Research Article

Results in Cone Metric Spaces and Related Fixed Point Theorems for Contractive Type Mappings

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The purpose of this article, is to establish some fixed point results for contractive type mappings in cone metric spaces. Examples are provided to support results and concepts presented herein. As an application of our results, we deduce other established fixed point theorems in cone metric spaces.

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1. Introduction

Banach's fixed point theorems for contraction mappings is one of the important results of mathematical analysis. Banach contraction principle [1] played a vital role in the development of a metric fixed point theory. This principle and its variants provide a useful apparatus in guaranteeing the existence and uniqueness of solutions of various nonlinear problems: differential equations, variational inequalities, optimization problems, integral equations. A host of this principle has been made modified and extended by several mathematicians in different perspectives, some of them are as follows:

Huang and Zhang [2] introduced the notion of cone metric space. In the paper, they replace the real numbers by ordering Banach space and define cone metric space. They also gave an example of a function which is a contraction in the category of cone metric but not contraction if considered over metric spaces and hence by proving fixed point theorem in cone metric spaces ensured that this map must have a unique fixed point. Later, Rezapour and Hamlbarani [3] omitted the assumption of normality in cone metric space. Subsequently, Aage and Salunke [4] introduced a generalized D^*-metric space. Furthermore, Malviya and Fisher [5] introduced the notion of N-cone metric space and proved fixed point

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theorems for asymptotically regular maps and sequence. This new notion generalized the notion of generalized G-cone metric space introduce in [6] and generalized D*-metric space [4]. For other generalizations, we refer to [7][8][9][10][11][12][13].

In view of the above considerations, we establish some fixed point results for contractive type mappings in cone metric spaces. Examples are provided to support results and concepts presented herein. As an application of our results, other established fixed point theorems in cone metric spaces are studied.

Throughout the article, we denoted E as a Banach space, P a cone in E with $intP \neq 0$, a cone $P \subset E$ and $\leq$ is partial ordering with respect to P by $x \leq y$ if and only if $y - x \in P$. We also write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, $intP$ denotes the interior of P.

2. Preliminaries

We start this section by presenting some relevant definitions and lemma.

Definition 1[14] Let E always be a real Banach space and P a subset of E. P is called a cone if and only if:

i. P is closed, nonempty, and $P \neq 0$;

ii. $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;

iii. $x \in P$ and $-x \in P \Rightarrow x = 0$.

Definition 2[2] The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$0 \leq x \leq y$and $\|x\| \leq K\|y\|$.

The least positive number satisfying above is called the normal constant of P.

Definition 3[2] The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$$

or some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0 (n \to \infty)$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

Definition 4[15][16][17] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(d1) $d(x, y) > 0$ and $d(x, y) = 0$ iff $x = y$;

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(d2) $d(x,y)=d(y,x)$;

(d3) $d(x,y)\leq d(x,z)+d(z,y)$ for all $x,y,z\in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space.

It is obvious that cone metric spaces generalize metric spaces.

Example 5[2] Let $E=\mathbb{R}^2$, $d(x,y)=\{(x,y)\in|x,y\geq 0\}\subset\mathbb{R}^2$, $X=R$ and $d:X\times X\to E$ such that $d(x,y)=(|x-y|,\alpha|x-y|)$, where $\alpha\geq 0$ is a constant. Then (X,d) is a cone metric space.

Definition 6[17] Let (X,d) be a cone metric space. Let x_n be a sequence in X and $x\in X$. If for every $c\in E$ with $0\ll c$ there is N such that for all $n>N$, $d(x_n,x)\ll c$, then x_n is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$.We denote this by

$$\lim_{n\to\infty}x_n=x$$

Lemma 7[2] Let (X,d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $d(x_n,x)\to 0(n\to 0)$.

Lemma 8[2] Let (X,d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x=y$. That is the limit of $\{x_n\}$ is unique

Definition 9[18] Let (X,d) be a cone metric space, x_n be a sequence in X. If for any $c\in E$ with $0\ll c$, there is N such that for all $n,m>N$, $d(x_n,x_m)\ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 10[19][20][21][22][23][3] Let (X,d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Lemma 11[2] Let (X,d) be a cone metric space, $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.

Lemma 12[2] Let (X,d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then x_n is a Cauchy sequence if and only if $d(x_n,x_m)\to 0(n,m\to 0)$.

Lemma 13[2] Let (X,d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n\to x$,$y_n\to y(n\to 0)$. Then $d(x_n,y_m)\to d(x,y)(n\to 0)$.

3. Main Results

In this section, we begin with the following definitions and Theorems.

Definition 3.1. Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. A mapping $T : X \to X$ is said to be type I contraction if for all $x, y \in X, x \neq y$ and $a_1, a_2, a_3 \geq 0$ with $a_1 + a_2 + a_3 < 1$ satisfying the following condition:

$$\begin{aligned} d(Tx, Ty) \leq a_1[d(Tx, x) + d(Ty, y)] + a_2 \frac{d(Tx, x)d(Ty, y)}{d(x, y)} + \\ a_3 \frac{d(Tx, x)d(Ty, y)}{d(x, y) + d(Tx, y) + d(Ty, x)} \end{aligned} \tag{3.1}$$

Definition 3.2. Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. A mapping $T : X \to X$ is said to be type II contraction if for all $x, y \in X, x \neq y$ and $a_1, a_2, a_3 \geq 0$ with $a_1 + a_2 + a_3 < 1$ satisfying the following condition:

$$\begin{aligned} d(Tx, Ty) \leq a_1[d(Tx, y) + d(Ty, x)] + a_2 \frac{d(Tx, x)d(Ty, y)}{d(x, y)} + \\ a_3 \frac{d(Tx, x)d(Ty, y)}{d(x, y) + d(Tx, y) + d(Ty, x)} \end{aligned} \tag{3.2}$$

Theorem 3.3 Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. A mapping $T : X \to X$ is said to be type I contraction. Then T has a unique fixed point in X and for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof Let $x_0 \in X$ be any arbitrary point in X. Define the iterate sequence $\{x_n\}$ by $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \ldots, x_{n+1} = Tx_n = T^{n+1}x_0$,.... If for some n, $x_{n+1} = x_n$, then x_n is a fixed point of T, the proof is complete. So, we assume that for all n, $x_{n+1} \neq x_n$. Then, by using (3.1), we get

$$\begin{aligned} d\left(x_{n+1}, x_n\right) &= d\left(Tx_n, Tx_{n-1}\right) \\ &\leq a_1\left[d\left(Tx_n, x_n\right) + d\left(Tx_{n-1}, x_{n-1}\right)\right] + a_2 \frac{d\left(Tx_n, x_n\right) d\left(Tx_{n-1}, x_{n-1}\right)}{d\left(x_n, x_{n-1}\right)} + \\ &\quad a_3 \frac{d\left(Tx_n, x_n\right) d\left(Tx_{n-1}, x_{n-1}\right)}{d\left(x_n, x_{n-1}\right) + d\left(Tx_n, x_{n-1}\right) + d\left(Tx_{n-1}, x_n\right)} \\ &= a_1\left[d\left(x_{n+1}, x_n\right) + d\left(x_n, x_{n-1}\right)\right] + a_2 \frac{d\left(x_{n+1}, x_n\right) d\left(x_n, x_{n-1}\right)}{d\left(x_n, x_{n-1}\right)} \\ &\quad + a_3 \frac{d\left(x_{n+1}, x_n\right) d\left(x_n, x_{n-1}\right)}{d\left(x_n, x_{n-1}\right) + d\left(x_{n+1}, x_{n-1}\right) + d\left(x_n, x_n\right)} \\ d\left(x_{n+1}, x_n\right) &\leq \frac{a_1 + a_3}{1 - (a_1 + a_2)} d\left(x_n, x_{n-1}\right) \qquad (3.3) \end{aligned}$$

Let $\lambda = \frac{a_1 + a_3}{1 - (a_1 + a_2)}$. Since $a_1 + a_2 + a_3 < 1$ implies that $\frac{a_1 + a_3}{1 - (a_1 + a_2)} < 1$. Hence,

$$d\left(x_{n+1}, x_n\right) \leq \lambda d\left(x_n, x_{n-1}\right) \text{ for all } n \in \mathbb{N}. \tag{3.4}$$

For any $m > n$ where $m, n \in \mathbb{N}$, we have,

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$$
\begin{aligned}
d\left(x_n, x_m\right) & \leq d\left(x_n, x_{n-1}\right)+d\left(x_{n-1}, x_{n-2}\right)+\cdots+d\left(x_{m+1}, x_m\right) \\
& \leq\left(\lambda^{n-1}+\lambda^{n-2}+\cdots+\lambda^m\right) d\left(x_1, x_0\right) \\
& \leq \frac{\lambda^m}{1-\lambda} d\left(x_1, x_0\right) \qquad (3.5)
\end{aligned}
$$

We get from (3.5) that $\|d(x_n, x_m)\| \leq \lambda^m/(1-\lambda)K\|d(x_1, x_0)\|$. Which implies $d(x_n, x_m) \rightarrow 0 (n, m \rightarrow \infty)$. This proves that $\{x_n\}$ is Cauchy sequence in X. Since X is a complete cone metric space, there exists $x^* \in X$ such that $x_n \rightarrow x^*(n \rightarrow \infty)$. Then

$$
\begin{aligned}
& d\left(T x^*, x^*\right) \leq d\left(T x_n, T x^*\right)+d\left(T x_n, x^*\right) \\
& \leq a_1\left[d\left(T x_n, x_n\right)+d\left(T x^*, x^*\right)\right]+a_2 \frac{d\left(T x_n, x_n\right) d\left(T x^*, x^*\right)}{d\left(x_n, x^*\right)}+ \\
& a_3 \frac{d\left(T x_n, x_n\right) d\left(T x^*, x^*\right)}{d\left(x_n, x^*\right)+d\left(T x_n, x^*\right)+d\left(T x^*, x_n\right)}+d\left(x_{n+1}, x^*\right) \\
& \quad d\left(T x^*, x^*\right) \leq \frac{1}{1-a_1}\left[a_1 d\left(T x_n, x_n\right)+d\left(x_{n+1}, x^*\right)\right] \\
& \left\|d\left(T x^*, x^*\right)\right\| \leq K \frac{1}{1-a_1}\left[a_1\left\|d\left(x_{n+1}, x_n\right)\right\|+\left\|d\left(x_{n+1}, x^*\right)\right\|\right] \rightarrow 0 \qquad 3.6
\end{aligned}
$$

Thus, from (3.6), we have $\|d(Tx^*, x^*)\| = 0$, that is, $Tx^* = x^*$. Which implies x^* is a fixed point of T.

If y^* is another fixed point of T, then $Ty^* = y^*$. Since T is type I contraction, we obtain

$$
\begin{aligned}
d\left(x^*, y^*\right)=d\left(T x^*, T y^*\right) \leq a_1\left[d\left(T x^*, x^*\right)+d\left(T y^*, y^*\right)\right]+a_2 \frac{d\left(T x^*, x^*\right) d\left(T y^*, y^*\right)}{d\left(x^*, y^*\right)}+ \\
a_3 \frac{d\left(T x^*, x^*\right) d\left(T y^*, y^*\right)}{d\left(x^*, y^*\right)+d\left(T x^*, y^*\right)+d\left(T y^*, x^*\right)} \qquad (3.7)
\end{aligned}
$$

Hence, from (3.7), we have $d(x^*, y^*) = 0$, that is, $x^* = y^*$. Therefore, the fixed point of T is unique.

Theorem 3.4 Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. A mapping $T: X \rightarrow X$ is said to be type II contraction. Then T has a unique fixed point in X and for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof Let $x_0 \in X$ be any arbitrary point in X. Define the iterate sequence $\{x_n\}$ by $x_1 = Tx_0, x_2 = Tx_1 = T^2 x_0, \ldots, x_{n+1} = Tx_n = T^{n+1} x_0$. Now, using (3.2), we get

$$
\begin{aligned}
& d\left(x_{n+1}, x_n\right)=d\left(T x_n, T x_{n-1}\right) \\
& \leq a_1\left[d\left(T x_n, x_{n-1}\right)+d\left(T x_{n-1}, x_n\right)\right]+a_2 \frac{d\left(T x_n, x_n\right) d\left(T x_{n-1}, x_{n-1}\right)}{d\left(x_n, x_{n-1}\right)}+ \\
& a_3 \frac{d\left(T x_n, x_n\right) d\left(T x_{n-1}, x_{n-1}\right)}{d\left(x_n, x_{n-1}\right)+d\left(T x_n, x_{n-1}\right)+d\left(T x_{n-1}, x_n\right)} \\
& =a_1\left[d\left(x_{n+1}, x_{n-1}\right)+d\left(x_n, x_n\right)\right]+a_2 \frac{d\left(x_{n+1}, x_n\right) d\left(x_n, x_{n-1}\right)}{d\left(x_n, x_{n-1}\right)} \\
& +a_3 \frac{d\left(x_{n+1}, x_n\right) d\left(x_n, x_{n-1}\right)}{d\left(x_n, x_{n-1}\right)+d\left(x_{n+1}, x_{n-1}\right)+d\left(x_n, x_n\right)}
\end{aligned}
$$

By triangular inequality, we have

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$$d\left(x_{n+1}, x_n\right) \leq \frac{a_1+a_3}{1-\left(a_1+a_2\right)} d\left(x_n, x_{n-1}\right) \tag{3.8}$$

Let $\lambda=\frac{a_1+a_3}{1-\left(a_1+a_2\right)}$. Since $a_1+a_2+a_3<1$ implies that $\frac{a_1+a_3}{1-\left(a_1+a_2\right)}<1$. Hence,

$$d\left(x_{n+1}, x_n\right) \leq \lambda d\left(x_n, x_{n-1}\right) \text{ for all } n \in \mathbb{N} \tag{3.9}$$

For any $m>n$ where $m, n \in \mathbb{N}$, we have,

$$\begin{aligned} d\left(x_n, x_m\right) & \leq d\left(x_n, x_{n-1}\right)+d\left(x_{n-1}, x_{n-2}\right)+\cdots+d\left(x_{m+1}, x_m\right) \\ & \leq\left(\lambda^{n-1}+\lambda^{n-2}+\cdots+\lambda^m\right) d\left(x_1, x_0\right) \\ & \leq \frac{\lambda^m}{1-\lambda} d\left(x_1, x_0\right) \qquad (3.10) \end{aligned}$$

We get from (3.10) that $\|d(x_n, x_m)\| \leq \frac{\lambda^m}{1-\lambda} K\|d(x_1, x_0)\|$. Which implies $d(x_n, x_m) \rightarrow 0(n, m \rightarrow \infty)$. This proves that $\{x_n\}$ is Cauchy sequence in X. Since X is a complete cone metric space, there exists $x^* \in X$ such that $x_n \rightarrow x^*(n \rightarrow \infty)$. Then

$$\begin{gathered} d\left(T x^*, x^*\right) \leq d\left(T x_n, T x^*\right)+d\left(T x_n, x^*\right) \\ \leq a_1\left[d\left(T x^*, x_n\right)+d\left(T x_n, x^*\right)\right]+a_2 \frac{d\left(T x^*, x^*\right) d\left(T x_n, x_n\right)}{d\left(x^*, x_n\right)}+ \\ a_3 \frac{d\left(T x^*, x^*\right) d\left(T x_n, x_n\right)}{d\left(x^*, x_n\right)+d\left(T x^*, x_n\right)+d\left(T x_n, x^*\right)}+d\left(x_{n+1}, x^*\right) \\ \leq a_1\left[d\left(T x^*, x^*\right)+d\left(x_n, x^*\right)+d\left(x_{n+1}, x^*\right)\right]+d\left(x_{n+1}, x^*\right) \\ d\left(T x^*, x^*\right) \leq \frac{1}{1-a_1}\left(a_1\left[d\left(x_n, x^*\right)+d\left(x_{n+1}, x^*\right)\right]+d\left(x_{n+1}, x^*\right)\right) \\ \left\|d\left(T x^*, x^*\right)\right\| \leq K \frac{1}{1-a_1}\left(a_1\left[\left\|d\left(x_n, x^*\right)\right\|+\left\|d\left(x_{n+1}, x^*\right)\right\|\right]+\left\|d\left(x_{n+1}, x^*\right)\right\|\right) \rightarrow 0 \\ (3.11) \end{gathered}$$

Thus, from (3.11), we have $\|d(Tx^*, x^*)\|=0$, that is, $Tx^*=x^*$. Which implies x^* is a fixed point of T.

If y^* is another fixed point of T, then $Ty^*=y^*$. Since T is type II contraction, we obtain

$$\begin{gathered} d\left(x^*, y^*\right)=d\left(T x^*, T y^*\right) \leq a_1\left[d\left(T x^*, y^*\right)+d\left(T y^*, x^*\right)\right]+a_2 \frac{d\left(T x^*, x^*\right) d\left(T y^*, y^*\right)}{d\left(x^*, y^*\right)}+ \\ a_3 \frac{d\left(T x^*, x^*\right) d\left(T y^*, y^*\right)}{d\left(x^*, y^*\right)+d\left(T x^*, y^*\right)+d\left(T y^*, x^*\right)} \\ =2 a_1 d\left(x^*, y^*\right) \qquad (3.12) \end{gathered}$$

Hence, from (3.12), we have $d(x^*, y^*)=0$, that is, $x^*=y^*$. Therefore, the fixed point of T is unique.

Corollary 3.5 Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Then a mapping $T: X \rightarrow X$ is said to be type I contraction for some positive integer n, if for all $x, y \in X, x \neq y$ and $a_1, a_2, a_3 \geq 0$ with $a_1+a_2+a_3<1$ satisfying the following condition:

$$d\left(T^{n}x,T^{n}y\right)\leq a_{1}[d(Tx,x)+d(Ty,y)]+a_{2}\frac{d(Tx,x)d(Ty,y)}{d(x,y)}+ \\ a_{3}\frac{d(Tx,x)d(Ty,y)}{d(x,y)+d(Tx,y)+d(Ty,x)} \tag{3.13}$$

Then T has a unique fixed point in X.

Proof From Theorem 3.3, T^{n}has a unique fixed point x^{*}. But $T^{n}(Tx^{*})=T(T^{n}x^{*})=Tx^{*}$, so Tx^{*} is also a fixed point of T^{n}. Hence $Tx^{*}=x^{*}$, x^{*} is a fixed point of T. Since the fixed point of T is also fixed point of T^{n}, the fixed point of $T$is unique.

Corollary 3.6 Let (X,d) be a complete cone metric space, P be a normal cone with normal constant K. Then a mapping $T:X\to X$ is said to be type II contraction for some positive integer n, if for all $x,y\in X,x\neq y$ and $a_{1},a_{2},a_{3}\geq 0$ with $a_{1}+a_{2}+a_{3}<1$ satisfying the following condition:

$$d\left(T^{n}x,T^{n}y\right)\leq a_{1}[d(Tx,y)+d(Ty,x)]+a_{2}\frac{d(Tx,x)d(Ty,y)}{d(x,y)}+ \\ a_{3}\frac{d(Tx,x)d(Ty,y)}{d(x,y)+d(Tx,y)+d(Ty,x)} \tag{3.14}$$

Then T has a unique fixed point in X.

Proof From Theorem 3.4, T^{n}has a unique fixed point x^{*}. But $T^{n}(Tx^{*})=T(T^{n}x^{*})=Tx^{*}$, so Tx^{*} is also a fixed point of T^{n}. Hence $Tx^{*}=x^{*}$, x^{*} is a fixed point of T. Since the fixed point of $T$is also fixed point of T^{n}, the fixed point of $T$is unique.

Corollary 3.7[2] Let (X,d) be a complete cone metric space and P be a normal cone with normal constant K. Suppose the mapping $T:X\to X$ satisfies the contractive condition$d(Tx,Ty)\leq a_{1}d(x,y)$, for all $x,y\in X$, where $a_{1}\in[0,1)$ is a constant. Then T has a unique fixed point in X and for any $x\in X$, iterative sequence $\{T^{n}x\}$ converges to the fixed point.

Corollary 3.8[2] Let (X,d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose a mapping $T:X\to X$ satisfies for some positive integer n, $d(T^{n}x,T^{n}y)\leq a_{1}d(x,y)$, for all $x,y\in X$, where $a_{1}\in[0,1)$ is a constant. Then T has a unique fixed point in X.

Example 3.9 Let $E=\mathbb{R}^{2}$, the Euclidean plane, and $P=\{(x,y)\in R^{2}|x,y\geq 0\}$ a normal cone in P. Let $X=\{(x,0)\in R^{2}|0\leq x\leq 1\}\cup\{(0,x)\in R^{2}|0\leq x\leq 1\}$. The mapping $d:X\times X\to E$ is defined by

$$d((x,0),(y,0))=\left(\frac{4}{3}|x-y|,|x-y|\right)$$
$$d((0,x),(0,y))=\left(|x-y|,\frac{2}{3}|x-y|\right)$$
$$d((x,0),(0,y))=d((0,y),(x,0))=\left(\frac{4}{3}x+y,x+\frac{2}{3}y\right)$$

Then (X,d) is a complete cone metric space.Let mapping $T:X\to X$ with $T(x,0)=(0,x)$ and $T(0,x)=(1/2x,0)$

Then T satisfies the type I contractive condition

$$\begin{gathered}d\left(T\left(x_1,x_2\right),T\left(y_1,y_2\right)\right)\leq a_1\left[d\left(T\left(x_1,x_2\right),\left(x_1,x_2\right)\right)+d\left(T\left(y_1,y_2\right),\left(y_1,y_2\right)\right)\right]\\ +a_2\frac{d\left(T\left(x_1,x_2\right),\left(x_1,x_2\right)\right)d\left(T\left(y_1,y_2\right),\left(y_1,y_2\right)\right)}{d\left(\left(x_1,x_2\right),\left(y_1,y_2\right)\right)}+\\ a_3\frac{d\left(T\left(x_1,x_2\right),\left(x_1,x_2\right)\right)d\left(T\left(y_1,y_2\right),\left(y_1,y_2\right)\right)}{d\left(\left(x_1,x_2\right),\left(y_1,y_2\right)\right)+d\left(T\left(x_1,x_2\right),\left(y_1,y_2\right)\right)+d\left(T\left(y_1,y_2\right),\left(x_1,x_2\right)\right)}\end{gathered}$$

for all $(x_1,x_2),(y_1,y_2)\in X$, with constant $a_1=2/30,a_2=3/40,a_3=1/30$. It is obvious that T has a unique fixed point $(0,0)\in X$. On the other hand, we see that T is not a contractive mapping in the Euclidean metric on X.

4. Application

In this section, as an application of our results, we establish that Theorem 3.3 and Theorem 3.4 can be utilized to derive the existence of fixed point results for some mappings in a cone metric space with different conditions. In the sequel, we begin with the following definitions.

Definition 4.1. Let (X,d) be a complete cone metric space and P be a normal cone with normal constant K. For $c\in E$ with $0\ll c,x_0\in X$, set $B(x_0,c)=\{x\in X|d(x_0,c)\leq c\}$. Then a mapping $T:X\to X$ is said to be type I contraction if for all $x,y\in X,x\neq y$ and $a_1,a_2,a_3\geq 0$ with $a_1+a_2+a_3<1$ satisfying the following condition:

$$\begin{gathered}d(Tx,Ty)\leq a_1[d(Tx,x)+d(Ty,y)]+a_2\frac{d(Tx,x)d(Ty,y)}{d(x,y)}+\\ a_3\frac{d(Tx,x)d(Ty,y)}{d(x,y)+d(Tx,y)+d(Ty,x)}\end{gathered}\tag{4.1}$$

Definition 4.2. Let (X,d) be a complete cone metric space and P be a normal cone with normal constant K. For $c\in E$ with $0\ll c,x_0\in X$, set $B(x_0,c)=\{x\in X|d(x_0,c)\leq c\}$. Then a mapping $T:X\to X$ is said to be type II contraction if for all $x,y\in X,x\neq y$ and $a_1,a_2,a_3\geq 0$ with $a_1+a_2+a_3<1$ satisfying the following condition:

$$\begin{gathered}d(Tx,Ty)\leq a_1[d(Tx,y)+d(Ty,x)]+a_2\frac{d(Tx,x)d(Ty,y)}{d(x,y)}+\\ a_3\frac{d(Tx,x)d(Ty,y)}{d(x,y)+d(Tx,y)+d(Ty,x)}\end{gathered}\tag{4.2}$$

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Theorem 4.3 Let (X,d) be a complete cone metric space and P be a normal cone with normal constant K. For $c \in E$ with $0 \ll c, x_0 \in X$, set $B(x_0,c) = \{x \in X | d(x_0,c) \leq c\}$. Then a mapping $T : X \to X$ is said to be type I contraction and $d(Tx_0,x_0) \leq (1-(a_1+a_2+a_3))c$. Then T has a unique fixed point in $B(x_0,c)$.

Proof We first prove that $B(x_0,c)$ is complete and then show that $Tx \in B(x_0,c)$ for all $x \in B(x_0,c)$. Suppose x_n is a Cauchy sequence in $B(x_0,c)$. Then $\{x_n\}$ is also a Cauchy sequence in X. By the completeness of X, there is $x \in X$such that $x_n \to x (n \to \infty)$. We have

$$d(x_0,x) \leq d(x_n,x_0) + d(x_n,x) \leq d(x_n,x) + c.$$

Since $x_n \to x, d(x_n,x) \to 0$. Hence $d(x_0,x) \leq c$, and $x \in B(x_0,c)$. Therefore $B(x_0,c)$is complete.

For every $x \in B(x_0,c)$,

$$\begin{gathered} d\left(x_0, Tx\right) \leq d\left(Tx_0, x_0\right) + d\left(Tx_0, Tx\right) \\ \leq \left(1-\left(a_1+a_2+a_3\right)\right) c + a_1\left[d\left(Tx_0, x_0\right) + d(Tx, x)\right] + a_2 \frac{d\left(Tx_0, x_0\right) d(Tx, x)}{d\left(x_0, x\right)} + \\ a_3 \frac{d\left(Tx_0, x_0\right) d(Tx, x)}{d\left(x_0, x\right) + d\left(Tx_0, x\right) + d\left(Tx, x_0\right)} \\ \leq \left(1-\left(a_1+a_2+a_3\right)\right) c \end{gathered}$$

Theorem 4.4 Let (X,d) be a complete cone metric space and P be a normal cone with normal constant K. For $c \in E$ with $0 \ll c, x_0 \in X$, set $B(x_0,c) = x \in X | d(x_0,c) \leq c$. Then a mapping $T : X \to X$ is said to be type II contraction and $d(Tx_0,x_0) \leq (1-(a_2+a_3-a_1))c$. Then T has a unique fixed point in $B(x_0,c)$.

Proof We prove that $B(x_0,c)$ is complete and $Tx \in B(x_0,c)$ for all $x \in B(x_0,c)$. Suppose x_n is a Cauchy sequence in $B(x_0,c)$. Then $\{x_n\}$ is also a Cauchy sequence in X. By the completeness of X, there is $x \in X$such that $x_n \to x (n \to \infty)$. We have

$$d(x_0,x) \leq d(x_n,x_0) + d(x_n,x) \leq d(x_n,x) + c.$$

Since $x_n \to x, d(x_n,x) \to 0$. Hence $d(x_0,x) \leq c$, and $x \in B(x_0,c)$. Therefore $B(x_0,c)$is complete.

For every $x \in B(x_0,c)$,

$$\begin{gathered} d\left(x_0, Tx\right) \leq d\left(Tx_0, x_0\right) + d\left(Tx_0, Tx\right) \\ \leq \left(1-\left(a_1+a_2+a_3\right)\right) c + a_1\left[d\left(Tx_0, x\right) + d\left(Tx, x_0\right)\right] + a_2 \frac{d\left(Tx_0, x_0\right) d(Tx, x)}{d\left(x_0, x\right)} + \\ a_3 \frac{d\left(Tx_0, x_0\right) d(Tx, x)}{d\left(x_0, x\right) + d\left(Tx_0, x\right) + d\left(Tx, x_0\right)} \\ \leq \left(1-\left(a_1+a_2+a_3\right)\right) c + a_1 2 d\left(x_0, x\right) \leq \left(1-\left(a_1+a_2+a_3\right)\right) c + 2 a_1 c \\ = \left(1-\left(a_2+a_3-a_1\right)\right) c. \end{gathered}$$

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Hence, $Tx \in B(x_0, c)$.

Corollary 4.5[2] Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. For $c \in E$ with $0 \ll c, x_0 \in X$, set $B(x_0, c) = x \in X | d(x_0, c) \leq c$. Suppose the mapping $T : X \to X$ satisfies the contractive condition $d(Tx, Ty) \leq a_1 d(x, y)$, for all $x, y \in B(x_0, c)$, where $a_1 \in [0, 1)$ is a constant and $d(Tx_0, x_0) \leq (1 - a_1)c$. Then T has a unique fixed point in $B(x_0, c)$.

5. Conclusion

In this paper, we establish some fixed point results for contractive type mappings in cone metric spaces. Examples are provided to support results and concepts presented herein. As an application of our results, cone metric spaces with different conditions are studied.

Statements and Declarations

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Data availability

No data availability in this paper.

Authors contributions

All authors contributed equally in the writing of this paper.

Conflicting interests

The authors declare that there are no conflicting interests

Compliance with ethical standards

Research involving human participants and/or animals: This article does not contain any studies with human participants or animals performed by the authors.

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Declarations

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