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Peer Review

Review of: "Fixing the Measure: Deriving $|\Psi|^2$ From Symmetry in Deterministic Geometry"

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The author presents a novel method for understanding why in quantum mechanics, Born's formula for the probability distribution of outcomes in a given state defined by a wave function or by a finite-dimensional vector in a Hilbert space, defined by the magnitude square of the wave function, should be valid. In particular, the author tries to justify Born's formula using the geometry of a Hilbert space and invariance of probabilities under a unitary evolution, rather than taking resort to probability theory. I find this geometric approach very elegant. However, the author should also give some emphasis to the following facts. Firstly, suppose the state space is C^n. Let psi be a vector in this space. Suppose U is a linear/nonlinear transformation on C^n such that <U.psi,U.psi≥=<psi,psi> for

all vectors psi in C^n, i.e., U preserves the length of vectors. Then it is generally not true that U must be a linear operator; only in certain special cases is this true. Hence, the linear dynamics of the Schrödinger equation cannot be justified only by looking at the conservation of total probabilities. Secondly, the author's claim that geometry implies the Born formula can also be stated in the following way: Suppose e_1,e_2,..., e_n is an orthonormal basis for C^n and that a map P maps a wave vector psi in C^n to a nonnegative real number which we call the probability of psi, in such a way that P(e_1)+P(e_2)+...+P(e_n)=1, irrespective of our choice of the orthonormal basis. Then, it is well known that P must be given by the Born formula, i.e., P(e_i)=|<e_i,psi>|^2 for some unit vector \psi in C^n. In fact, this result is a special case of the celebrated Gleason's theorem in any Hilbert space according to which if

H is a Hilbert space and P_1,P_2,... is any orthogonal projective resolution of the identity and mu is a map that associates with any orthogonal projection P a nonnegative real number mu(P) in such a way that irrespective of our choice of the orthogonal projective resolution P_1,P_2,... of the identity, we always have mu(P_1)+mu(P_2)+...=1, then, mu must be given by the formula mu(P)=Trace(rho.P) where rho is a pure/mixed state, i.e., a positive semidefinite operator in H having unit trace. I think Gleason's theorem is the first major result that provides the far-reaching generalization of Born's formula that is at the heart of the mathematical foundations of quantum mechanics. Before Gleason's theorem, it was Dirac and John Von Neumann who described how probabilities in quantum mechanics must be computed, especially for mixed states, but it was Gleason who gave mathematical rigor to the intuitive computations of Dirac. In quantum mechanics, an event is described by an orthogonal projection onto a subspace (PVM stands for projective measurements, and these are defined by orthogonal projective resolutions of the identity), but in modern quantum information theory, one speaks of positive operator-valued measurements (POVM) which are generalizations of projective valued measurements. For such generalized measurements (POVM), Born's rule for computing probabilities has to be interpreted in a different way. The author then speaks about justifying Born's formula in terms of unitary invariance. In the more general setting of mixed states rho, if U(t) is the quantum unitary evolution in the Hilbert space, then the state rho(0) after time t evolves to rho(t)=U(t).rho(0).U(t)^* and if P is an orthogonal projection, then probabilities are not the same before and after evolution, i.e., Trace(rho(t)P) is not the same as Tr(rho(0).P), although these probabilities will be the same if U(t) commutes with P. It will be the same for all P only when U(t) commutes with rho(0), a classic example of which is when U(t) is generated by a Hermitian operator H called the Hamiltonian and when H commutes with rho(0), as is the case for Gibbs states rho(0)=C.exp(-H/kT). I think that with these additional remarks, the author's work will stand as a very good contribution to the foundations of quantum mechanics.

If U(psi)=exp(iH(psi)).psi where H(psi) is any Hermitian matrix-valued function of the vector psi, then U will define a nonlinear operator that preserves the length of the vector psi. However, it will not preserve the angle or inner product between two vectors psi and phi.

When the quantum system is coupled to a noisy quantum bath, with both the system and the bath initially in a pure state, then Born's formula for the joint probabilities of the system and the bath will be valid in computing probabilities, but probabilities for the system alone would require partial tracing over the bath, and hence Born's formula would require a generalization to mixed states. Moreover, if there is classical randomness in the pure state of the system, then Born's formula for the system probabilities would have to be further averaged over this classical randomness, which would lead to Gleason's formula for mixed states. Moreover, in quantum mechanics, at times we have to deal with dissipative quantum systems in which the particle escapes away to infinity and the total probability is not conserved. In this

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case, we have to add a skew-Hermitian matrix to the Hermitian Hamiltonian, and the evolution would then not be unitary; it would not conserve total probabilities defined by Born's formula.

If U(psi) is a unitary operator depending upon the state psi on which it acts, then total probabilities are conserved: <U(psi).psi,U(psi)psi=<psi,psi> but the inner products between different states psi, phi would not be conserved: <U(psi)psi,U(phi)phi> is not equal to <psi,phi>.

Declarations

Potential competing interests: No potential competing interests to declare.

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