid [\psi])$ of the prepared state and the chosen measurement operation. We impose invariance under transformations that correspond to changes in description or experimental implementation without observable consequence. These include global phase transformations of state representatives and unitary changes of measurement context, under which corresponding outcomes and states are related covariantly. Together with normalisation and additivity over exclusive outcomes, these operational symmetries uniquely constrain the admissible probability assignments.

This section does not attempt to derive the Schrödinger equation, the linearity of quantum dynamics, or any autonomous evolution equation for $|\psi|^2$. Phase information is essential for quantum dynamics and is not removed by the present analysis. The result obtained here concerns only the unique probability functional compatible with the stated operational symmetries and consistency requirements.

3.1. States, Outcomes, and Probability Assignments

We work in a finite-dimensional complex Hilbert space $\mathcal{H} \cong \mathbb{C}^n$. A physical state preparation is represented by a ray $[\psi]$, defined as an equivalence class of nonzero vectors $\psi \in \mathcal{H}$ under multiplication by a nonzero complex scalar. This reflects the operational irrelevance of global phase and overall normalization for probability assignments.

Measurement outcomes are represented by projectors P acting on $\mathcal{H}$ in the case of sharp measurements, or more generally by positive operators E satisfying $0 \le E \le \mathbb{I}$, corresponding to effects in a positive-operator-valued measure. A measurement context is specified by a set of mutually exclusive outcomes $\{E_i\}$ satisfying $\sum_i E_i = \mathbb{I}$.

The central object of this section is a probability assignment

$$p(E \mid [\psi]) \in [0,1],$$

giving the probability of observing outcome E when the system is prepared in state $[\psi]$. This assignment is assumed to depend only on the ray $[\psi]$ and on the operationally defined measurement outcome E, and not on the particular choice of state representative or on details of the experimental implementation. For each measurement context $\{E_i\}$, the probabilities are required to satisfy normalisation,

$$\sum_i p(E_i \mid [\psi]) = 1,$$

without yet specifying their functional form.

3.2. Operational Symmetry Postulates

We now specify the operational symmetries that constrain admissible probability assignments. These symmetries express the requirement that probabilities be invariant under transformations corresponding to physically irrelevant changes in description or experimental implementation.

First, probabilities must be invariant under global phase transformations of the state representative. Since global phase has no operational significance, the probability assignment satisfies

$$p(E \mid [\psi]) = p(E \mid [e^{i\theta}\psi])$$

for all real θ.

Second, probabilities must be covariant under unitary transformations that represent changes of measurement context or descriptive basis. If a state preparation and a measurement outcome are transformed jointly by a unitary operator U, the corresponding probabilities are required to be preserved:

$$p(E \mid [\psi]) = p(UEU^{\dagger} \mid [U\psi]).$$

Here, unitary transformations are taken as the symmetry operations that relate experimentally equivalent implementations, not as dynamical evolutions derived within the present framework.

These symmetry postulates ensure that probability assignments depend only on operationally meaningful features of the preparation and measurement, and not on arbitrary representational choices.

3.3. Additivity, Normalisation, and Contextual Consistency

In addition to the operational symmetry requirements specified above, admissible probability assignments must satisfy basic consistency conditions reflecting the structure of measurement outcomes.

For any measurement context defined by a finite set of mutually exclusive outcomes $\{E_i\}$ satisfying $\sum_i E_i = \mathbb{I}$, probabilities are required to be normalised,

$$\sum_i p(E_i \mid [\psi]) = 1.$$

Probabilities must also be additive over mutually exclusive outcomes within a given measurement context. If two effects E and F correspond to exclusive outcomes in the same context, so that $EF = 0$, then the probability assigned to their coarse-grained outcome satisfies

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$$p(E+F \mid [\psi]) = p(E \mid [\psi]) + p(F \mid [\psi]).$$

These additivity and normalisation conditions are imposed as operational consistency requirements on probability assignments within each measurement context. They do not presuppose any specific functional form for $p(E \mid [\psi])$, nor do they impose relations between probabilities associated with incompatible measurement contexts.

3.4. Uniqueness of the Probability Rule

We now combine the operational symmetry and consistency conditions specified above to constrain the admissible probability assignments. The result is a uniqueness statement for the probability rule associated with sharp measurement outcomes.

Consider a fixed measurement context defined by a finite set of mutually orthogonal projectors $\{P_i\}$ satisfying $\sum_i P_i = \mathbb{I}$. The assignment $p(P_i \mid [\psi])$ defines a frame function on the lattice of projectors, satisfying normalisation and finite additivity within each context. Together with unitary covariance, this ensures that the probability assignment depends on the state only through the ray $[\psi]$ and transforms consistently under changes of measurement basis.

For Hilbert spaces of dimension $n \geq 3$, these conditions fall within the scope of Gleason-type representation theorems. Under the standard assumptions of finite additivity, boundedness, and noncontextuality within measurement contexts, such theorems imply that there exists a unique density operator $\rho_{[\psi]}$ associated with each ray $[\psi]$ such that

$$p(P \mid [\psi]) = Tr(\rho_{[\psi]} P)$$

for all projectors P. For pure state preparations, $\rho_{[\psi]} = |\psi\rangle\langle\psi|$, yielding the familiar Born form

$$p(P \mid [\psi]) = \langle\psi|P|\psi\rangle.$$

This result is invoked as a constraint on admissible probability assignments given the stated operational symmetries and consistency requirements. No claim is made here regarding the dynamics of quantum states or the ontological status of ψ.

3.5. The Qubit Case and POVM Extension

The uniqueness result of the previous subsection relies on representation theorems that apply directly to Hilbert spaces of dimension $n \geq 3$. The two-dimensional case therefore requires separate consideration.

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For qubit systems, restriction to projective measurements alone is insufficient to uniquely fix the probability rule. However, if one enlarges the class of admissible measurement outcomes to include positive-operator-valued measures, the same operational symmetry and consistency requirements may be imposed. In this setting, probabilities are assigned to effects E satisfying $0 \leq E \leq \mathbb{I}$, with normalisation enforced at the level of each POVM.

Allowing POVMs constitutes an operational strengthening of the measurement framework rather than a change of physical assumptions. Under this extension, Gleason-type representation results apply also in dimension two, implying that admissible probability assignments admit a trace representation,

$$p(E \mid [\psi]) = Tr(\rho_{[\psi]} E),$$

for all effects E. For pure state preparations, $\rho_{[\psi]} = |\psi\rangle\langle\psi|$, yielding the Born form for qubit measurements as well.

Thus, once operational symmetry and consistency requirements are imposed uniformly across all admissible measurement procedures, the probability rule is uniquely fixed in all finite dimensions.

3.6. Mixed States and Subsystems

The preceding analysis has focused on probability assignments associated with pure state preparations. In experimental practice, however, preparations may involve classical uncertainty or access to only a subsystem of a larger composite system. It is therefore necessary to extend the probability rule to mixed states.

Operationally, a mixed state is represented by a density operator ρ, encoding either a classical mixture of preparation procedures or the effective description of a subsystem obtained by tracing over inaccessible degrees of freedom. Given a measurement outcome represented by an effect E, the probability assignment is

$$p(E \mid \rho) = Tr(\rho E).$$

This extension is consistent with the operational symmetry and additivity requirements imposed above. In particular, convex combinations of preparation procedures correspond to convex combinations of probability assignments,

$$p(E \mid \lambda\rho_1 + (1-\lambda)\rho_2) = \lambda p(E \mid \rho_1) + (1-\lambda) p(E \mid \rho_2),$$

reflecting classical uncertainty at the level of preparation rather than any additional quantum structure.

The trace form of the probability rule thus extends uniformly to mixed states and subsystems without introducing new assumptions beyond operational consistency.

4. Discussion

The derivation presented above shows that the Born rule follows necessarily from a deterministic, symmetry-constrained geometric framework: outcome weights emerge from volume ratios over a compact, invariant state space, with no requirement for probabilistic or interpretive inputs. In this section, we discuss the broader implications of this result.

First, we clarify how volume-typicality connects to observable outcome frequencies, and why no probability postulate is needed (Section 4.1). Next, we compare this approach with the conventional axioms of quantum mechanics, identifying which postulates are replaced or made redundant (Section 4.2). Finally, we outline the limitations of the current framework and identify key open questions for future work (Section 4.3).

Throughout, the focus remains on what this derivation requires and what it does not. The success of the volume-weighted approach does not depend on philosophical stance. It is a structural result grounded in symmetry, continuity, and the intrinsic geometry of the state space.

4.1. Volume-Typicality and Frequencies

The outcome weights $w_i = \mu(\Omega_i)/\mu(\Omega_0)$ derived in Sections 2 and 3 have the form of a probability distribution but arise from purely geometric and deterministic considerations. No stochastic process, random collapse, or statistical sampling is involved in their definition. Instead, these weights describe the relative volume of each outcome region under a measure-preserving flow on a compact state space.

This supports the notion of volume-typicality introduced in[3] , where observed outcome frequencies are understood as the long-run frequencies of macroscopic events generated by a deterministic system, given uncertainty about the precise microstate. If the system is prepared repeatedly under the same macroscopic conditions, and the set of corresponding microstates is uniformly distributed with respect to the symmetry-invariant measure μ, then the observed frequency of each outcome will, in the limit, approach w_i.

This approach to frequency is entirely deterministic:

- The system evolves predictably from each microstate via the flow φ_t.

- The partition $\{\Omega_i\}$ is defined geometrically by the observable.
- The outcome label i is a deterministic function of the initial microstate.
- The outcome weights arise solely from the relative measure of the outcome regions.

What appears probabilistic at the macroscopic level is thus the consequence of two factors: our ignorance of the specific microstate, and the fact that outcome labels arise from coarse-graining a continuous state space. The term "typicality" reflects this: most microstates consistent with a given preparation lie in outcome regions in proportion to their volume.

Importantly, this view differs sharply from conventional probabilistic interpretations of quantum mechanics. Here, frequencies emerge not from intrinsic indeterminism, but from the geometric structure of the dynamical system and its symmetries.

This closes the explanatory loop: the Born rule is recovered not by assumption or statistical postulate, but by identifying how deterministic systems produce stable ensemble frequencies under well-defined geometric constraints.

4.2. Comparison with Quantum Postulates

Standard formulations of quantum mechanics rely on several core postulates, including:

1. State Postulate: A system is fully described by a unit vector $|\Psi\rangle$ in a Hilbert space $\mathcal{H}$.
2. Observable Postulate: Physical observables correspond to Hermitian operators on $\mathcal{H}$, with eigenvalues representing possible outcomes.
3. Evolution Postulate: The state evolves unitarily via the Schrödinger equation.
4. Measurement Postulate (Born Rule): Upon measurement in basis $\{|\phi_i\rangle\}$, the probability of outcome i is given by $P(i) = |\langle\phi_i|\Psi\rangle|^2$.
5. Collapse Postulate: After measurement, the system collapses to the eigenstate corresponding to the observed outcome.

The framework presented in this paper recovers the outcome statistics implied by Postulate 4 but does so without invoking it. Instead, the result $w_i = |\langle\phi_i|\Psi\rangle|^2$ emerges from deterministic dynamics and symmetry-constrained geometry alone.

Specifically:

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- The Born rule is not assumed; it is shown to be the unique outcome of volume-based typicality under $SU(n)$ symmetry.
- No probabilistic interpretation is invoked. Frequencies arise from volume ratios over coarse-grained outcome regions, not from stochastic collapse or epistemic randomness.
- The derivation is agnostic to ontological commitments. It does not rely on hidden variables, wavefunction branching, or subjective belief updating.

This geometric account thus provides an explanatory foundation for outcome frequencies that is logically prior to the standard postulates. It suggests that if the quantum formalism is an emergent statistical description of an underlying deterministic system, then the Born rule need not be fundamental, it can be derived.

Moreover, by grounding outcome statistics in measurable geometric volumes on a symmetric state space, the approach sidesteps long-standing controversies over collapse and interpretation. It offers a clear distinction between what is observed (frequencies) and what is explained (volume typicality), without needing to assert how or whether the wavefunction itself is ontic.

In this way, the approach complements the operational success of standard quantum theory while offering a deeper structural account of one of its most fundamental statistical rules.

4.3. Limitations and Open Questions

While this paper establishes that the Born rule follows uniquely from deterministic volume geometry under symmetry constraints, several important limitations remain. These concern the scope of applicability, the treatment of composite systems, and the possible extension to more general observables.

Finite-Dimensional Systems

The derivation presented here is restricted to systems represented by finite-dimensional state spaces. The symmetry constraints used in Section 3 rely on the compactness of the projective state space and the unitary invariance of the measure μ.

While our derivation has focused on finite-dimensional quantum systems with compact projective state space $S \cong \mathbb{C}P^{n-1}$, the principles outlined here may extend to continuous or infinite-dimensional systems by coarse-graining. In such cases, effective discretization methods, such as those used in lattice

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field theory[4], allow the continuum to be approximated by a finite number of distinguishable degrees of freedom within a compact geometry. These discretized state spaces can preserve the same geometric symmetries (e.g. unitary invariance) and support a derived volume measure that converges to the continuum form in the limit of finer resolution. Thus, while the present analysis does not address infinite-dimensional Hilbert spaces directly, the volume-based symmetry argument may remain valid in physically relevant limits.

Beyond Projective Measurements

Only sharp measurements, i.e., those defined by orthonormal projective bases, are considered in this framework.

Empirical Signatures

Although the volume-based framework reproduces the Born rule under well-defined assumptions, it currently makes no novel empirical predictions. This limits its immediate falsifiability.

Summary

This paper establishes that deterministic evolution on a symmetry-constrained state space yields outcome frequencies matching the Born rule, with no need for stochastic postulates. What remains is to extend, test, and refine the model:

- Can the framework accommodate entanglement, decoherence, or information flow?
- Can or should it be generalised to infinite dimensions or continuous spectra?
- Are there regimes, cosmological, mesoscopic, or gravitational, where deviations might appear?

These questions define the natural next steps for a volume-based approach to quantum foundations.

5. Falsifiability and Empirical Tests

Any proposed explanation of the Born rule must be subject to empirical scrutiny. While the standard quantum formalism takes $P(i) = |\langle\phi_i|\Psi\rangle|^2$ as an axiom, the present framework derives it from deterministic geometry. This opens the door to testing the assumptions under which the derivation holds.

The key empirical claim is this:

If macroscopic preparation procedures select microstates uniformly (with respect to a symmetry-invariant, volume-preserving measure μ) over the state space S, and if observable outcomes correspond to well-defined partitions $\{\Omega_i\}$, then the frequency of each outcome should converge to $w_i = \mu(\Omega_i)/\mu(\Omega_0) = |\langle\phi_i|\Psi\rangle|^2$.

Any persistent deviation from this rule, under controlled, repeatable conditions, would falsify one or more assumptions of the volume-typicality framework.

For example, Interferometry as a Test of the Measure:

The geometric symmetry-based framework predicts outcome frequencies proportional to $|\Psi|^2$ for all quantum systems.

One feasible test platform is quantum interferometry, such as a Mach–Zehnder interferometer with adjustable phase delay. A single-photon source prepares quantum states with tunable path amplitudes, and detectors at the output arms record relative frequencies.

To test for deviations, one can prepare a known input state $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$, vary the phase ϕ, and record output detection rates. By comparing empirical frequencies one can detect or constrain non-quadratic scaling in the effective probability rule.

Current photon-counting experiments can detect deviations in balanced configurations ($|\alpha| = |\beta| = 1/\sqrt{2}$). No such deviations have been observed, but improved precision could probe closer to the symmetry-derived limit.

Another example, Experimental Implementation in Superconducting Qubit Arrays:

Here high-fidelity unitary operations and repeated state preparation are now routine. These systems allow precise engineering of states $|\Psi\rangle \in \mathbb{C}^d$, with control over initialization, entanglement, and measurement axes via microwave gate sequences.

To test the volume-based prediction $\mu \propto |\Psi|^2$, one can:

- Prepare superpositions of known amplitudes (e.g. $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$),
- Measure in multiple bases via tomographic rotation gates,
- Compare the empirical frequencies of outcomes across repeated trials to the expected Born-rule predictions.

This experimental route tests the core claim of the paper: that the $|\Psi|^2$ rule follows from geometric symmetry constraints, not from axiomatic postulates or probabilistic collapse.

5.1. What Could Be Falsified

There are several concrete ways in which the framework could, in principle, fail:

- Asymmetry in Microstate Sampling: If physical preparation procedures do not yield an invariant ensemble of microstates, but instead bias toward specific regions of Σ, the resulting outcome frequencies would deviate from those predicted by $\mu(\Omega_i)$. High-precision preparation and tomography could test this.
- Violation of Volume Preservation: If the dynamics φ_t fail to preserve volume (e.g. due to dissipation, decoherence, or external interaction), then weights computed from $\mu(\Omega_i)$ would not remain stable. Time-dependent statistical drift could indicate such effects.
- Failure of Coarse-Graining Consistency: If the macroscopic observable does not yield a sharp partition of Σ, or if microscopic ambiguity leads to smeared outcomes, then deviations from the sharp-volume rule may arise. This could be probed by increasing measurement resolution.
- Incompatibility in Composite Systems: If experiments on entangled systems reveal correlations inconsistent with any $SU(n)$-invariant volume assignment across tensor products of Σ, the framework would require modification.

5.2. Prospects and Constraints

At present, no empirical deviation from the Born rule has been confirmed. The volume-based derivation aligns perfectly with standard outcome statistics wherever they have been tested. However, the framework provides a conceptual advantage: its assumptions are visible, structural, and testable. This is in contrast to postulates treated as fundamental or axiomatic.

The challenge for future work is to identify realistic experimental contexts, possibly involving decoherence, gravitational effects, or high-precision state preparation, where deviations from the assumed geometry might occur.

6. Conclusion

This paper has shown how Born weights can be fixed in a finite-dimensional setting from symmetry and operational consistency, without taking probabilistic postulates as primitive. The geometric starting point is that pure states are rays, so the operational pure-state space is complex projective space CP^{n-1}. Basis-independence is implemented as invariance under the natural $SU(n)$ action on CP^{n-1}, which

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uniquely selects the normalised Fubini–Study measure μ_{FS} as the invariant probability measure on state space.

Symmetry alone does not determine how weights attach to particular outcome projectors. To fix the functional form of outcome weights, we introduced standard operational consistency requirements for a probability assignment $p(P \mid \psi)$ on projectors, including normalisation on orthogonal resolutions, additivity on orthogonal projectors, noncontextuality, and unitary covariance. For $n \geq 3$ these conditions imply the quadratic form $p(P \mid \psi) = \langle\psi| P |\psi\rangle$ by a Gleason-class theorem, with the qubit case handled by a standard strengthening such as extension to POVMs. In this sense the Born form is not assumed, but singled out by symmetry together with operational coherence.

The link from weights to observed long-run frequencies is supplied by the deterministic volume-typicality framework developed in the companion work (Paper A). There, measurement contexts induce measurable outcome partitions of an underlying microstate space, and under measure-preserving dynamics typical initial conditions yield empirical frequencies equal to the corresponding volume ratios. The present paper identifies the unique operational weight rule to which those volume-based frequencies should be compared.

The framework is intentionally finite-dimensional and makes no claims of an infinite-dimensional uniqueness theorem, nor of an immediate extension to relativistic quantum field theory. Its empirical content resides in the stated invariance and operational assumptions and in the typicality and measure-preservation conditions required for the frequency link. Within that scope, the result supports a deterministic foundation in which Born statistics arise from symmetry and operational consistency, and observed randomness is understood as typicality under measure-preserving evolution rather than as a fundamental stochastic postulate.

Appendix A. Comparison with Other Approaches

Numerous attempts have been made to derive or justify the Born rule from deeper principles. While differing in assumptions, scope, and interpretation, these approaches share the goal of explaining quantum outcome statistics without postulating them. Below, we briefly compare the present volume-based framework with several leading alternatives.

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A.1. Everettian (Many-Worlds) Derivations

In Everettian frameworks, all outcomes occur, and probabilities are interpreted as branch weights. Modern versions appeal to decision theory[5][6] or typicality over branches[7].

- Strength: Retains the full formalism of quantum theory.
- Limitation: Requires interpretational commitments (e.g. branching reality, rational agents).
- Comparison: The present framework fixes the Born form as the unique operationally coherent weight rule in finite dimensions (Gleason-class, with a qubit strengthening) and treats the link to observed frequencies via deterministic volume-typicality in the companion work (Paper A).

A.2. Envariance and Decoherence-Based Arguments

Zurek's envariance program[8] uses entanglement symmetry to argue that amplitudes must square to produce consistent reduced states. Decoherence is invoked to explain effective outcome separation.

- Strength: Tied to physical processes like entanglement and environment.
- Limitation: Envariance is subtle, and derivations often assume partial trace properties or reduced-state normalisation.
- Comparison: The volume-based approach obtains the Born form from symmetry plus operational coherence constraints in finite dimensions (Gleason-class, with a qubit strengthening) and treats the frequency link separately via deterministic volume-typicality in the companion work (Paper A).

A.3. Gleason-Type Theorems

Gleason's theorem[2] and its generalisations show that, under certain conditions, any probability measure on projectors must match the Born rule.

- Strength: Rigorous mathematical constraint from Hilbert space axioms.
- Limitation: Assumes the Hilbert space formalism and measure additivity from the outset.
- Comparison: This paper adopts these coherence constraints but embed them in a symmetry-fixed projective geometry, using $SU(n)$ invariance to select μ_{FS} and Paper A typicality to interpret the resulting weights as long-run frequencies.

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A.4. Hidden-Variable Theories

In deterministic models like Bohmian mechanics[9][10], the Born rule is typically imposed as an initial condition on the distribution of hidden variables.

- Strength: Provides a deterministic ontology.
- Limitation: Requires a postulated equilibrium distribution, typically not derived.
- Comparison: Our framework does not introduce additional ontic variables beyond the operational state description, and instead fix the Born form from symmetry plus operational coherence (Gleason-class, with a qubit strengthening), with frequencies linked via deterministic volume-typicality in Paper A.

The volume-typicality framework uniquely combines:

- Determinism,
- No hidden structure,
- No appeal to rational agents,
- And a self-contained derivation of the Born rule from geometry and symmetry.

Appendix B. Geometric Interpretation of the Probability Rule

This appendix provides a geometric interpretation of the probability rule established operationally in Section 3. The purpose of this construction is explanatory rather than deductive. It illustrates how the Born form naturally arises when probability assignments are realised as measures on state space that respect the same operational symmetries imposed in the main text.

No new assumptions are introduced here, and no independent derivation of the probability rule is claimed. The uniqueness of the probability assignment follows from the operational analysis of Section 3. The geometric picture presented below is compatible with that result and provides an intuitive realisation of it.

B.1. Projective State Space and Invariant Measure

Let $\mathcal{H} \cong \mathbb{C}^n$ be a finite-dimensional Hilbert space. Pure physical states are represented by rays $[\psi]$, forming the complex projective space $\mathbb{CP}^{n-1}$. This space carries a natural Riemannian structure given by the Fubini–Study metric, which is invariant under the action of the unitary group $SU(n)$.

Associated with this metric is a unique unitarily invariant volume measure μ_{FS} on $\mathbb{CP}^{n-1}$, normalised so that $\mu_{FS}(\mathbb{CP}^{n-1}) = 1$. This measure provides a canonical notion of uniformity over pure states, consistent with the operational symmetries imposed in Section 3.

B.2. Measurement Outcomes as Geometric Partitions

Consider a sharp measurement defined by a set of mutually orthogonal projectors $\{P_i\}$ satisfying $\sum_i P_i = \mathbb{I}$. Each projector defines a subset of projective space associated with the corresponding outcome.

Operationally, a measurement context induces a partition of $\mathbb{CP}^{n-1}$ into disjoint regions $\{\Omega_i\}$, where each region Ω_i corresponds to outcome P_i. These regions are defined relative to the measurement context and transform covariantly under unitary transformations. No physical dynamics on $\mathbb{CP}^{n-1}$ is assumed or required.

B.3. Probability as Relative Geometric Measure

Within this geometric framework, the probability assigned to outcome P_i for a prepared state $[\psi]$ may be interpreted as the relative measure of the region Ω_i with respect to the invariant measure μ_{FS}, conditioned on the preparation.

Imposing unitary invariance and additivity over mutually exclusive outcomes uniquely fixes the dependence of this measure on the prepared ray. The resulting probability assignment takes the form

$$p(P_i \mid [\psi]) = \langle \psi | P_i | \psi \rangle,$$

in agreement with the Born rule obtained operationally in Section 3.

B.4. Scope and Interpretation

The geometric construction presented here does not posit an autonomous dynamics for probabilities, nor does it interpret quantum evolution as motion on projective space. Geometry enters solely as a means of representing operationally invariant probability assignments.

Accordingly, this appendix should be read as providing an intuitive geometric realisation of the probability rule already fixed by operational symmetry and consistency, rather than as an independent derivation or physical model.

Appendix C. Deterministic Volume Typicality and Outcome Robustness

This appendix collects the deterministic and measure-theoretic assumptions underlying the volume-based interpretation of outcome weights used in this paper. These assumptions are not required for the operational uniqueness result established in Section 3. Rather, they provide a concrete physical realisation of how the uniquely fixed probability rule may arise from deterministic dynamics on an underlying state space.

C.1. Deterministic State Space and Invariant Measure

Let Σ be a smooth, compact, finite-dimensional manifold representing the microstate space of the system. We assume the existence of a deterministic flow

$$\varphi_t : \Sigma \to \Sigma$$

generated by a smooth vector field and preserving a normalised measure μ,

$$\mu(\varphi_t(A)) = \mu(A)$$

for all measurable subsets $A \subset \Sigma$ and all times t. No assumption of ergodicity is required. The invariant measure μ defines a notion of typicality on Σ.

C.2. Outcome Regions and Coarse-Graining

Measurement is modelled as a coarse-graining map

$$f : \Sigma \to O,$$

where $O = \{o_i\}$ is a finite set of macroscopically distinguishable outcomes. Each outcome o_i corresponds to a measurable region

$$\Omega_i = f^{-1}(o_i) \subset \Sigma,$$

with the regions $\{\Omega_i\}$ forming a partition of a reference region $\Omega_0 \subset \Sigma$.

Outcome weights are defined as relative volumes,

$$w_i = \frac{\mu(\Omega_i)}{\mu(\Omega_0)}.$$

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C.3. Typicality and Emergence of Frequencies

Under deterministic, measure-preserving dynamics, standard typicality arguments from statistical mechanics imply that for μ-almost all initial conditions, long-run empirical frequencies of outcomes converge to the corresponding volume fractions w_i. The appearance of probabilistic frequencies thus reflects the predominance of typical microstates rather than intrinsic stochasticity.

C.4. Robustness Under Environmental Coupling

In realistic settings, systems interact with additional degrees of freedom. We model this by extending the state space to

$$\Sigma_{tot} = \Sigma_{sys} \times \Sigma_{env},$$

with joint dynamics preserving the product measure $\mu_{tot} = \mu_{sys} \times \mu_{env}$.

Outcome regions extend to product regions

$$\widetilde{\Omega}_i = \Omega_i \times E_i,$$

where $E_i \subset \Sigma_{env}$ represent correlated environmental configurations. Provided the system-level regions Ω_i remain disjoint and macroscopically stable, the relative volume ratios remain approximately invariant,

$$\frac{\mu_{tot}(\widetilde{\Omega}_i)}{\mu_{tot}(\widetilde{\Omega}_0)} \approx \frac{\mu(\Omega_i)}{\mu(\Omega_0)}.$$

C.5. Relation to the Operational Probability Rule

When the invariant measure on the effective state space is identified with a unitarily invariant measure on projective Hilbert space, and outcome regions are defined compatibly with quantum observables, the volume-based weights w_i coincide numerically with the probability rule fixed operationally in Section 3.

Accordingly, this appendix provides a deterministic realisation of the operationally established probability rule, without introducing additional probabilistic postulates or modifying the symmetry-based analysis of the main text.

Appendix D. Glossary of Key Terms

Ψ (psi):

A complex vector representing the state of a finite-dimensional quantum system. In this paper, $\Psi \in \mathbb{C}^n$ is treated as an operational descriptor of system preparation, not a fundamental ontological object.

Outcome region, Ω_i:

A measurable, disjoint subset of the unit sphere in state space corresponding to a macroscopically distinguishable outcome. Each Ω_i is constructed to align with a specific observable result.

Volume-typicality:

The principle that, in a deterministic system with branching, long-run frequencies of outcomes are determined by the invariant volume of each outcome region. A “typical” trajectory lies in the largest such region compatible with its history.

Branching:

The partitioning of state space into disjoint regions as the system evolves, representing distinct future outcomes. Branching is deterministic and geometric, not probabilistic.

Invariant measure, μ:

A volume measure on state space preserved under the system’s deterministic dynamics. Frequencies of outcomes are given by $\mu\left(\Omega_i\right)/\mu\left(\Omega_0\right)$, where Ω_0 is the full accessible state region.

Symmetry constraints:

The physical invariances imposed on the system, specifically:

- Complex-scaling symmetry: $\Psi \rightarrow \lambda\Psi$ for $\lambda \in \mathbb{C}^*$
- Unitary covariance: $\Psi \rightarrow U\Psi$ for $U \in SU(n)$

These ensure the measure respects physical indistinguishability of rescaled, rotated, or phase-shifted states.

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Deterministic flow, φ_t:

A smooth, volume-preserving evolution map $\varphi_t : \Sigma \to \Sigma$ that governs how states evolve over time. Used in blore[3] to track branching and outcome region volumes.

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