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Research Article

Results in Cone Metric Spaces and Related Fixed Point Theorems for Contractive Type Mappings with Application

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In this article, we prove some new fixed point theorems for contractive type mappings in the setting of complete cone metric spaces and provide examples to illustrate the concepts and results developed in the article. We consider some consequences of our results to establish fixed point theorems in the context of cone metric spaces. As an application of our results, periodic point results for the contractive type mappings are proved.

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1. Introduction

Banach's fixed point theorems for contraction mappings are among the important results of mathematical analysis. The Banach contraction principle [1] played a vital role in the development of metric fixed point theory. This principle and its variants provide a useful apparatus for guaranteeing the existence and uniqueness of solutions to various nonlinear problems: differential equations, variational inequalities, optimization problems, integral equations. A host of this principle has been modified and extended by several mathematicians from different perspectives; some of them are as follows:

Huang and Zhang [2] introduced the notion of a cone metric space. In the paper, they replace the real numbers with an ordered Banach space and define a cone metric space. They also gave an example of a function that is a contraction in the category of cone metric but not a contraction if considered over metric spaces and hence, by proving a fixed point theorem in cone metric spaces, ensured that this map must have a unique fixed point. Later, Rezapour and Hamlbarani [3] omitted the assumption of normality in a cone metric space. Subsequently, Aage and Salunke [4] introduced a generalized D*-metric space. Furthermore, Wadei et. al. [5] obtained common fixed point results in the neutrosophic cone metric space and also used the notion of (ϕ,ψ)-weak contraction defined in the neutrosophic cone metric space by using the idea of an altering distance function. This new notion generalized the notion of a generalized G-cone metric space introduced in [6] and a generalized D*-metric space [4]. For other generalizations, we refer to [7][8][9][10][11][12][13].

In view of the above considerations, we establish some fixed point results for contractive type mappings in cone metric spaces. Examples are provided to support the results and concepts presented herein. We consider some consequences of our results to establish fixed point theorems in the context of cone metric spaces. As an application of our results, periodic point results for the contractive type mappings are proved.

Throughout the article, we shall denote E as a Banach space, P a cone in E with $int\ P \neq \{0\}$, a cone $P \subset E$ and $\leq$ as a partial ordering with respect to P. Thus, for any $x, y \in P$, $x \leq y$ if and only if $y - x \in P$, $x < y$ if and only if $x \leq y$ but $x \neq y$, and $x \ll y$ if and only if $y - x \in int\ P$, $int\ P$ denotes the interior of P.

2. Preliminaries

We start this section by presenting some relevant definitions and lemmas.

Definition 1[14] Let E always be a real Banach space and P a subset of E. P is called a cone if and only if:

(i) P is closed, nonempty, and $P \neq \{0\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Definition 2[2] The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$0 \leq x \leq y$ implies $\|x\| \leq K\,\|y\|$.

The least positive number satisfying the above is called the normal constant of P.

Definition 3[2] The cone P is called regular if every increasing sequence which is bounded above is convergent. That is, if $\{x_n\}$ is a sequence such that

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq y$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0 (n \to \infty)$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded below is convergent.

Definition 4[15][16][17] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(d1) $0 < d(x,y)$ and $d(x,y) = 0$ iff $x = y$;

(d2) $d(x,y) = d(y,x)$;

(d3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space.

Example 5[1] Let $E = \mathbb{R}^2$, $d(x,y) = \{(x,y) \in E | x,y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$ and $d : X \times X \to E$ such that $d(x,y) = (|x-y|, \alpha|x-y|)$, where $0 \leq \alpha$ is a constant. Then (X,d) is a cone metric space.

Definition 6[17][1] Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$.We denote this by

$$\lim_{n \to \infty} x_n = x.$$

Lemma 7[2] Let (X,d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0 \ (n \to \infty)$.

Lemma 8[2] Let (X,d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$. That is the limit of $\{x_n\}$ is unique

Definition 9[18] Let (X,d) be a cone metric space, $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 \ll c$, there is N such that for all $n,m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 10[19][20][21][22][23][3] Let (X,d) be a cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Lemma 11[2] Let (X,d) be a cone metric space, $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.

Lemma 12[2] Let (X,d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0 \ (n,m \to \infty)$.

Lemma 13[2] Let (X,d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \to x, y_n \to y \quad (n \to \infty)$. Then $d(x_n, y_m) \to d(x,y)(n \to \infty)$.

3. Main Results

In this section, we begin with the following definitions and theorems.

Definition 3.1. Let $(X,\ d)$ be a complete cone metric space and P be a normal cone with normal constant K. $T : X \to X$ is said to be a type I contraction if for all $x,y \in X$, with $x \neq y$ where $a_1, a_2, a_3 \geq 0$ and $2a_1 + a_2 + a_3 < 1$ satisfying the following condition:

$$d(Tx,Ty) \leq a_1 [d(Tx,x) + d(Ty,y)] + a_2 \frac{d(Tx,x)d(Ty,y)}{d(x,y)} + a_3 \frac{d(Tx,x)d(Ty,y)}{d(x,y)+d(Tx,y)+d(Ty,x)} \tag{3.1}$$

Definition 3.2. Let $(X,\ d)$ be a complete cone metric space and P be a normal cone with normal constant K. $T : X \to X$ is said to be a type II contraction if for all $x,y \in X$, with $x \neq y$ where $a_1, a_2, a_3 \geq 0$, and $2a_1 + a_2 + a_3 < 1$ satisfying the following condition:

$$d(Tx,Ty) \leq a_1 [d(Tx,y) + d(Ty,x)] + a_2 \frac{d(Tx,x)d(Ty,y)}{d(x,y)} + a_3 \frac{d(Tx,x)d(Ty,y)}{d(x,y)+d(Tx,y)+d(Ty,x)} \tag{3.2}$$

Theorem 3.3 Let $(X,\ d)$ be a complete cone metric space and P be a normal cone with normal constant K. Let $T : X \to X$ be a type I contraction. Then T has a unique fixed point in X and for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof Let $x_0 \in X$ be any arbitrary point in X. Define the iterative sequence $\{x_n\}$ by $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \ldots,\ x_{n+1} = Tx_n = T^{n+1}x_0$,.... If for some n, $x_{n+1} = x_n$, then x_n is a fixed point of T, and the proof is complete. So, we assume that for all n, $x_{n+1} \neq x_n$. Then, by using (3.1), we get

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq a_1 [d(Tx_n, x_n) + d(Tx_{n-1}, x_{n-1})] + a_2 \frac{d(Tx_n,x_n)d(Tx_{n-1},x_{n-1})}{d(x_n,x_{n-1})} \\ &\quad + a_3 \frac{d(Tx_n,x_n)d(Tx_{n-1},x_{n-1})}{d(x_n,x_{n-1})+d(Tx_n,x_{n-1})+d(Tx_{n-1},x_n)} \\ &= a_1 [d(x_{n+1}, x_n) + d(x_n, x_{n-1})] + a_2 \frac{d(x_{n+1},x_n)d(x_n,x_{n-1})}{d(x_n,x_{n-1})} \\ &\quad + a_3 \frac{d(x_{n+1},x_n)d(x_n,x_{n-1})}{d(x_n,x_{n-1})+d(x_{n+1},x_{n-1})+d(x_n,x_n)} \end{aligned}$$

implies

$$d(x_{n+1}, x_n) \leq \frac{a_1 + a_3}{1 - (a_1 + a_2)} d(x_n, x_{n-1}) \tag{3.3}$$

Let $\lambda = \frac{a_1+a_3}{1-(a_1+a_2)}$. Since $2a_1 + a_2 + a_3 < 1$ and $a_1 + a_2 < 1$ imply that $\frac{a_1+a_3}{1-(a_1+a_2)} < 1$. Hence,

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}) \text{ for all } n \in \mathbb{N} \tag{3.4}$$

For any $m > n$ where $m, n \in \mathbb{N}$, we have,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \ldots \\ &\quad + d(x_{m+1}, x_m) \\ &\leq (\lambda^{n-1} + \lambda^{n-2} + \ldots + \lambda^m) d(x_1, x_0) \leq \frac{\lambda^m}{1-\lambda} d(x_1, x_0) \end{aligned} \tag{3.5}$$

We get from (3.5) that $\|d(x_n, x_m)\| \leq \frac{\lambda^m}{1-\lambda} K \|d(x_1, x_0)\|$. Which implies $d(x_n, x_m) \to 0 \ (n,m \to \infty)$. This proves that $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete cone metric space, there exists $x^* \in X$ such that $x_n \to x^*(n \to \infty)$. Then

$$d(Tx^*, x^*) \leq d(Tx_n, Tx^*) + d(Tx_n, x^*)$$

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$$\leq a_1\left[d\left(Tx_n,x_n\right)+d\left(Tx^*,x^*\right)\right]+a_2\frac{d(Tx_n,x_n)d(Tx^*,x^*)}{d(x_n,x^*)}$$
$$+a_3\frac{d(Tx_n,x_n)d(Tx^*,x^*)}{d(x_n,x^*)+d(Tx_n,x^*)+d(Tx^*,x_n)}+d\left(x_{n+1},x^*\right)$$

$$d\left(Tx^*,x^*\right)\leq\frac{1}{1-a_1}\left[a_1d\left(Tx_n,x_n\right)+d\left(x_{n+1},x^*\right)\right]$$

$$\left\|d\left(Tx^*,x^*\right)\right\|\leq K\frac{1}{1-a_1}\left[a_1\left\|d\left(x_{n+1},x_n\right)\right\|+\left\|d\left(x_{n+1},x^*\right)\right\|\right]\rightarrow 0. \quad (3.6)$$

Thus, from (3.6), we have $\|d(Tx^*,x^*)\|=0$, that is, $Tx^*=x^*$. Which implies x^* is a fixed point of T.

If y^* is another fixed point of T, then $Ty^*=y^*$. Since T is a type I contraction, we obtain

$$d\left(x^*,y^*\right)=d\left(Tx^*,Ty^*\right)\leq a_1\left[d\left(Tx^*,x^*\right)+d\left(Ty^*,y^*\right)\right]+a_2\frac{d(Tx^*,x^*)d(Ty^*,y^*)}{d(x^*,y^*)}+a_3\frac{d(Tx^*,x^*)d(Ty^*,y^*)}{d(x^*,y^*)+d(Tx^*,y^*)+d(Ty^*,x^*)} \quad (3.7)$$

Hence, from (3.7), we have $d(x^*,y^*)=0$, that is, $x^*=y^*$. Therefore, the fixed point of T is unique.

Theorem 3.4 Let $(X,\ d)$ be a complete cone metric space and P be a normal cone with normal constant K. Let $T:X\rightarrow X$ be a type II contraction. Then T has a unique fixed point in X and for any $x\in X$, the iterative sequence $\{T^nx\}$ converges to the fixed point.

Proof Let $x_0\in X$ be any arbitrary point in X. Define the iterative sequence $\{x_n\}$ by $x_1=Tx_0,x_2=Tx_1=T^2x_0,\ldots,\ x_{n+1}=Tx_n=T^{n+1}x_0$. Assume for all n, $x_{n+1}\neq x_n$ and using (3.2), we get

$$d\left(x_{n+1},x_n\right)=d\left(Tx_n,Tx_{n-1}\right)$$
$$\leq a_1\left[d\left(Tx_n,x_{n-1}\right)+d\left(Tx_{n-1},x_n\right)\right]+a_2\frac{d(Tx_n,x_n)d(Tx_{n-1},x_{n-1})}{d(x_n,x_{n-1})}$$
$$+a_3\frac{d(Tx_n,x_n)d(Tx_{n-1},x_{n-1})}{d(x_n,x_{n-1})+d(Tx_n,x_{n-1})+d(Tx_{n-1},x_n)}$$
$$=a_1\left[d\left(x_{n+1},x_{n-1}\right)+d\left(x_n,x_n\right)\right]+a_2\frac{d(x_{n+1},x_n)d(x_n,x_{n-1})}{d(x_n,x_{n-1})}$$
$$+a_3\frac{d(x_{n+1},x_n)d(x_n,x_{n-1})}{d(x_n,x_{n-1})+d(x_{n+1},x_{n-1})+d(x_n,x_n)}$$

By the triangular inequality, we have

$$d\left(x_{n+1},x_n\right)\leq\frac{a_1+a_3}{1-(a_1+a_2)}d\left(x_n,x_{n-1}\right) \quad (3.8)$$

Let $\lambda=\frac{a_1+a_3}{1-(a_1+a_2)}$. Since $2a_1+a_2+a_3<1$ and $a_1+a_2<1$ imply that $\frac{a_1+a_3}{1-(a_1+a_2)}<1$. Hence,

$$d\left(x_{n+1},x_n\right)\leq\lambda d\left(x_n,x_{n-1}\right) for all n\in\mathbb{N}. \quad (3.9)$$

For any $m>n$ where $m,n\in\mathbb{N}$, we have,

$$d\left(x_n,x_m\right)\leq d\left(x_n,x_{n-1}\right)+d\left(x_{n-1},x_{n-2}\right)+\ldots+d\left(x_{m+1},x_m\right)$$
$$\leq\left(\lambda^{n-1}+\lambda^{n-2}+\ldots+\lambda^m\right)d\left(x_1,x_0\right)$$
$$\leq\frac{\lambda^m}{1-\lambda}d\left(x_1,x_0\right) \quad (3.10)$$

We get from (3.10) that $\|d(x_n,x_m)\|\leq\frac{\lambda^m}{1-\lambda}K\|d(x_1,x_0)\|$, which implies $d(x_n,x_m)\rightarrow 0\ (n,m\rightarrow\infty)$. This proves that $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete cone metric space, there exists $x^*\in X$ such that $x_n\rightarrow x^*(n\rightarrow\infty)$. Then

$$d\left(Tx^*,x^*\right)\leq d\left(Tx_n,Tx^*\right)+d\left(Tx_n,x^*\right)$$
$$\leq a_1\left[d\left(Tx^*,x_n\right)+d\left(Tx_n,x^*\right)\right]+a_2\frac{d(Tx^*,x^*)d(Tx_n,x_n)}{d(x^*,x_n)}+$$
$$a_3\frac{d(Tx^*,x^*)d(Tx_n,x_n)}{d(x^*,x_n)+d(Tx^*,x_n)+d(Tx_n,x^*)}+d\left(x_{n+1},x^*\right)$$
$$\leq a_1\left[d\left(Tx^*,x^*\right)+d\left(x_n,x^*\right)+d\left(x_{n+1},x^*\right)\right]+d\left(x_{n+1},x^*\right)$$
$$d\left(Tx^*,x^*\right)\leq\frac{1}{1-a_1}\left(a_1\left[d\left(x_n,x^*\right)+d\left(x_{n+1},x^*\right)\right]+d\left(x_{n+1},x^*\right)\right)$$
$$\left\|d\left(Tx^*,x^*\right)\right\|\leq K\frac{1}{1-a_1}\left(a_1\left[\left\|d\left(x_n,x^*\right)\right\|+\left\|d\left(x_{n+1},x^*\right)\right\|\right]+\left\|d\left(x_{n+1},\right.\right.\right.$$
$$\rightarrow 0$$

Thus, from (3.11), we have $\|d(Tx^*,x^*)\|=0$, that is, $Tx^*=x^*$, which implies x^* is a fixed point of T.

If y^* is another fixed point of T, then $Ty^*=y^*$. Since T is a type II contraction, we obtain

$$d\left(x^*,y^*\right)=d\left(Tx^*,Ty^*\right)\leq a_1\left[d\left(Tx^*,y^*\right)+d\left(Ty^*,x^*\right)\right]+a_2\frac{d(Tx^*,x^*)d(Ty^*,y^*)}{d(x^*,y^*)}+a_3\frac{d(Tx^*,x^*)d(Ty^*,y^*)}{d(x^*,y^*)+d(Tx^*,y^*)+d(Ty^*,x^*)}=2a_1d\left(x^*,y^*\right) \quad (3.12)$$

Hence, from (3.12), we have $d(x^*,y^*)=0$, that is, $x^*=y^*$. Therefore, the fixed point of T is unique.

We now consider a type I contraction mapping and a type II contraction mapping for some positive integer as corollaries 3.5 and 3.6.

Corollary 3.5 Let $(X,\ d)$ be a complete cone metric space, P be a normal cone with normal constant K. Let $T:X\rightarrow X$ be a type I contraction for some positive integer n, if for all $x,y\in X$, with $x\neq y$ where $a_1,a_2,a_3\geq 0$ and $2a_1+a_2+a_3<1$ satisfying the following condition:

$$d\left(T^nx,T^ny\right)\leq a_1\left[d(Tx,x)+d(Ty,y)\right]+a_2\frac{d(Tx,x)d(Ty,y)}{d(x,y)}+a_3\frac{d(Tx,x)d(Ty,y)}{d(x,y)+d(Tx,y)+d(Ty,x)} \quad (3.13)$$

Then T has a unique fixed point in X.

Proof From Theorem 3.3, T^n has a unique fixed point x^*. But $T^n(Tx^*)=T(T^nx^*)=Tx^*$, so Tx^* is also a fixed point of T^n. Hence $Tx^*=x^*$, x^* is a fixed point of T. Since the

fixed point of T is also a fixed point of T^n, the fixed point of T is unique.

Corollary 3.6 Let $(X,\ d)$ be a complete cone metric space, P be a normal cone with normal constant K. Let $T:X\rightarrow X$ be a type II contraction for some positive integer n, if for all $x,y\in X$, with $x\neq y$ where $a_1,a_2,a_3\geq 0$ and $2a_1+a_2+a_3<1$ satisfying the following condition:

$$d\left(T^nx,T^ny\right)\leq a_1\left[d(Tx,y)+d(Ty,x)\right]+a_2\frac{d(Tx,x)d(Ty,y)}{d(x,y)}+a_3\frac{d(Tx,x)d(Ty,y)}{d(x,y)+d(Tx,y)+d(Ty,x)} \quad (3.14)$$

Then T has a unique fixed point in X.

Proof From Theorem 3.4, T^n has a unique fixed point x^*. But $T^n(Tx^*) = T(T^n x^*) = Tx^*$, so Tx^* is also a fixed point of T^n. Hence $Tx^* = x^*$, x^* is a fixed point of T. Since the fixed point of T is also a fixed point of T^n, the fixed point of T is unique.

Corollary 3.7[2] Let $(X,\ d)$ be a complete cone metric space and P be a normal cone with normal constant K. Suppose the mapping $T: X \to X$ satisfies the contractive condition

$d(Tx, Ty) \leq a_1 d(x, y)$, for all $x, y \in X$,

where $a_1 \in [\,0, 1)$ is a constant. Then T has a unique fixed point in X and for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Corollary 3.8[2] Let $(X,\ d)$ be a complete cone metric space, P be a normal cone with normal constant K. Suppose a mapping $T: X \to X$ satisfies for some positive integer n,

$d(T^n x, T^n y) \leq a_1 d(x, y)$, for all $x, y \in X$,

where $a_1 \in [\,0, 1)$ is a constant. Then T has a unique fixed point in X.

Example 3.9 Let $E=\mathbb{R}^2$, the Euclidean plane, and $P = \left\{(x, y) \in \mathbb{R}^2 | x, y \geq 0\right\}$ a normal cone in P.

Let $X = \left\{(x, 0) \in \mathbb{R}^2 | 0 \leq x \leq 1\right\} \cup \left\{(0, x) \in \mathbb{R}^2 | 0 \leq x \leq 1\right\}$. The mapping $d: X \times X \to E$ is defined by

$$d((x,0),(y,0)) = \left(\frac{4}{3}|x-y|, |x-y|\right)$$

$$d((0,x),(0,y)) = \left(|x-y|, \frac{2}{3}|x-y|\right)$$

$$d((x,0),(0,y)) = d((0,y),(x,0)) = \left(\frac{4}{3}x+y, x+\frac{2}{3}y\right)$$

Then $(X,\ d)$ is a complete cone metric space.

Let the mapping $T: X \to X$ with

$T(x, 0) = (0, x)$ and $T(0, x) = \left(\frac{1}{2}x, 0\right)$

Then T satisfies the type I contractive condition

$$\begin{aligned} &d(T(x_1,x_2), T(y_1,y_2)) \\ &\leq a_1 [d(T(x_1,x_2),(x_1,x_2)) + d(T(y_1,y_2),(y_1,y_2))] \\ &+a_2 \frac{d(T(x_1,x_2),\ (x_1,x_2))d(T(y_1,y_2),\ (y_1,y_2))}{d((x_1,x_2),(y_1,y_2))} + \\ &a_3 \frac{d(T(x_1,x_2),\ (x_1,x_2))d(T(y_1,y_2),\ (y_1,y_2))}{d((x_1,x_2),(y_1,y_2))+d(T(x_1,x_2),(y_1,y_2))+d(T(y_1,y_2),(x_1,x_2))} \end{aligned}$$

For all $(x_1, x_2), (y_1, y_2) \in X$, with constant $a_1 = \frac{2}{30}$, $a_2 = \frac{3}{40}$, $a_3 = \frac{1}{30}$. It is obvious that T has a unique fixed point $(0, 0) \in X$. On the other hand, we see that T is not a contractive mapping in the Euclidean metric on X.

4. Some consequences

In this section, we consider some consequences of our results and establish that Theorem 3.3 and Theorem 3.4 can be utilized to derive the existence of fixed point results for some mappings in a cone metric space with different conditions. In the sequel, we begin with the following definitions.

Definition 4.1. Let $(X,\ d)$ be a complete cone metric space and P be a normal cone with normal constant K. For $c \in E$ with $0 \ll c,\ x_0 \in X$, set $B(x_0, c) = \{x \in X | d(x_0, x) \leq c\}$. $T: X \to X$ is said to be a type I contraction if for all $x, y \in X,\ x \neq y$ and $a_1, a_2, a_3 \geq 0$ with $2a_1 + a_2 + a_3 < 1$ satisfying the following condition:

$$\begin{aligned} d(Tx, Ty) \leq a_1 [d(Tx, x) + d(Ty, y)] + a_2 \frac{d(Tx,\ x)d(Ty,\ y)}{d(x,y)} \\ +a_3 \frac{d(Tx,\ x)d(Ty,\ y)}{d(x,y)+d(Tx,y)+d(Ty,x)} \end{aligned} \tag{4.1}$$

Definition 4.2. Let $(X,\ d)$ be a complete cone metric space and P be a normal cone with normal constant K. For $c \in E$ with $0 \ll c,\ x_0 \in X$, set $B(x_0, c) = \{x \in X | d(x_0, x) \leq c\}$. $T: X \to X$ is said to be a type II contraction if for all $x, y \in X,\ x \neq y$ and $a_1, a_2, a_3 \geq 0$ with $2a_1 + a_2 + a_3 < 1$ satisfying the following condition:

$$\begin{aligned} d(Tx, Ty) \leq a_1 [d(Tx, y) + d(Ty, x)] + a_2 \frac{d(Tx,\ x)d(Ty,\ y)}{d(x,y)} \\ +a_3 \frac{d(Tx,\ x)d(Ty,\ y)}{d(x,y)+d(Tx,y)+d(Ty,x)} \end{aligned} \tag{4.2}$$

Theorem 4.3 Let $(X,\ d)$ be a complete cone metric space and P be a normal cone with normal constant K. For $c \in E$ with $0 \ll c,\ x_0 \in X$, set $B(x_0, c) = \{x \in X | d(x_0, x) \leq c\}$. Let $T: X \to X$ be a type I contraction and $d(Tx_0, x_0) \leq (1 - (a_1 + a_2 + a_3))\, c$. Then T has a unique fixed point in $B(x_0, c)$.

Proof We first prove that $B(x_0, c)$ is complete and then show that $Tx \in B(x_0, c)$ for all $x \in B(x_0, c)$.

Suppose $\{x_n\}$ is a Cauchy sequence in $B(x_0, c)$. Then $\{x_n\}$ is also a Cauchy sequence in X. By the completeness of X, there is $x \in X$ such that $x_n \to x (n \to \infty)$. We have

$$d(x_0, x) \leq d(x_n, x_0) + d(x_n, x) \leq d(x_n, x) + c.$$

Since $x_n \to x,\ d(x_n, x) \to 0$, hence $d(x_0, x) \leq c$, and $x \in B(x_0, c)$. Therefore $B(x_0, c)$ is complete.

For every $x \in B(x_0, c)$,

$$\begin{aligned} &d(x_0, Tx) \leq d(Tx_0, x_0) + d(Tx_0, Tx) \\ &\leq (1 - (a_1 + a_2 + a_3))\, c + a_1 [d(Tx_0, x_0) + d(Tx, x)] \\ &+ a_2 \frac{d(Tx_0,\ x_0)d(Tx,\ x)}{d(x_0,x)} \\ &+a_3 \frac{d(Tx_0,\ x_0)d(Tx,\ x)}{d(x_0,x)+d(Tx_0,x)+d(Tx,x_0)} \\ &\leq (1 - (a_1 + a_2 + a_3))\, c. \end{aligned}$$

Hence $Tx \in B(x_0, c)$.

Theorem 4.4 Let $(X,\ d)$ be a complete cone metric space and P be a normal cone with normal constant K. For $c \in E$ with $0 \ll c,\ x_0 \in X$, set $B(x_0, c) = \{x \in X | d(x_0, x) \leq c\}$. Let $T: X \to X$ be a type II contraction and $d(Tx_0, x_0) \leq (1 - (a_2 + a_3 - a_1))\, c$. Then T has a unique fixed point in $B(x_0, c)$.

Proof We prove that $B(x_0, c)$ is complete and that $Tx \in B(x_0, c)$ for all $x \in B(x_0, c)$. Suppose $\{x_n\}$ is a Cauchy sequence in $B(x_0, c)$. Then $\{x_n\}$ is also a Cauchy sequence in X. By the completeness of X, there is $x \in X$ such that $x_n \to x (n \to \infty)$. We have

$$d(x_0, x) \leq d(x_n, x_0) + d(x_n, x) \leq d(x_n, x) + c.$$

Since $x_n \to x,\ d(x_n, x) \to 0$, hence $d(x_0, x) \leq c$, and $x \in B(x_0, c)$. Therefore, $B(x_0, c)$ is complete.

For every $x \in B(x_0, c)$,

$$d(x_0, Tx) \leq d(Tx_0, x_0) + d(Tx_0, Tx)$$

$$\leq (1-(a_1+a_2+a_3))\,c + a_1\left[d(Tx_0, x) + d(Tx, x_0)\right] + a_2\frac{d(Tx_0, x_0)d(Tx, x)}{d(x_0,x)} + a_3\frac{d(Tx_0, x_0)d(Tx, x)}{d(x_0,x)+d(Tx_0,x)+d(Tx,x_0)}$$

$$\leq (1-(a_1+a_2+a_3))\,c + a_1 2d(x_0, x) \leq (1-(a_1+a_2+a_3))\,c + 2a_1 c$$

$$= (1-(a_2+a_3-a_1))\,c.$$

Hence, $Tx \in B(x_0, c)$.

Corollary 4.5[2] Let $(X,\ d)$ be a complete cone metric space and P be a normal cone with normal constant K. For $c \in E$ with $0 \ll c,\ x_0 \in X,$ set $B(x_0, c) = \{x \in X | d(x_0, x) \leq c\}$. Suppose the mapping $T : X \to X$ satisfies the contractive condition

$d(Tx, Ty) \leq a_1 d(x, y)$, for all $x, y \in B(x_0, c)$,

where $a_1 \in [\,0, 1)$ is a constant and $d(Tx_0, x_0) \leq (1-a_1)\,c$. Then T has a unique fixed point in $B(x_0, c)$.

5. Application

In this section, as an application of our results, we establish the existence of a periodic point result for a self-mapping on a complete cone metric space. In the sequel, we begin with the following definition.

Definition 5.1 A mapping $T\ : X \to X$ is said to have property (P) if $Fix(T^n) = Fix(T)$ for every $n \in \mathbb{N}$, where

$$Fix(T) : \{x \in X : Tx = x\}. \tag{5.1}$$

For further details on this property, we refer to [24][25].

Theorem 5.3 Let $(X,\ d)$ be a complete cone metric space and P be a normal cone with normal constant K. Suppose that the mapping $T : X \to X$ satisfies the conditions in Theorem 3.3. Then T has property P.

Proof From Theorem 3.3, T has a unique fixed point. Let $x^* \in F(T^n)$. Then, by using (3.1), we get

$$d(x^*, Tx^*) = d\left(T\left(T^{n-1}x^*\right), T(T^n x^*)\right)$$

$$\leq a_1\left[d\left(T^n x^*, T^{n-1}x^*\right) + d\left(T^{n+1}x^*, T^n x^*\right)\right] + a_2\frac{d\left(T^n x^*, T^{n-1}x^*\right)d\left(T^{n+1}x^*, T^n x^*\right)}{d\left(T^{n-1}x^*, T^n x^*\right)} + a_3\frac{d\left(T^n x^*, T^{n-1}x^*\right)d\left(T^{n+1}x^*, T^n x^*\right)}{d\left(T^{n-1}x^*, T^n x^*\right)+d\left(T^{n-1}x^*, T^{n+1}x^*\right)+d\left(T^{n-1}x^*, T^n x^*\right)}$$

$$\leq a_1\left[d\left(x^*, T^{n-1}x^*\right) + d(Tx^*, x^*)\right] + a_2\frac{d\left(x^*, T^{n-1}x^*\right)d(Tx^*, x^*)}{d\left(T^{n-1}x^*, x^*\right)} + a_3\frac{d\left(x^*, T^{n-1}x^*\right)d(Tx^*, x^*)}{d\left(T^{n-1}x^*, x^*\right)+d\left(x^*, T^{n-1}x^*\right)+d(Tx^*, x^*)}$$

implies

$$d(x^*, Tx^*) \leq \delta d\left(x^*, T^{n-1}x^*\right)$$

where $\delta = \frac{a_1+a_3}{1-(a_1+a_2)} < 1$.

$$d(x^*, Tx^*) = d\left(T^n x^*, T^{n+1}x^*\right) \leq \delta d\left(T^n x^*, T^{n-1}x^*\right) \leq \ldots \leq \delta^n d(x^*, Tx^*). \tag{5.2}$$

Using (5.2) and definition 2, we have

$$\|d(x^*, Tx^*)\| \leq \delta^n K \|d(x^*, Tx^*)\|. \tag{5.3}$$

Taking the limit as $n \to \infty$ in (5.3), we get

$$\|d(x^*, Tx^*)\| = 0.$$

Hence, $x^* = Tx^*$ and $Fix(T^n) = Fix(T)$.

Theorem 5.4 Let $(X,\ d)$ be a complete cone metric space and P be a normal cone with normal constant K. Suppose that the mapping $T : X \to X$ satisfies the conditions in Theorem 3.4. Then T has property P.

Proof From Theorem 3.4, T has a unique fixed point. Let $x^* \in F(T^n)$. Then, by using (3.2), we get

$$d(x^*, Tx^*) = d\left(T\left(T^{n-1}x^*\right), T(T^n x^*)\right)$$

$$\leq a_1\left[d\left(T^{n-1}x^*, T^{n+1}x^*\right) + d\left(T^{n-1}x^*, T^n x^*\right)\right] + a_2\frac{d\left(T^n x^*, T^{n-1}x^*\right)d\left(T^{n+1}x^*, T^n x^*\right)}{d\left(T^{n-1}x^*, T^n x^*\right)} + a_3\frac{d\left(T^n x^*, T^{n-1}x^*\right)d\left(T^{n+1}x^*, T^n x^*\right)}{d\left(T^{n-1}x^*, T^n x^*\right)+d\left(T^{n-1}x^*, T^{n+1}x^*\right)+d\left(T^{n-1}x^*, T^n x^*\right)}$$

$$\leq a_1\left[d\left(x^*, T^{n-1}x^*\right) + d(Tx^*, x^*)\right] + a_2\frac{d\left(x^*, T^{n-1}x^*\right)d(Tx^*, x^*)}{d\left(T^{n-1}x^*, x^*\right)} + a_3\frac{d\left(x^*, T^{n-1}x^*\right)d(Tx^*, x^*)}{d\left(T^{n-1}x^*, x^*\right)+d\left(x^*, T^{n-1}x^*\right)+d(Tx^*, x^*)}$$

implies

$$d(x^*, Tx^*) \leq \alpha d\left(x^*, T^{n-1}x^*\right)$$

where $\alpha = \frac{a_1+a_3}{1-(a_1+a_2)} < 1$.

$$d(x^*, Tx^*) = d\left(T^n x^*, T^{n+1}x^*\right) \leq \alpha d\left(T^n x^*, T^{n-1}x^*\right) \leq . \leq \alpha^n d(x^*, Tx^*).$$

Using (5.4) and definition 2, we have

$$\|d(x^*, Tx^*)\| \leq \alpha^n K \|d(x^*, Tx^*)\|. \tag{5.5}$$

Taking the limit as $n \to \infty$ in (5.5), we get

$$\|d(x^*, Tx^*)\| = 0.$$

Thus, $x^* = Tx^*$ and $Fix(T^n) = Fix(T)$.

6. Conclusion

In this paper, we prove some new fixed point theorems for contractive type mappings in complete cone metric spaces. We also present examples to illustrate the concepts and results developed in the article. We consider some consequences of our results to establish fixed point theorems in the context of cone metric spaces. Finally, as an application of our results, periodic point results for the contractive type mappings are proved to demonstrate how our theorems can be used to generalize existing fixed point theorems for cone metric spaces, highlighting the significance and applicability of our results.

Statements and Declarations

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Data availability

No data availability in this paper.

Authors' contributions

All authors contributed equally to the writing of this paper.

Conflicting interests

The authors declare that there are no conflicting interests.

Research involving human participants and/or animals

This article does not contain any studies with human participants or animals performed by the authors.

Other References

- Arif A., Nazam M., Al-Sulami HH, Hussain A., Mahmood H., Fixed Point and Homotopy Methods in Cone A-Metric Spaces and Application to the Existence of Solutions to Urysohn Integral Equation, Symmetry 14 (7) (2022) 1328.
- Raji M., Generalized α-ψ Contractive Type Mappings and Related Coincidence Fixed Point Theorems with Applications, The Journal of Analysis, 31(2), (2023), 1241-1256.
- Azam, A., Mehmood, N., Rashid, M., & Radenović, S. Fuzzy fixed point theorems in ordered cone metric spaces. Filomat, 29(4), (2015), 887-896.

References

1. a, b, c *Banach, S.: Sur les operations dans les ensambles abstraits et leur application aux equations integrals. Fund. Math. 3, (1922),133–181.*
2. a, b, c, d, e, f, g, h, i, j, k *Huang, L.G., Zhang, X.: Cone metric spaces and fixed point theorems for contractive mappings. J. Math. Anal. Appl. 332,2, (2007),1468–1476.*
3. a, b *Rezapour, Sh, Hamlbarani, R.: Some notes on the paper, "Cone metric spaces and fixed point theorems of contractive mappings"J. Math. Anal. Appl. 345, (2008),719–724.*
4. a, b *Aage, C.T., Salunke, J.N.: Some fixed point theorem in generalized D*-metric spaces. Appl. Sci. 12, (2010),1–13.*
5. ^ *Wadei F. A., Saeid J., and Florentin S. (ϕ,ψ)-Weak Contractions in Neutrosophic Cone Metric Spaces via Fixed Point Theorems, Journal Mathematical Problems in Engineering, 8, (2020), Pages 1-8.*
6. ^ *Ismat, Beg, Mujahid, Abbas, Talat, Nazir: Generalized cone metric spaces. J. Nonlinear Sci. Appl. 3, (2010),21–31.*
7. ^ *Mujahid, A., Rhoades, B.E.: Fixed and periodic point results in cone metric spaces. Appl. Math. Lett. 22, (2009),511–515.*
8. ^ *Stojan, Radenovic: Common fixed points under contractive conditions in cone metric spaces. Comput. Math. Appl. 58, (2009),1273–1278.*
9. ^ *Feng Y., Mao W., The equivalence of cone metric spaces and metric spaces, Fixed Point Theory, 11(2) (2010), 259-264.*
10. ^ *Kadelburg Z., Radenovic S., Rakocevic V., A note on the equivalence of some metric and cone metric fixed point results, Appl. Math. Lett., 24 (2011), 370-374.*
11. ^ *Kadelburg Z., Pavlovic M., Radenovic S., Common fixed point theorems for ordered contractions and quasi-contractions in ordered cone metric spaces, Comput. Math. Appl., 59 (2010), 3148-3159.*
12. ^ *Wang T., Yin J., Yan Q., Fixed point theorems on cone 2-metric spaces over Banach algebras and an application, Fixed Point Theory Appl., (2015) 2015:204.*
13. ^ *Wadei F. A., Saeid J., Smarandache F., Neutrosophic Fixed Point Theorems and Cone Metric Spaces, Journal Neutrosophic Sets and Systems, 31, (2020), 250-265.*
14. ^ *Abbas M., Jungck G., Common fixed point results for non commuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl., 341 (2008), 416-420.*
15. ^ *Cho, S.H., Bae, J.S.: Common fixed point theorems for mappings satisfying property (E.A.) on cone metric space. Math. Comput. Model. 53, (2011),945–951.*
16. ^ *Huaping, Huang, Shaoyuan, Xu: Fixed point theorems of contractive mappings in cone b-metric spaces and applications. Fixed Point Theory Appl. 10, (2012),220.*
17. a, b *Hussain, N., Shah, M.H.: KKM mapping in cone b-metric spaces. Comput. Math. Appl. 62, (2011),1677–1684.*
18. ^ *Babu, G.V.R., Alemayehu, G.N., Vara Prasand, K.N.V.V.: Common fixed point theorems of generalized contraction, Zamfirescu pair of maps in cone metric spaces. Albenian J. Math. 4,1, (2010),19–29.*
19. ^ *Badshah V., Bhagat P., Shukla S., Some fixed point theorems for α-ϕ-contractive mappings in cone 2-metric spaces, Appl. Math. Inform. Mech., 8,2, (2016), 187-199.*
20. ^ *Jankovic, S., Kodelburg, Z., Radenovic, S.: On cone metric spaces: a survey. Nonlinear Anal. 4,7, (2011),2591–2601.*
21. ^ *Di Bari C., Vetro P., φ-Pairs and common fixed points in cone metric space, Rend. Circ. Mat. Palermo, 57 (2008) 279-285.*
22. ^ *Malviya, N., Chouhan, P.: Fixed point of expansive type mapping in N-cone metric space. Math. Theory Model. 3,7, (2013)*
23. ^ *Du W.S., A note on cone metric fixed point theory and its equivalence, Nonlinear Anal., 72,5, (2010), 2259-2261.*
24. ^ *Abbas M, Ali B, Romaguera S. Fixed and periodic points of generalized contractions in metric space. Fixed Point Theory Appl, 2013, 2013: 243*
25. ^ *Raji M., Ibrahim M.A. Fixed point theorems for modified F-weak contractions via α- admissible mapping with application to periodic points, Annals of Mathematics and Computer Science, (2024),20, 82-97.*

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