

Research Article

Fixing the Measure: Symmetry, Projective Volume and the Born Rule in Deterministic Geometry

Zayn Blore¹

1. Independent researcher

We give a finite-dimensional geometric account of the Born weights that separates three logically distinct questions: selection of the projective measure, geometric realisation of the outcome weights, and their interpretation as empirical frequencies.

Pure states are rays $[\psi] \in \mathbb{CP}^{N-1}$. Unitary symmetry fixes a unique invariant Borel probability measure on this homogeneous space, the Fubini–Study measure μ_{FS} . This uniqueness result is now machine-checked in Lean 4 at general finite N .

The Born weights then follow by a separate geometric step. Under the toric moment map,

$$m([\psi]) = (|\langle e_0, \psi \rangle|^2, \dots, |\langle e_{N-1}, \psi \rangle|^2),$$

the normalised Fubini–Study measure induces the uniform Dirichlet $(1, \dots, 1)$ measure on the probability simplex. For a preparation $b = m([\psi])$, the state-dependent barycentric region associated with outcome i has relative Fubini–Study volume

$$\mu_{\text{FS}}(R_i(\psi)) = |\langle e_i, \psi \rangle|^2.$$

The formal result holds for all finite dimensions and includes vanishing Born weights.

Symmetry therefore fixes the measure, not the measurement regions themselves. The latter are explicit geometric realisations whose physical implementation belongs to the measurement layer of CSD. A second, independent route based on Gleason-type operational consistency is retained as corroborating support rather than as the load-bearing derivation. Finally, the deterministic volume-typicality result of Paper A connects these weights to repeated-trial frequencies under its stated preparation assumptions.

The resulting account derives the finite-dimensional Born weights without postulating them as a fundamental stochastic law.

1. Introduction

This paper separates the Born-rule problem into three mathematical layers.

First, pure states form the projective space

$$\mathbb{CP}^{N-1},$$

on which the unitary group acts transitively. Requiring the projective probability measure to respect this symmetry uniquely selects the Fubini–Study measure μ_{FS} .

Second, measure uniqueness alone does not determine which measurable regions represent the outcomes of a given preparation and measurement. The direct geometric result proved here uses the toric moment map and its induced simplex geometry. For a preparation $[\psi]$, an explicit state-dependent barycentric subdivision has Fubini–Study region weights

$$|\langle e_i, \psi \rangle|^2.$$

The measure and the regions therefore play distinct roles.

Third, Paper A supplies the deterministic repeated-preparation theorem by which appropriate ontic measure weights appear as long-run frequencies. Paper B supplies the finite-dimensional projective weight layer that makes those frequencies take the Born form.

An independent operational route through Gleason-type consistency conditions is retained for comparison. It reaches the same trace rule but is no longer load-bearing for the direct geometric result.

The paper is restricted to finite-dimensional quantum mechanics. Pure states are rays, not unit vectors; vectors $|\psi\rangle$ are representatives used for calculation.

1.1. Framework Overview

We work with a finite-dimensional complex Hilbert space $\mathcal{H} = \mathbb{C}^n$. Pure states are rays $[\psi] \in \mathbb{CP}^{n-1}$, while kets $|\psi\rangle$ are representatives used for calculation.

The Born-weight argument has three logically distinct components.

First, unitary symmetry fixes the projective reference measure. The unitary action on $\mathbb{CP}^{n-1}$ is transitive, and the unique invariant Borel probability measure is the Fubini–Study measure μ_{FS} .

Second, measure uniqueness alone does not determine outcome regions. The direct geometric construction of Section 3 uses the toric moment map and a state-dependent barycentric subdivision of the probability simplex. The resulting projective regions satisfy

$$\mu_{\text{FS}}(R_i(\psi)) = |\langle e_i, \psi \rangle|^2.$$

This is the load-bearing Born-weight result of the paper.

Third, Paper A supplies the separate deterministic typicality theorem connecting appropriate volume weights to repeated-trial frequencies under its stated preparation assumptions.

A Gleason-type operational route is retained as an independent characterisation of the same trace rule. It is not a premise of the direct geometric derivation.

The division of labour is therefore

$$\text{unitary symmetry} \longrightarrow \mu_{\text{FS}},$$

$$\mu_{\text{FS}} + \text{moment geometry} \longrightarrow \text{Born weights},$$

followed separately by

$$\text{Born weights} + \text{Paper A typicality} \longrightarrow \text{repeated-trial frequencies}.$$

1.2. Postulates Addressed

This paper retains the finite-dimensional kinematic structure of quantum mechanics. Pure states are represented by rays in a complex Hilbert space, and measurement contexts are represented by finite spectral decompositions or their associated projectors.

The specific postulate addressed here is the Born probability rule. For a normalised pure-state representative $|\psi\rangle$ and a rank-one outcome projector

$$P_i = |e_i\rangle\langle e_i|,$$

standard quantum mechanics assigns

$$p_i = \langle \psi | P_i | \psi \rangle = |\langle e_i, \psi \rangle|^2.$$

Paper B shows that these weights arise as Fubini–Study volumes of explicit state-dependent projective regions once the invariant projective measure and moment-map geometry are fixed. The Born weights are therefore not introduced as an independent stochastic law.

The map from these weights to repeated-trial frequencies is a separate result supplied by Paper A. Measurement dynamics, definite outcomes and post-measurement update belong to the measurement papers and are not rederived here.

The scope is finite-dimensional. No claim about infinite-dimensional measure uniqueness, relativistic quantum field theory or continuum limits is made in this paper.

2. Symmetry and the invariant measure

This section fixes the natural reference measure on the finite-dimensional projective pure-state space.

The key principle is basis-independence. The projective probability measure should not depend on a choice of coordinates in $\mathcal{H} = \mathbb{C}^n$. This is expressed by invariance under the natural unitary action on $\mathbb{CP}^{n-1}$.

The result of this section is measure selection only:

$$\mu = \mu_{\text{FS}}.$$

Nothing in the symmetry argument uniquely determines the regions associated with measurement outcomes. Their explicit state-dependent geometric construction is supplied separately in Section 3.

2.1. Projective State Space and Measurement Contexts

Let $H = \mathbb{C}^n$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We restrict attention to pure states. A nonzero vector $\psi \in H \setminus \{0\}$ represents the same physical pure state as $\lambda\psi$ for any $\lambda \in \mathbb{C}^*$, so pure states are identified with rays. The resulting operational pure-state space is complex projective space

$$CP^{n-1} = (H \setminus \{0\}) / \mathbb{C}^*. \quad (1)$$

Equivalently, one may work on the unit sphere $S^{2n-1} = \{\psi \in H : \|\psi\| = 1\}$ and quotient by global phase $U(1)$, yielding

$$CP^{n-1} \cong S^{2n-1} / U(1). \quad (2)$$

We write $[\psi] \in \mathbb{CP}^{n-1}$ for the ray containing ψ , and denote by

$$q_n : S^{2n-1} \rightarrow \mathbb{CP}^{n-1}, q_n(\psi) = [\psi],$$

the canonical projective quotient map.

A measurement context is represented by an orthogonal resolution of the identity $\{P_i\}_{i=1}^m$ on H , where $P_i P_j = \delta_{ij} P_i$ and $\sum_i P_i = I$. When convenient, we also refer to a corresponding orthonormal basis of outcome rays $\{|\phi_i\rangle\}$ in the nondegenerate case, with $P_i = |\phi_i\rangle\langle\phi_i|$.

Our central geometric object is a Borel probability measure μ on $\mathbb{CP}^{n-1}$, used as the projective reference measure.¹

For a measurable set $A \subset \mathbb{CP}^{n-1}$, $\mu(A)$ denotes its weight. In the next subsection we impose symmetry requirements that single out μ uniquely as the Fubini–Study measure.

2.2. Unitary covariance and measure invariance

The unitary group $U(n)$ acts on $\mathcal{H} = \mathbb{C}^n$ by $\psi \mapsto U\psi$, preserving the inner product. This induces a well-defined action on rays,

$$U \cdot [\psi] := [U\psi], U \in U(n), [\psi] \in \mathbb{CP}^{n-1},$$

since $U(\lambda\psi) = \lambda(U\psi)$ for every $\lambda \in \mathbb{C}^*$.

We interpret this action as expressing basis-independence of the projective description: a unitary change of coordinates should not change the measure assigned to a Borel set of rays.

A Borel probability measure μ on $\mathbb{CP}^{n-1}$ is unitarily invariant when

$$\mu(U \cdot A) = \mu(A)$$

for every Borel set A and every $U \in U(n)$.

The same projective action may equivalently be formulated using $SU(n)$, since the additional $U(1)$ phase acts trivially on rays.

Operational basis-independence is therefore implemented as unitary invariance of μ . Since the unitary action on $\mathbb{CP}^{n-1}$ is transitive, the homogeneous-space invariant-measure theorem gives

$$\boxed{\mu = \mu_{\text{FS}}}$$

for every unitarily invariant Borel probability measure μ on $\mathbb{CP}^{n-1}$.

The result may equivalently be expressed through

$$\mathbb{CP}^{n-1} \simeq U(n)/(U(1) \times U(n-1)),$$

or through the corresponding $SU(n)$ homogeneous-space representation.

This proof requires no power-law ansatz, radial integration, or prior Born probability rule. The general finite-dimensional characterisation is machine-checked in Lean 4 as `fubiniStudyMeasure_unique`.

2.3. Coordinate cross-check: why the quadratic norm is distinguished

The invariant-measure theorem above is coordinate-free and does not require a power-law ansatz. Nevertheless, Alsing pointed out two useful coordinate-level checks.

Consider

$$Q_k(c) = \sum_j |c_j|^k.$$

A unitary may map

$$e_1 \mapsto \frac{e_1 + e_2}{\sqrt{2}}.$$

If Q_k is unitary invariant, then

$$1 = 2 \left(\frac{1}{\sqrt{2}} \right)^k = 2^{1-k/2},$$

and hence

$$k = 2.$$

Equivalently, the ℓ^p norm of e_1 is one, whereas the transformed vector has norm

$$2^{1/p-1/2}.$$

Unitary invariance therefore requires $p = 2$.

These arguments explain the distinguished quadratic exponent in amplitude coordinates. They are supporting checks rather than the load-bearing proof, which is the invariant-measure uniqueness theorem above.

2.4. Summary

Pure states are rays in $\mathbb{CP}^{n-1}$, and basis-independence is implemented through the unitary action on projective space. The corresponding invariant Borel probability measure is uniquely the Fubini–Study measure μ_{FS} .

The coordinate checks of Section 2.3 independently illustrate why the quadratic norm is distinguished by unitary symmetry, but they are not required for the invariant-measure theorem.

Crucially, measure uniqueness does not determine outcome regions. Section 3 therefore introduces the additional moment-map geometry needed to construct explicit state-dependent regions with Born volumes.

3. Projective Volume and the Born Rule

3.1. Moment coordinates

Fix an orthonormal measurement frame $\{e_1, \dots, e_n\}$. The toric moment map is

$$m : \mathbb{CP}^{n-1} \rightarrow \Delta_{n-1},$$

with

$$m([\phi])_i = \frac{|\langle e_i, \phi \rangle|^2}{\|\phi\|^2}, \sum_i m([\phi])_i = 1.$$

The expression is well defined on rays and does not depend on the chosen representative $|\phi\rangle$.

3.2. Fubini–Study measure in moment space

Under the normalised Fubini–Study measure, the joint moment coordinates have the uniform Dirichlet distribution

$$\text{Dirichlet}(1, \dots, 1).$$

More precisely, on the free coordinates of the open probability simplex,

$$(m_{\text{free}})_* \mu_{\text{FS}} = (n-1)! \lambda \upharpoonright_{\Delta_{n-1}^\circ}.$$

where λ denotes Lebesgue measure. Since the simplex boundary has measure zero, this is equivalent to the uniform Dirichlet(1, ..., 1) probability law on Δ_{n-1} .

This is the finite-dimensional projective instance of Duistermaat–Heckman moment-map geometry ^[1].

In the current Lean 4 corpus the statement is proved directly, rather than imported, as `fs_moment_joint_dirichlet_N` in `CsdLean4/LF4/MomentDirichletN.lean`.

This theorem supplies the measure-theoretic input used by the direct Born-volume construction of the following subsection.

3.3. Direct Born-volume theorem

Let the prepared ray be $[\psi]$ and set

$$b = m([\psi]) = (b_1, \dots, b_n).$$

Subdivide the probability simplex barycentrically at b . Let $S_i(b)$ be the sub-simplex associated with vertex i , and define its projective pullback

$$R_i(\psi) = m^{-1}(S_i(b)).$$

The relative simplex volume of $S_i(b)$ is the barycentric coordinate b_i . Consequently,

$$\boxed{\mu_{\text{FS}}(R_i(\psi)) = b_i = |\langle e_i, \psi \rangle|^2}$$

for a normalised representative $|\psi\rangle$.

The current Lean 4 theorem `fs_born_volume_ratio_N_uncond` establishes this result for arbitrary finite dimension, including vanishing Born weights.

The logical scope is important. The regions $R_i(\psi)$ are state-dependent. Unitary symmetry uniquely fixes μ_{FS} but does not uniquely fix these, or any other, physical measurement regions. The result proves that this explicit geometric construction has the Born volumes. The physical implementation of measurement contexts belongs to the separate measurement layer of CSD.

3.4. General observables

Let $A = A^\dagger$ be an arbitrary finite-dimensional observable. Choose an orthonormal eigenbasis and write

$$A = \sum_k a_k |e_k\rangle\langle e_k|,$$

where repeated eigenvalues are allowed.

The direct Born-volume result gives the rank-one weights

$$p_k = |\langle e_k, \psi \rangle|^2,$$

and therefore

$$\mathbb{E}_\psi[A] = \sum_k a_k p_k = \langle \psi | A | \psi \rangle.$$

If an eigenvalue λ is degenerate, with spectral projector

$$P_\lambda = \sum_{k:a_k=\lambda} |e_k\rangle\langle e_k|,$$

then finite additivity gives

$$p(\lambda) = \sum_{k:a_k=\lambda} p_k = \langle \psi | P_\lambda | \psi \rangle.$$

For mixed preparations the standard affine extension gives

$$\mathbb{E}_\rho[A] = \text{Tr}(\rho A),$$

and for a POVM $\{E_i\}$ the corresponding operational extension is

$$p_i = \text{Tr}(\rho E_i).$$

The Lean 4 theorem `hermitian_observable_correspondence` in `CsdLean4/LF4/ObservableCorrespondenceN.lean` machine-checks this finite-dimensional self-adjoint observable correspondence by spectral unitary transport. Its genericity hypothesis requires the prepared state to have nonzero overlap with every eigenvector of the observable. The underlying Born-volume theorem `fs_born_volume_ratio_N_uncond` separately removes the nonvanishing-amplitude restriction at the level of the Born-volume result itself.

3.5. Independent operational consistency route

A logically separate route reaches the same probability rule.

For $n \geq 3$, Gleason's representation theorem ^[2] shows that a bounded, normalised and finitely additive assignment on projectors, noncontextual in the standard Gleason sense, has trace form

$$p(P|\rho) = \text{Tr}(\rho P).$$

Dimension two requires an operational strengthening. This may be obtained by extending the domain to effects and POVMs ^{[3][4]}, or through suitable mixtures of projective measurements ^[5].

This route remains mathematically valuable as an independent operational characterisation of the Born rule. It is not a premise of the direct Fubini–Study volume theorem above.

4. Discussion

The preceding sections separate the finite-dimensional Born-weight result into two geometric statements. Unitary symmetry fixes the invariant projective measure, while the moment-map construction supplies explicit state-dependent regions having the Born volumes.

The remaining interpretive step is the connection between these geometric weights and empirical frequencies. That step is not another Born-rule derivation: it is supplied by the repeated-preparation typicality result of Paper A.

This section therefore clarifies the frequency interpretation, compares the result with the standard quantum postulates, and states the exact scope of the finite-dimensional construction. The independent Gleason-type route of Section 3.5 remains supporting evidence rather than part of the load-bearing chain.

4.1. Volume-Typicality and Frequencies

Section 3 establishes geometric Born weights:

$$\mu_{\text{FS}}(R_i(\psi)) = |\langle e_i, \psi \rangle|^2.$$

This statement concerns projective volume. It does not by itself assert that one physical trajectory spends a corresponding fraction of time in each region, nor does it assume ergodicity.

The connection to empirical repeated-trial frequencies is supplied by Paper A. There, repeated preparations are represented on the appropriate product preparation space, and under the stated typicality and sampling assumptions the empirical frequency of outcome i converges to its measure-theoretic weight.

This use of a distinguished measure to characterise typical statistical behaviour has precedents in deterministic foundations, notably the quantum-equilibrium and typicality programme in Bohmian mechanics [\[6\]](#). The assumptions are not identical. Paper A explicitly models repeated trials by independent preparation draws from the conditional preparation measure; it does not derive a universal equilibrium distribution from the dynamics.

Paper B therefore supplies the value of the finite-dimensional weight; Paper A supplies the separate deterministic frequency interpretation.

The distinction prevents two different claims from being conflated:

Born as projective volume

is established here, whereas

volume weight as repeated-trial frequency

is established in the companion typicality result.

4.2. Comparison with Quantum Postulates

Standard finite-dimensional quantum mechanics takes as part of its formal structure:

1. pure states represented by rays in a complex Hilbert space;
2. observables represented by self-adjoint operators;
3. unitary Schrödinger evolution for closed systems;
4. Born weights

$$p_i = \langle \psi | P_i | \psi \rangle;$$

5. a separate measurement-update rule.

Paper B addresses item 4.

The Hilbert-space and projective kinematics are retained. Unitary symmetry fixes the Fubini–Study reference measure, and the moment-map barycentric construction then gives

$$\mu_{\text{FS}}(R_i(\psi)) = \langle \psi | P_i | \psi \rangle$$

for the corresponding rank-one outcomes, with the general finite-dimensional observable structure obtained by spectral decomposition.

Thus the Born weights are no longer introduced as a primitive stochastic assignment. Their repeated-trial interpretation is supplied separately by Paper A.

This paper does not attempt to rederive Schrödinger dynamics, measurement dynamics or post-measurement conditioning. Those are separate components of the wider CSD programme.

4.3. Scope

The result is finite-dimensional.

Several distinctions delimit its claim.

First, unitary symmetry uniquely fixes the invariant projective probability measure μ_{FS} . It does not uniquely determine the physical outcome regions.

Second, the direct Born-volume theorem uses explicit state-dependent barycentric regions in moment space and their projective pullbacks. It proves that these regions have the Born volumes. It is not a uniqueness theorem for all possible measurement partitions.

Third, the general CSD bridge from ontic Liouville measure to Fubini–Study measure requires an invariant group action on Σ together with an equivariant projection

$$\pi : \Sigma \rightarrow \mathbb{CP}^{n-1}.$$

The concrete product

$$\Sigma = \mathbb{CP}^{n-1} \times T^2$$

is a witness. Its Fubini–Study marginal is not itself a derivation because μ_{FS} is already present in the product measure.

Fourth, arbitrary finite-dimensional self-adjoint observables are covered by spectral decomposition and unitary transport. No infinite-dimensional invariant-measure or continuum theorem is claimed here.

These are scope boundaries of the finite-dimensional result rather than unresolved steps in the derivation.

5. Empirical Status and Falsifiability

Paper B reproduces the standard finite-dimensional Born weights exactly. It therefore does not, by itself, predict an experimental deviation from quantum mechanics.

Its empirical status is conditional. If the structural assumptions of the construction apply to a physical sector, the observed repeated-trial frequencies must take the Born form. A reproducible violation of the Born rule in such a sector would therefore falsify at least one premise of the account, for example the relevant projective symmetry, the measure bridge, the geometric Born-volume construction, or the repeated-preparation assumptions used to connect weights to frequencies.

Such a deviation would not by itself identify which premise had failed. Conversely, continued experimental confirmation of the Born rule does not distinguish CSD from standard quantum mechanics.

The contribution of Paper B is therefore explanatory and structural rather than a prediction of a modified probability law. Empirical discrimination between CSD and standard quantum mechanics must, if it exists, arise from additional physical structure elsewhere in the theory.

6. Conclusion

Paper B establishes a direct finite-dimensional geometric route to the Born weights.

Pure states are rays in $\mathbb{CP}^{n-1}$. Unitary symmetry uniquely fixes the Fubini–Study probability measure on this projective space. Under the toric moment map, that measure becomes the uniform Dirichlet measure on the probability simplex. Barycentric subdivision at the prepared state then gives state-dependent projective regions whose Fubini–Study volumes are exactly

$$|\langle e_i, \psi \rangle|^2.$$

The invariant-measure characterisation and the direct general- n Born-volume result are machine-checked in Lean 4, including vanishing weights. A Gleason-type operational route gives the same trace rule independently and is retained as supporting confirmation.

The limits of the claim are explicit. Symmetry fixes the measure, not the physical measurement regions. The general ontic measure bridge requires equivariance. The product construction $\mathbb{CP}^{n-1} \times T^2$ is a witness rather than the general CSD ontology. Paper A supplies the separate repeated-preparation theorem that connects the resulting weights to observed frequencies.

Within these conditions, the finite-dimensional Born weights are consequences of projective symmetry and geometry rather than an independently postulated stochastic law.

Appendix A. Relation to Other Structural Derivations

The attempt to derive or reconstruct quantum probabilities from structural principles has a substantial literature.

Caves, Fuchs and Schack develop a Bayesian account in which the quantum probability rule constrains coherent probability assignments [7]. Wright and Weigert derive Gleason-type trace rules from additive consistency conditions and provide finite-dimensional routes relevant also to the qubit case [8]. Auffèves and Grangier develop a realist reconstruction based on contexts and modalities [9]. Helland develops an

epistemic reconstruction in which conceptual variables, group actions and epistemic processes lead to Hilbert-space quantum structure and Born's formula [\[10\]](#).

The present construction shares the use of symmetry and consistency but gives them a different division of labour. Unitary symmetry fixes the projective reference measure. Moment-map geometry then supplies a direct Fubini–Study-volume realisation of the Born weights. Paper A separately connects those weights to deterministic repeated-trial frequencies.

The Gleason and POVM route retained in Section 3 provides an independent operational check on the same trace rule rather than the primary geometric derivation.

No claim of priority is made for the general programme of deriving quantum probabilities from symmetry or operational consistency. The claim of Paper B is the specific projective-measure and moment-map Born-volume construction developed here, together with the separate deterministic typicality result supplied by Paper A.

Appendix B. Glossary of Key Terms

$[\psi]$: A pure-state ray in $\mathbb{CP}^{n-1}$.

$|\psi\rangle$: A nonzero vector representative of the ray $[\psi]$, usually chosen normalised for calculation.

μ_{FS} : The normalised Fubini–Study measure, the unique unitarily invariant Borel probability measure on finite-dimensional complex projective space.

Moment map m : The map from a projective state to its simplex coordinates relative to an orthonormal measurement frame,

$$m([\psi])_i = |\langle e_i, \psi \rangle|^2$$

for a normalised representative.

Born-volume region $R_i(\psi)$: The state-dependent projective pullback of the i th barycentric simplex region. Its Fubini–Study volume is the Born weight associated with outcome i .

Σ : The ontic state space used in the wider CSD framework. Paper B does not identify Σ in general with projective Hilbert space.

π : The ontic-to-projective map $\pi : \Sigma \rightarrow \mathbb{CP}^{n-1}$. In the symmetry-compatible measure bridge, equivariance of π is required.

Volume typicality: The measure-theoretic framework developed in Paper A that connects preparation-space volume weights to repeated-trial frequencies under its stated sampling assumptions.

Acknowledgements

I thank Paul Alsing for identifying a weakness in an earlier coordinate-level argument and for suggesting the quadratic and L^p invariance checks included in Section 2. I thank Inge S. Helland for comments concerning the relation to existing Born-rule reconstructions, the treatment of general observables, and the relevant literature. I also thank Oliver Nash for independently checking the invariant-measure characterisation. These comments materially improved the scope and presentation of the paper.

Footnotes

¹ In the broader CSD programme, the corresponding ontic-to-projective bridge is conditional. If Σ carries an invariant Liouville measure μ_L and $\pi : \Sigma \rightarrow \mathbb{CP}^{n-1}$ is an equivariant projection for the relevant unitary action, then $\pi_*\mu_L = c\mu_{\text{FS}}$. Equivariance is load-bearing: invariance of μ_L alone does not force the pushforward to be Fubini–Study. The concrete witness $\Sigma = \mathbb{CP}^{n-1} \times T^2$ with $\mu_L = \mu_{\text{FS}} \otimes \text{vol}_{T^2}$ is a consistency witness, not an independent derivation of μ_{FS} .

References

1. [△]Duistermaat J, Heckman G (1982). "On the Variation in the Cohomology of the Symplectic Form of the Reduced Phase Space." *Invent Math.* **69**(2):259–268. doi:[10.1007/BF01399506](https://doi.org/10.1007/BF01399506).
2. [△]Gleason A (1957). "Measures on the Closed Subspaces of a Hilbert Space." *J Math Mech.* **6**:885–893.
3. [△]Busch P (2003). "Quantum States and Generalized Observables: A Simple Proof of Gleason's Theorem." *Phys Rev Lett.* **91**:120403. doi:[10.1103/PhysRevLett.91.120403](https://doi.org/10.1103/PhysRevLett.91.120403).
4. [△]Caves C, Fuchs C, Manne K, Renes J (2004). "Gleason-Type Derivations of the Quantum Probability Rule for Generalized Measurements." *Found Phys.* **34**(2):193–209. doi:[10.1023/B:FOOP.0000019581.00318.a5](https://doi.org/10.1023/B:FOOP.0000019581.00318.a5).
5. [△]Wright V, Weigert S (2019). "A Gleason-Type Theorem for Qubits Based on Mixtures of Projective Measurements." *J Phys A Math Theor.* **52**(5):055301. doi:[10.1088/1751-8121/aaf93d](https://doi.org/10.1088/1751-8121/aaf93d).
6. [△]Dürr D, Goldstein S, Zanghì N (1992). "Quantum Equilibrium and the Origin of Absolute Uncertainty." *J Stat Phys.* **67**(5–6):843–907. doi:[10.1007/BF01049004](https://doi.org/10.1007/BF01049004).

7. [△]Caves C, Fuchs C, Schack R (2002). "Quantum Probabilities as Bayesian Probabilities." *Phys Rev A*. **65**:022305. doi:[10.1103/PhysRevA.65.022305](https://doi.org/10.1103/PhysRevA.65.022305).
8. [△]Wright V, Weigert S (2019). "Gleason-Type Theorems from Cauchy's Functional Equation." *Found Phys*. **49**: 594–606. doi:[10.1007/s10701-019-00275-x](https://doi.org/10.1007/s10701-019-00275-x).
9. [△]Auffèves A, Grangier P (2019). "A Generic Model for Quantum Measurements." *Entropy*. **21**(9):904. doi:[10.3390/e21090904](https://doi.org/10.3390/e21090904).
10. [△]Helland I (2021). *Epistemic Processes: A Basis for Statistics and Quantum Theory*. Springer. doi:[10.1007/978-3-030-81923-1](https://doi.org/10.1007/978-3-030-81923-1).

Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.