# Classical and Quantum Boltzmann Equations in the presence of bath noise with quantum filtering applied to scattered field analysis

Harish Parthasarathy
ECE division, NSUT

October 16, 2024

https://doi.org/10.32388/4QMM4S

## Abstract

We formulate the classical and quantum Boltzmann equations for the one particle distribution function and density operator corresponding to the motion of several indistinguisable point charges in motion with mutual electromagnetic interaction attached to central potentials and also interacting with an external electromagnetic field. These equations are used to derive formulas for the scattered electromagnetic field and analyze their properties like frequency content. Using the Wigner distribution to draw an analogy between classical probability distribution functions in phase space and corresponding quantum analogues, we compare the classical and quantum Boltzmann equations showing explicitly how the quantum Boltzmann equation for the Wigner distribution is the same as the classical Boltzmann equation but with extra quantum corrections expressible as a power series in Planck'c constant. We then proceed to formulate the quantum Boltzmann equation for open quantum systems, ie, systems connected to a noisy bath and explain how to estimate the Lindblad noisy coupling parameters from the nonlinear quantum Boltzmann state evolution between successive POV measurements. We then proceed to formulate the quantum Boltzmann equation for the one particle density operator derived from the quantum Belavkin filter for N-particles taking into account non-demolition measurement noise. Partial tracing of the Belavkin filter with the molecular chaos hypothesis for the N-particle state yields this nonlinear one particle filtered quantum Boltzmann equation. We then apply Lec-Bouten's method of control unitaries based on the measurement process to cancel out part of Lindblad noise from the filtered quantum Boltzmann equation. Finally, we explain how to perform a second order perturbative analysis of the classical Boltzmann equation in order to arrive at the frequency spectrum of the scattered field from the frequency spectrum of the incident field. The last section is devoted to the derivation of the two particle quantum Boltzmann equation for the Belavkin quantum filter. During the process of derivation, we explain how to derive the general $r^{th}$ order quantum Boltzmann equation for the Belavkin quantum filter on $N$ particles when the bath acts symmetrically, ie, in a permutation invariant way on all the particles.

## 0.1 Calculating the scattered electromagnetic field produced by a slowly spatially varying incident electromagnetic field on a set large charged particles with several identical smaller charged particles moving in the vicinity of the former

Consider a lattice of $M$ (not very large) charged particles with each particle having a charge $Ze$ and each such particle having $N$ (very large) charge particles, each of charge -$e$. When an electromagnetic field $E_c(t,r), B_c(t,r)$ is incident upon this system, it causes the charged particles to execute motion and thus generated a scattered electromagnetic field. Our aim is to calculate this scattered field approximately assuming that the incident field varies very slowly over the region defined by the charged particles of charge $-e$ surrounding each centre. The potential generated by each centre is

$$Ze/|r-Q_k| - - - (1)$$

where $Q_k$ is the position of the centre. Thus, the potential experience by the charge $-e$ located at $Q_k+\xi$ is

$$V_0(\xi) = Ze/|\xi| - - - (2)$$

Apart from this, the interaction between two charges $-e$ located at $\xi_1, \xi_2$ relative to the centre $Q_k$ is given by

$$V_{12} = e^2/|\xi_1-\xi_2| - - - (3)$$

Taking this into account, the classical Boltzmann equation in the molecular chaos approximation for a single charge attached to the centre $Q=Q_k$ is given by

$$\partial_t f(t,\xi,v)+(v,\nabla_\xi)f(t,\xi,v)+(e/m)(\nabla V_0(\xi),\nabla_v)f(t,\xi,v)-(e/m)(E_c(t,Q)+v\times B_c(t,Q))f(t\xi,v)+$$

$$((N-1)e^2\int f(t,\xi',v')(\xi-\xi')d^3vd^3\xi')/|\xi-\xi'|^3,\nabla_v)f(t,\xi,v)+(f_0(\xi,v)-f(t,\xi,v))/\tau(v)=0---(4)$$

Herein, we are neglecting the effects of the force of the magnetic fields produced by the $N-1$ remaining charges in each central region on a given charge within the same region. The last term involving the relaxation time constant is a replacement of the collision term. If we wish to take into account the effects of the magnetic fields generated by the motion of the charges in the non-relativistic approximation, then we would have to add the term

$$e^2(N-1)(\int d^3\xi' d^3v' v\times(v'\times(\xi-\xi'))f(t,\xi',v')d^3\xi' d^3v'/|\xi-\xi'|^3,\nabla_v)f(t,\xi,v)---(5)$$

It should also be noted that each centre of charge $Ze$ will also generally carry a spin described by an operator in a $2J+1$ Hilbert space where $J$ is an integer or a half integer. The effect of the magnetic field produced by this spin on the charges $-e$ is a purely quantum mechanical effect and can only be accounted in a quantum mechanical formulation of the Boltzmann equation.

In summary, the chaos hypothesis leads to approximating the pairwise interaction terms between the charges by bilinear forms in the Boltzmann distribution function. For the present, we neglect these bilinear terms assuming that all of their effects can be absorbed within the relaxation time term. We note that the equilibrium Boltzmann density is

$$f_0(\xi,v) = NZ(\beta)^{-1}exp(-\beta(mv^2/2 - eV_0(\xi))) - - - (6)$$

For each charged particle $-e$ Writing the perturbation as

$$\delta f(t,\xi,v) = f(t,\xi,v - f_0(\xi,v) - - - (7)$$

we obtain the following first order approximation to the Boltzmann equation assuming that the external em field is of the first order of smallness as compared to the electrostatic field generated by the centres:

$$\partial_t \delta f(t,\xi,v) + (v,\nabla_\xi)\delta f(t,\xi,v) - (e/m)(E_0(\xi),\nabla_v)\delta f(t,\xi,v)$$

$$-(e/m)(E_c(t,Q) + v \times B_c(t,Q)),\nabla_v)f_0(\xi,v) + \delta f(t,\xi,v)/\tau(v) = 0 - - - (8)$$

where

$$E_0(\xi) = -\nabla V_0(\xi) - - - (9)$$

To be precise, we should write $f(t,\xi,v|Q)$ and $\delta f(t,\xi,v|Q)$, the dependence upon $Q$ arising from the external field This equation simplifies to

$$\partial_t \delta f(t,\xi,v) + (v,\nabla_\xi)\delta f(t,\xi,v) - (e/m)(E_0(\xi),\nabla_v)\delta f(t,\xi,v)$$

$$+\beta e(E_c(t,Q),v)f_0(\xi,v) + \delta f(t,\xi,v)/\tau(v) = 0 - - - (10)$$

It should be noted that the total charge and current density generated by the motion of the charges caused by the external field are respectively given by

$$\rho(t,Q+\xi) = -e\int \delta f(t,\xi,v|Q)d^3v - - - (11)$$

$$J(t,r) = -e\int v\delta f(t,\xi,v|Q)d^3v - - - (12)$$

and the total scattered electromagnetic four potential produced by lattice is given to first order by the formulas

$$A_s(t,r) = -e\sum_{k=1}^{M}\int v\delta f(t,\xi,v|Q_k).d^3\xi.d^3v/|r - Q_k - \xi| - - - (13)$$

and

$$V_s(t,r) = -e\sum_{k=1}^{M}\int \delta f(t,\xi,v|Q_k).d^3\xi.d^3v/|r-Q_k-\xi| --- (14)$$

These potentials admit a multipole expansion, with the zeroth order term in this expansion being the most significant given by

$$A_s(t,r) \approx -e\sum_{k=1}^{M}(v\delta f(t,\xi,v|Q_k).d^3\xi.d^3v)/|r-Q_k| --- (15)$$

and

$$V_s(t,r) = -e(\sum_{k=1}^{M}\int \delta f(t,\xi,v|Q_k).d^3\xi.d^3v)/|r-Q_k| --- (16)$$

Defining space-time Fourier transforms by

$$X(\omega,K) = \int X(t,r)exp(-i\omega t + K.r)dtd^3K --- (17)$$

we obtain from the above linearized Boltzmann equation in the four wave vector domain:

$$[i(\omega-(K,v)) + 1/\tau(v)]\delta f(\omega,K,v) + \beta e(E_c(t,Q),v)f_0(K,v)$$

$$-(2\pi)(e/m)f\int(E_0(K-K').\nabla_v)\delta f(\omega,K',v)d^3K' = 0 --- (18)$$

Note that we have the normalizations

$$\int f_0(r,v)d^3rd^3v = \int f(t,r,v)d^3rd^3v = N --- (19)$$

so that

$$\int \delta f(t,r,v)d^3rd^3v = 0 --- (20)$$

## 0.2 A generalized derivation of the classical Boltzmann equation

Consider a system of $N$ indistinguishable particles described by position-velocity pairs $(\xi_k, v_k), k = 1, 2, ..., N$ where $\xi_k, v_k \in \mathbb{R}^3$. Assume that the external fields cause a force $F_{ext}(t, \xi_k, v_k)$ to be exerted on the $k^{th}$ particle while the internal particle interactions cause a force $F_{int}(t, \xi_k, v_k|\xi_j, v_j)$ to be exerted by the $j^{th}$ particle on the $k^{th}$ particle. Boltzmann's equation for the $N^{th}$ order particle distribution function $f_N(t,\xi_k,v_k,k=1,2,...,N)$ is the same as Liouville's equation in mechanics assuming that the dynamics of the particles described by the above forces keeps the total phase volume $\Pi_{k=1}^N d^3\xi_k d^3 v_k$ invariant. This means that the phase space divergence of the force field vanishes, ie,

$$\sum_k (div_{\xi_k} F_{ext}(t,\xi_k,v_k)+div_{v_k} F_{ext}(t,\xi_k,v_k))+\sum_{k\neq j}(div_{\xi_k}(F_{int}(t,\xi_k,v_k|\xi_j,v_j))+div_{v_k}(F_{int}(t,\xi_k,v_k|\xi_j,v_j)) = 0--$$

This amounts to requiring separately that

$$div_{\xi_1} F_{ext}(t,\xi_1,v_1)+div_{v_1} F_{ext}(t,\xi_1,v_1) = 0 - - - (22)$$

$$div_{\xi_1} F_{int}(t,\xi_1,v_1|\xi_2,v_2)+div_{v_1} F_{int}(t,\xi_1,v_1|\xi_2,v_2)$$
$$+div_{\xi_2} F_{int}(t,\xi_2,v_2|\xi_1,v_1)+div_{v_2} F_{int}(t,\xi_2,v_2|\xi_1,v_1) = 0 - - - (23)$$

The N-particle Boltzmann equation is merely a statement of the the conservation of the total number of particles in 6N-dimensional phase volume:

$$\partial_t f_N + \sum_{k=1}^N (v_k, \nabla_{\xi_k}) f_N +$$

$$\sum_{k=1}^N (F_{ext}(t,\xi_k,v_k),\nabla_{v_k}) f_N + \sum_{k\neq j}(F_{int}(t,\xi_k,v_k|\xi_j,v_j),\nabla_{v_k}) f_N = 0 - - - (24)$$

Integrating over $\xi_k, v_k, k=2,3,...,N$ and using the indistinguishability of the particles gives us

$$\partial_t f_1(t,\xi_1,v_1)+(v_1,\nabla_{\xi_1})f_1(t,\xi_1,v_1)+(F_{ext}(t,\xi_1,v_1),\nabla_{v_1})f_1(t,\xi_1,v_1)$$

$$+(N-1)\int (F_{int}(t,\xi_1,v_1|\xi_2,v_2),\nabla_{v_1})f_{12}(t,\xi_1,v_1,\xi_2,v_2)d^3\xi_2 d^3 v_2 = 0 - - - (25)$$

provided that we assume the additional conditions

$$div_{v_1} F_{ext}(t,\xi_1,v_1) = 0 - - - (26)$$

$$div_{v_1} F_{int}(t,\xi_1,v_1|\xi_2,v_2)+div_{v_2} F_{ext}(t,xi_2,v_2|\xi_1,v_1) = 0 - - - (27)$$

For example, if the forces are produced by the electromagnetic fields generated by the external sources and the $N$ moving charges in the non-relativistic approximation, then

$$F_{ext}(t,\xi_1,v_1) = (-e/m)(E_{ext}(t,\xi_1)+v_1\times B_{ext}(t,\xi_1)) - - - (28)$$

and clearly, we have

$$div_{v_1} F_{ext}(t,\xi_1,v_1) = div_{v_1}(v_1\times B_{ext}(t,\xi_1)) = 0 - - - (29)$$

and further,

$$F_{int}(t,\xi_1,v_1|\xi_2,v_2) = (e^2/m)(\xi_1-xi_2)/|\xi_1-\xi_2|^3)+(e^2/m)(v_1\times(v_2\times(\xi_1-\xi_2))/|\xi_1-\xi_2|^3))---(30)$$

Now, if $w$ is a vector not involving $v_1, v_2$, then

$$v_1 \times (v_2 \times w) = (v_1, w)v_2 - (v_1, v_2)w - - - (31)$$

gives

$$div_{v_1}(v_1 \times (v_2 \times w)) = (w, v_2) - (w, v_2) = 0 - - - (32)$$

so the second condition is also trivially satisfied. In the molecular chaos approximation, we approximate $f_{12}(t, \xi_1, v_1, \xi_2, v_2)$ by the product $f_1(t, \xi_1, v_1).f_1(t, \xi_2, v_2)$ and thus obtain a simple version of the Boltzmann equation for the one particle distribution function:

$$\partial_t f_1(t, \xi_1, v_1) + (v_1, \nabla_{\xi_1})f_1(t, \xi_1, v_1) + (F_{ext}(t, \xi_1, v_1), \nabla_{v_1})f_1(t, \xi_1, v_1)$$

$$+(N-1)(\int f_1(t, \xi_2, v_2)F_{int}(t, \xi_1, v_1|\xi_2, v_2)d^3\xi_2 d^3 v_2), \nabla_{v_1})f_1(t, \xi_1, v_1) = 0 - - - (33)$$

In the general case, without making any approximations, we can write

$$f_{12}(t, \xi_1, v_1, \xi_2, v_2) = f_1(t, \xi_1, v_1)f(t, \xi_2, v_2) + g_{12}(t, \xi_1, v_1, \xi_2, v_2) - - - (34)$$

abbreviated in the obvious way as

$$f_{12} = f_1 f_2 + g_{12} - - - (35)$$

where since

$$\int f_{12} d(2) = f_1 - - - (36)$$

we must have

$$\int g_{12} d(2) = 0 - - - (37)$$

and of course, by indistinguishability,

$$\int g_{12} d(1) = 0 - - - (38)$$

also. Here, $d(k)$ is an abbreviation for $d^3\xi_k.d^3v_k$. Again, integrating the $N$-particle Boltzmann equation over $3, 4, ..., N$ gives us the two particle Boltzmann equation:

$$\partial_t f_{12} + ((v_1, \nabla_{\xi_1}) + (v_2, \nabla_{\xi_2}))f_{12} +$$

$$(F_{ext}(t, \xi_1, v_1|\xi_2, v_2), \nabla_{v_1}) + (F_{ext}(t, \xi_2, v_2|\xi_1, v_1), \nabla_{v_2}))f_{12}$$

$$+(N-2)\int (F_{ext}(t, \xi_1, v_1|\xi_3, v_3), \nabla_{v_1}) + (F_{ext}(t, \xi_2, v_2|\xi_3, v_3), \nabla_{v_2}))f_{123} d(3) = 0 - - - (39)$$

Likewise, we write

$$f_{123} = f_1 f_2 f_3 + f_1 g_{23} + f_2 g_{13} + f_3 g_{12} + g_{123} - - - (40)$$

and the integrating over 3 gives us

$$f_{12} = f_1 f_2 + g_{12} - - - (41)$$

as required, provided that

$$\int g_{123} d(3) = 0 - - - (42)$$

## 0.3 The quantum version of calculating the field scattered by the charged particles in the phonon lattice

The approximate quantum Boltzmann equation in the molecular chaos approximation for the one particle density operator reads

$$ih\partial_t \rho_1(t) = [H_0 + \delta H(t), \rho_1(t)] + (N-1)Tr_2[V_{12}, \rho_1(t) \otimes \rho_1(t)] - - - (43)$$

where

$$\rho_1(t) = \rho_1(t|Q), Q \in \{Q_1, ..., Q_M\} - - - (44)$$

Note that in above example, in the position representation,

$$V_{12}(\xi, \xi') = e^2/|\xi - \xi'| - - - (45)$$

Here,

$$H_0 + \delta H(t) = (-h^2/2m)(\nabla_\xi + ieA_c(t, Q+\xi)/h)^2 - eV_0(\xi) - eV_c(t, Q+\xi) - - - (46)$$

$$H_0 = (-h^2/2m)\nabla_\xi^2 - eV_0(\xi) - - - (47)$$

and then to $O(e)$, we get the first order perturbation to the one particle Hamiltonian (assuming $divA_c = 0$, ie, we are operating in the Coulomb gauge for the incident em wave so that $V_c = 0$ too)

$$\delta H(t) = (-ieh/m)(A_c(t, Q+\xi), \nabla_\xi) - - - (48)$$

Let $|n>, n = 1, 2, ...$ denote the stationary states of the centres, ie,

$$H_0|n> = E(n)|n>, n = 1, 2, ... - - - (49)$$

Then, the scattered electromagnetic four potential is calculated as follows.

$$A_s(t, r) = -Nem^{-1} \sum_{k=1}^{M} Tr(\rho_1(t|Q_k)(-ih\nabla_\xi + eA_c(Q_k+\xi))/|r - Q_k - \xi|) - - - (50)$$

$$V_s(t, r) = -Nem^{-1} \sum_{k=1}^{M} Tr(\rho_1(t|Q_k)/|r - Q_k - \xi|) - - - (51)$$

where in the first expression, symmetrization of the operator involved is assumed. It should be noted that in a non-relativistic approximation, the scattered electromagnetic field can also be directly written down without having to go through the calculation of the four potential:

$$E_s(t,r) = -Ne\sum_{k=1}^{M} Tr(\rho_1(t|Q_k)(r-Q_k-\xi)/|r-Q_k-\xi|^3) - - - (52)$$

$$B_s(t,r) = -Ne\sum_{k=1}^{M} Tr(\rho_1(t|Q_k)v_k \times (r-Q_k-\xi)/|r-Q_k-\xi|^3) - - - (53)$$

where $v_k$ is velocity operator

$$v_k = (-ih\nabla_\xi + eA_c(Q_k+\xi))/m - - - (54)$$

Note that for an observable $X$, we calculate in the position representation

$$Tr(\rho_1(t|Q)X) = \int \rho_1(t,\xi',\xi''|Q)X(\xi'',\xi')d^3\xi'.d^3\xi'' - - - (55)$$

where

$$\rho_1(t,\xi',\xi''|Q) =< \xi'|\rho_1(t|Q)|\xi'' > - - -(56)$$
$$X(\xi'',\xi') =< \xi''|X|\xi' > - - -(57)$$

## 0.4 Approximate first order solution to the quantum Boltzmann equation

We write

$$\rho_1(t) = \rho_1^{(0)} + \delta\rho_1(t) - - - (58)$$

Then, taking

$$\rho_1^{(0)} = Z(\beta)^{-1}exp(-\beta H_0) = \sum_n |n > p(n) < n| - - - (59)$$

where

$$p(n) = exp(-\beta E(n))/Z(\beta), Z(\beta) = \sum_m exp(-\beta E(m)) - - - (60)$$

we see that $\rho_1^{(0)}$ trivially satisfies the zeroth order Boltzmann equation:

$$0 = ih\partial_t\rho_1^{(0)} = [H_0, \rho_1^{(0)}] - - - (61)$$

The first order Boltzmann equation is then given by

$$\partial_t\delta\rho_1(t) = [H_0, \delta\rho_1(t)] + [\delta H(t), \rho_1^{(0)}] + (N-1)Tr_2[V_{12}, \rho_1^{(0)} \otimes \rho_1^{(0)}] - - - (62)$$

We are herein assuming that the nonlinear term in the quantum Boltzmann equation is of the first order of smallness, ie, of the same order of smallness as the external em field or equivalently, of the same order of smallness as the perturbing Hamiltonian $\delta H(t)$. Note that writing

$$u_n(\xi') =< \xi'|n >?, n = 1, 2, ... - - - (63)$$

we have

$$V_{12} = e^2 \sum_{n,m,r,s} |n > |m >< r| < s| \int \bar{u}_n(\xi')\bar{u}_m(\xi'')u_r(\xi')u_s(\xi'')d^3\xi' d^3\xi''/|\xi'-\xi''|$$

$$= \sum_{n,m,r,s} |n > |m >< r| < s|a(nm|rs) = \sum_{nmrs} a(nm|rs)(|n >< r|\otimes|m >< s|) - - -(64)$$

where

$$a(nm|rs) = \int \bar{u}_n(\xi')\bar{u}_m(\xi'')u_r(\xi')u_s(\xi'')d^3\xi' d^3\xi''/|\xi' - \xi''| - - - (65)$$

is symmetric and real w.r.t interchange of the ordered pairs $(nm)$ and $(rs)$. It is also symmetric w.r.t interchange of the ordered pairs $(nr)$ and $(ms)$. Therefore, by the spectral theorem, it can be diagonalized as

$$a(nm|rs) = \sum_p \lambda(p)w_p(nr)w_p(ms) - - - (66)$$

Writing therefore

$$W_p = \sum_{nr} w_p(nr)|n >< r| - - - (67)$$

we get

$$V_{12} = \sum_p \lambda(p)W_p \otimes W_p - - - (68)$$

where $W_p's$ are Hermitian operators in the Hilbert space $L^2(\mathbb{R}^3)$ and $\lambda(p)$ are real numbers. Note that the symmetry of $a(nm|rs)$ under intrerchange of $(nm)$ and $(rs)$ implies that we can choose $w_p(nr)$ so that it is real and symmetric under interchange of $(nr)$. This in turn implies that $W_p$ is real symmetric and in particular Hermitian. The first order perturbed qbe given by

$$\partial_t \delta\rho_1(t) = -i[H_0, \delta\rho_1(t)] - i[\delta H(t), \rho_1^{(0)}] - i(N-1)\sum_p \lambda(p)Tr(W_p\rho_1^{(0)})[W_p, \rho_1^{(0)}] - - -(69)$$

can now be expressed in terms of matrix elements w.r.t the unperturbed eigenstates of the one particle Hamiltonian s

$$\partial_t < n|\delta\rho_1(t)|m >=$$

$$-iE(nm) < n|\delta\rho_1(t)|m > -i(p(m)-p(n)) < n|\delta H(t)|m > -i(N-1)\sum \lambda(p)Tr(W_p\rho_1^{(0)})(p(m)-p(n)) < n|W_p|n$$

where $E(nm) = E(n) - E(m)$. Note that

$$Tr(W_p\rho_1^{(0)}) = \sum_n p(n) < n|E_p|n > ---(71)$$

This linear differential equation is easily solved to give

$$< n|\delta\rho_1(t)|m >=$$

$$\int_0^t exp(-iE(nm)(t-s))[-i(p(m)-p(n)) < n|\delta H(s)|m > -i(N-1)\sum_p \lambda(p)Tr(W_p\rho_1^{(0)})(p(m)-p(n)) < n|W_p|m$$

$$= -ip(mn)\int_0^t exp(-iE(nm)(t-s)) < n|\delta H(s)|m > ds - i(N-1)p(mn)((1-exp(-iE(nm)t))/iE(nm))\sum_p \lambda(p)$$

where

$$p(mn) = p(m) - p(n) --- (73)$$

## 0.5 Comparison of the quantum Boltzmann equation with the classical Boltzmann equation using the Wigner distribution

For a given density operator $\rho(t)$ in one particle Hilbert space, ie, in $L^2(\mathbb{R}^3)$, we define its position space continuous matrix kernel:

$$\rho(t,\xi',\xi'') =< \xi'|\rho(t)|\xi'' > ---(74)$$

Its Wigner transform is the complex valued function of $(\xi, P) \in \mathbb{R}^6$ defined by

$$\hat{\rho}(t,\xi,P) = C.\int \rho(t,\xi+q/2,\xi-q/2)exp(iP.q/h)d^3q --- (75)$$

where $h$ equals Planck's constant divided by $2\pi$. Clearly, for appropriate choice of the normalization constant $C$, we have

$$\int \hat{\rho}(t,\xi,P)d^3P = \rho(t,\xi,\xi) --- (76)$$

namely, the probability density of the position of the particle in the state $\rho$ and moreover,

$$\int \hat{\rho}(t,\xi,P)d^3\xi = \int \rho(t,\xi',\xi'')exp(iP.(\xi'-\xi'')/h)d^3\xi'.d^3\xi'' --- (77)$$

namely, the probability density of the momentum of the particle in the state $\rho$. Further, we have the inversion formula:

$$\rho(t,\xi+q/2,\xi-q/2)=C'.\int\hat{\rho}(t,\xi,P).exp(-iP.q/h)d^3P---(78)$$

or equivalently,

$$\rho(t,\xi',\xi'')=C'\int\hat{\rho}(t,(\xi'+\xi'')/2,P)exp(-iP.(\xi'-\xi'')/h)d^3P---(79)$$

In order to compare the quantum Boltzmann equation with the classical Boltzmann equation, we transform the former to give an equation for the Wigner distribution function $\hat{\rho}(t,\xi,P)$ and then expand this equation in powers of Planck's constant to show that the zeroth order term is simply the classical Boltzmann equation for the one particle distribution in phase space and that higher order terms in powers of Planck's constant give quantum corrections to the classical Boltzmann equation.

We write down the non-relativistic quantum Boltzmann equation for charged particles interacting mutually on a pairwise basis and also with an external electromagnetic field as

$$ih\partial_t\rho(t)=[H_0+\delta H(t),\rho(t)]+(N-1)Tr_2[V_{12},\rho(t)\otimes\rho(t)]---(80)$$

where

$$H_0=(-h^2/2m)\nabla_\xi^2-eV_0(\xi),\delta H(t)=-(ieh/m)(A(t,\xi),\nabla_\xi)-eV(t,\xi)---(81)$$

In the position representation, the term $[\delta H(t),\rho(t)]$ is represented by the kernel

$$(-ieh/m)[(A(t,\xi'),\nabla_{\xi'})\rho(t,\xi',\xi'')+div_{\xi''}(A(t,\xi'')\rho(t,\xi',\xi''))]$$

$$-e[(V(t,\xi')-V(t,\xi''))\rho(t,\xi',\xi'')]---(82)$$

because for any function $f$ of position, we have using integration by parts,

$$\int\rho(t,\xi',\xi'')(A(t,\xi''),\nabla_{\xi''})f(\xi'')d\xi''$$

$$=-\int div_{\xi''}(\rho(t,\xi',\xi'')A(t,\xi''))f(\xi'')d\xi''---(83)$$

Now making the Coulomb gauge choice $divA=0$, we get the result that the position space representation of $[\delta H(t),\rho(t)]$ is given by

$$[\delta H(t),\rho(t)](\xi',\xi'')=$$

$$(-ieh/m)[(A(t,\xi'),\nabla_{\xi'})\rho(t,\xi',\xi'')+(A(t,\xi''),\nabla_{\xi''})\rho(t,\xi',\xi''))]$$

$$-e[(V(t,\xi')-V(t,\xi''))\rho(t,\xi',\xi'')]---(84)$$

Now we compute the Wigner transform of this quantity. First observe that the Wigner transform of the second term, namely of the kernel $F(t,\xi',\xi'') = (V(t,\xi') - V(t,\xi''))\rho(t,\xi',\xi'')$ is given upto $O(h)$ terms by

$$\hat{F}(t,\xi,Q) = \int F(t,\xi+hq/2,\xi-hq/2)exp(-iP.q)dq$$

$$= \int (V(t,\xi+hq/2) - V(t,\xi-hq/2))\rho(t,\xi+hq/2,\xi-hq/2)exp(-iP.q)dq$$

$$= h\int (q,\nabla_\xi V(t,\xi))\rho(t,\xi+hq/2,\xi-hq/2)exp(-iP.q)dq$$

$$= (-\nabla_\xi V(t,\xi)), ih\nabla_P)\hat{\rho}(t,\xi,P) - - - (85)$$

since

$$\hat{\rho}(t,\xi,P) = \int \rho(t,\xi+hq/2,\xi-hq/2)exp(-iP.q)dq - - - (86)$$

Thus, the Wigner transform of $F_1(t,\xi',\xi'') = (-i/h)F(t,\xi',\xi'') = (-i/h)(V(t,\xi') - V(t,\xi''))\rho(t,\xi',\xi'')$ is given by

$$\hat{F}_1(t,\xi,P) = (-\nabla_\xi V(t,\xi), \nabla_P)\hat{\rho}(t,\xi,P) - - - (87)$$

which has the desired form of the contribution of the electrostatic field to the classical Boltzmann equation in position-momentum space. Next we observe that the Wigner transform of the first term, namely of the kernel

$$G(t,\xi',\xi'') = (A(t,\xi'),\nabla_{\xi'})\rho(t,\xi',\xi'') + (A(t,\xi''),\nabla_{\xi''})\rho(t,\xi',\xi'')) - - - (88)$$

is given by

$$\hat{G}(t,\xi,P) = \int (A(t,\xi+hq/2),\nabla_1)\rho(t,\xi+hq/2,\xi-hq/2) + (A(t,\xi-hq/2),\nabla_2)\rho(t,\xi+hq/2,\xi-hq/2))exp(-iP.q)dq$$

where $\nabla_1$ denotes gradient w.r.t $\xi_1 = \xi + hq/2$ and $\nabla_2$ w.r.t $\xi_2 = \xi - hq/2$. We can write

$$\nabla_\xi = \nabla_1 + \nabla_2 - - - (90)$$

$$\nabla_q = (h/2)(\nabla_1 - \nabla_2) - - - (91)$$

and hence,

$$(h/2)\nabla_1 = (h/2)\nabla_\xi + \nabla_q, (h/2)\nabla_2 = (h/2)\nabla_\xi - \nabla_q - - - (92)$$

Thus,

$$(h/2)\hat{G}(t,\xi,P) =$$

$$\int [(A(t,\xi+hq/2),(h/2)\nabla_\xi+\nabla_q)\rho(t,\xi+hq/2,\xi-hq/2) + (A(t,\xi-hq/2),(h/2)\nabla_\xi-\nabla_q)\rho(t,\xi+hq/2,\xi-hq/2)]exp($$

Expanding the vector potential upto $O(h^2)$ terms, we see that the term involving gradient of the vector potential is given by

$$(h^2/2)\int (q,\nabla_\xi)A(t,\xi),\nabla_q)\rho(t,\xi+hq/2,\xi-hq/2)exp(-iP.q)dq$$

$$=(ih^2/2)\int [((q,\nabla_\xi)A(t,\xi),P)\rho(t,\xi+hq/2,\xi-hq/2))].exp(-iP.q)dq$$

$$=(A_{,j}(t,\xi),P)\partial_{P_j}\hat{\rho}(t,Q,P)=A_{k,j}(t,\xi)P_k\partial_{P_j}\hat{\rho}(t,Q,P)---(94)$$

except for a proportionality constant.

Now, we assume that that magnetic field is nearly a constant in space over each central region. This means that if we denote the position of the centre by $xi_0$, then

$$A(t,\xi)\approx B(t,\xi_0)\times(\xi-\xi_0)/2---(95)$$

or equivalently, in terms of components,

$$A_k(t,\xi)=\epsilon(krs)B_r(t,\xi_0)(\xi_s-\xi_{0s})/2---(96)$$

so that

$$A_{k,j}(t,\xi)=\epsilon(krj)B_r(t,\xi_0)\approx\epsilon(krj)B_r(t,\xi)/2---(97)$$

and hence,

$$A_{k,j}(t,\xi)P_k\partial_{P_j}\hat{\rho}(t,Q,P)=\epsilon(krj)B_r(t,\xi)P_k\partial_{P_j}\hat{\rho}(t,Q,P)$$

$$=(P\times B(t,\xi),\nabla_P)\hat{\rho}(t,Q,P)---(98)$$

which is exactly the contribution of the magnetic field to the classical Boltzmann equation.

## 0.6 Identifying the quantum corrections to Boltzmann's equation coming from the nonlinear term

qbe stands for quantum Boltzmann equation. The main reason behind these calculations is to show that if some basic parameters of the optical fibre are known like the positions of the centres of charges, then, in principle, we can compute the scattered classical electromagnetic field produced by the charges around these centres and hence design the counter potential or counter TPCP map to effect its cancellation.

Before however doing so, we first discuss a method for estimating the Lindblad noise term in the simplest form of the qbe. This equation reads

$$\partial_t\rho(t)=(-i/h)[H_0+\delta H(t),\rho(t)]+\theta(\rho(t))-(i/h)(N-1)Tr_2[V_{12},\rho(t)\otimes\rho(t)]---(99)$$

where $\theta$ is the Lindblad term which is to be estimated. We assume that

$$\theta(\rho) = \sum_{k=1}^{p} \alpha_k \theta_k(\rho) --- (100)$$

where the $\theta_k' s$ are known maps on the space of density matrices and the $\alpha_k' s$ are parameters to be estimated. Before however introducing an algorithm based on the maximum likelihood method for estimating the $\alpha_k' s$, we shall explain how to transform the Lindblad term and the nonlinear term to the Wigner distribution domain. First consider the nonlinear term. In the position domain, it is expressible as

$$F(\xi_1, \xi_1') = Tr_2[V_{12}, \rho \otimes \rho](\xi_1, \xi_1') =$$

$$\int V(\xi_1, \xi_2)\rho(\xi_1, \xi_1')\rho(\xi_2, \xi_2)d\xi_2 --- (101)$$

and hence its Wigner transform is

$$\hat{F}(xi_1, P_1) = \int F(xi_1 + hq/2, \xi_1 - hq/2)exp(-iP_1.q)dq =$$

$$\int V(\xi_1 + hq/2, \xi_2)\rho(\xi_1 + hq/2, \xi_1 - hq/2)\hat{\rho}(xi_2, P_2)exp(-iP_1.q)dP_2 d\xi_2 dq$$

$$= (\int V(\xi_1, \xi_2)\hat{\rho}(\xi_2, P_2)d]xi_2 dP_2)\hat{\rho}(\xi_1, P_1)$$

$$+ \sum_{n\geq 1}(n!)^{-1}(h/2)^n(\int V_1^{(n)}(\xi_1, \xi_2)\hat{\rho}(\xi_2, P_2)d\xi_2 dP_2)\int q^n \rho(\xi_1 + hq/2, \xi_1 - hq/2)exp(-iP_1.q)dq$$

$$= (\int V(\xi_1, \xi_2)\hat{\rho}(\xi_2, P_2)d]xi_2 dP_2)\hat{\rho}(\xi_1, P_1)$$

$$+ \sum_{n\geq 1}(n!)^{-1}(1/2)^n(\int V_1^{(n)}(\xi_1, \xi_2)\hat{\rho}(\xi_2, P_2)d\xi_2 dP_2)(ih\partial_{P_1})^n \hat{\rho}(\xi_1, P_1) --- (102)$$

In this way, it becomes a long but straightforward calculation to obtain quantum corrections to the classical Boltzmann equations coming from the nonlinear terms.

## 0.7 Quantum corrections to Boltzmann's equation coming from the external scalar potential field terms

We now become more explicit by evaluating exactly all the higher order quantum correction terms caused by external field and mutual interaction effects.

Specifically, to all orders, the Wigner transform of the external electostatic potential term $F_1(t,\xi',\mathbf{x}'')=(-i/h)(V(t,\xi')-V(t,\xi''))\rho(t,\xi',\xi'')$ is given by

$$\hat{F}_1(t,\xi,P)=(-i/h)\int(V(t,\xi+hq/2)-V(t,\xi-hq/2))\rho(t,\xi+hq/2,\xi-hq/2)exp(-iP.q)dq$$

$$=(-2i/h)\sum_{n\geq 0}((2n+1)!)^{-1}(h/2)^{2n+1}(\partial_\xi^{2n+1}V(t,\xi))\int q^{2n+1}\rho(t,\xi+hq/2,\xi-hq/2)exp(-iP.q)dq$$

$$(-2i/h)\sum_{n\geq 0}((2n+1)!)^{-1}(-ih/2)^{2n+1}(\partial_\xi^{2n+1}V(t,\xi))(-i\partial_P)^{2n+1}\hat{\rho}(t,\xi,P)$$

$$=-\sum_{n\geq 0}((2n+1)!)^{-1}(-1)^n h^{2n}/2^{2n+1}(\partial_\xi^{2n+1}V(t,\xi))(-i\partial_P)^{2n+1}\hat{\rho}(t,\xi,P)---(103)$$

The zeroth degree term in Planck's constant is $(-\partial_\xi V(t,\xi),\partial_P)\hat{\rho}(t,\xi,P)$ which may be identified with the classical contribution while the higher degree terms in Planck's constant are proportional to $h^{2n}(\partial_\xi^{\otimes 2n},\partial_P^{\otimes 2n})\hat{\rho}(t\,xi,P),n\geq 1$ and can be identified with quantum corrections to the classical contribution.

## 0.8 Quantum Boltzmann equation taking into account Lindblad coupling to a noisy bath:Estimating the Lindblad parameters based on sequential POVM's

We now express the qbe with Lindblad term in the form

$$\partial_t\rho(t)=\delta.\theta(\rho(t)|\alpha)+F(\rho(t))---(104)$$

where $\delta$ is small perturbation parameter and $F(\rho)=-i[H,\rho]-i\delta.(N-1)Tr_2[V_{12},\rho\otimes\rho]$. In doing thus, we are assuming that the contribution of the nonlinear mutual interaction term in the qbe is of the same order of smallness $O(\delta)$ as the contribution coming from the Lindlbad term. A more satisfactory way that avoids such an assumption is to assume first that the nonlinear term is absent and then obtain the perturbation series $T_0$ for the linear differential equation defined by the Hamiltonian and Lindblad term alone and then include the nonlinear term and obtain the perturbation series with respect to this term assuming the zeroth order solution being given by $T_0$. Yet another way to obtain the perturbation series that avoids doing two perturbation expansions is to assume that the Lindblad term is $O(\delta)$ while the nonlinear term is much smaller of $O(\delta^m)$ where $m$ is a positive integer and then expand the solution density in powers of $\delta$.

The Lindlbad term has the expansion

$$\theta(\rho|\alpha)=\sum_k\alpha(k)\theta_k(\rho)---(105)$$

Using perturbation theory, we can express the solution in the form

$$\rho(t) = T_{t,s}(\rho(s)|\alpha) = T_{t,s}(\alpha)[\rho(s)], t > s - - - (106)$$

where $T_{t,s}$ is a nonlinear evolution operator expressed as a power series in the parameter vector $\alpha$. Specifically, we can write upto $O(\delta)$. Assume that a POV measurement $\{M(a) : a = 1, 2, ..., K\}$ is performed on the system at times $t_1 < t_2 < ... < t_N$ with the outcomes noted at each stage. Then, according to the collapse postulate of quantum mechanics after a measurement, the sequential probability of obtaining outcomes $a_1, ..., a_N$ at these measurements allowing for qbe plus Lindblad evolution in-between two measurements is given by

$$P(a_1, ..., a_N; t_1, ..., t_N|\alpha) =$$

$$Tr[\sqrt{M(a_N)})T_{t_N,t_{N-1}}(\alpha)[\sqrt{M(a_{N-1}}T_{t_{N-1},t_{N-2}}(\alpha)[\sqrt{M(a(N-2))}...$$

$$T_{t_2,t_1}(\alpha)[\sqrt{M(a_1)}T_{t_1,0}(\alpha)[\rho(0)]\sqrt{M(a_1)}]...\sqrt{M(a_{N-2}}]\sqrt{M(a_{N-1})}]\sqrt{M(a_N)}]---(107)$$

The maximum likelihood method then involves maximizing this joint sequential probability w.r.t $\alpha$. There appears to be no other satisfactory method for estimating the Lindlbad noise terms coming from the random perturbing Hamiltonian produced by the lattice of charged phonons with charges surrounding them in order to generate the counter potential or counter TPCP term for noise cancellation.

## 0.9 Quantum Boltzmann equation taking into account non-demolition noisy measurements using the Belavkin filter

Another more complex way to estimate the Lindblad parameters is based on the continuous non-demolition method developed by V.P.Belavkin. This involves first dilating the Lindblad dynamics into a unitary evolution on the joint state of the system and bath phonons and then estimating the evolving state of the system alone on a real time basis using non-demolition measurements. It should be noted here that the Belavkin equation for the system state evolution is being applied to the joint state of all the $N$ indistinguishable particles of the system. We then partially trace out this Belavkin equation over all but one of the $N$ particles to obtain a quantum Boltzmann equation for one particle that takes into account non-demolition measurement noise. Specifically, the Belavkin equation for the state estimate $\rho_B(t)$ of the $N$ particles is given by

$$d\rho_B(t) = -i[H, \rho_B(t)]dt + \theta(\rho_B(t)|\alpha)dt + F_1(t, \rho_B(t))(dY(t) - F_2(t, \rho_B(t))dt), H = \sum_{k=1}^{N} H_k + \sum_{1\leq k<j\leq N} V_{kj} - - - (10$$

where $dY$ is the measurement noise differential, $F_1(t, \rho_B)$ is of the form $M\rho_B + \rho_B M^* - Tr(\rho_B(M + M^*))\rho_B$, or in short, $F_1$ is a linear-quadratic function of

$\rho_B$ and $F_2(t,\rho_B) = Tr(\rho_B(M+M^*))$, or in short, $F_2$ is linear in $\rho_B$. Note that $\rho_B, M$ are operators in the $N$-fold tensor product of the one particle Hilbert space $\mathcal{H}_1$ and that $M$ is invariant under any permutation of the particles.

Partial tracing out this equation over $\mathcal{H}_1^{\otimes N-1}$ results in an equation of the form

$$d\rho_{B1}(t) = -i[H_1,\rho_{B1}(t)]dt - i(N-1)Tr_2[V_{12},\rho_{B1}(t)\otimes\rho_{B1}(t)]dt$$

$$+\sum_k f_{1k}(t,\rho_{B1}(t)|\alpha).\theta_{1k}(\rho_{B1}(t)|\alpha)dt+\sum_k f_{2k}(t,\rho_{B1}(t)|\alpha).\theta_{2k}(\rho_{B1}(t)|\alpha)dY(t)---(109)$$

Note that in arriving at this approximate equation, we use the molecular chaos approximation:

$$\rho_B(t) \approx \rho_{B1}(t)^{\otimes N} --- (110)$$

and we also use the assumption that the Lindblad noise coupling operators for the particles to the bath are symmetric w.r.t permutations of the particles. $\theta_{1k}, \theta_{2k}$ are linear maps depending on the parameters $\alpha$ acting in the space of one particle system operators. $f_{1k}$ is a polynomial function of $\rho_{B1}$ of degree $N$, specifically,

$$\sum_k f_{1k}(t,\rho_{B1}|\alpha)\theta_{1k}(\rho_{B1}|\alpha) = Tr_{23...N-1}(F_1(t,\rho_{B1}^{\otimes N}))$$

$$= Tr_{23...N-1}(M\rho_{B1}^{\otimes N} + \rho_{B1}^{\otimes N}M^*) - Tr(\rho_{B1}^{\otimes N}(M+M^*))\rho_{B1}) --- (111)$$

is a polynomial in $\rho_{B1}$ containing only $N^{th}$ and $(N+1)^{th}$ degree terms and

$$\sum_k f_{2k}(t,\rho_{B1}(t)|\alpha)\theta_{2k}(\rho_{B1}|\alpha) = -Tr_{23...N}(F_1(t,\rho_{B1}^{\otimes N})F_2(t,\rho_{B1}^{\otimes N})) --- (112)$$

is a polynomial in $\rho_{B1}$ containing only $(2N)^{th}$ and $(2N+1)^{th}$ degree terms. Thus, $f_{1k}$ is a polynomial in $\rho_{B1}$ containing only $(N-1)^{th}$ and $N^{th}$ degree terms while $f_{2k}$ is a polynomial in $\rho_{B1}$ containing only $(2N-1)^{th}$ and $(2N)^{th}$ degree terms.

It should be noted that once the non-demolition measurements $Y(.)$ upto time $t$ are made, we get access to the one particle state $\rho_{B1}$, and we can then using this state and the above Belavkin-Boltzmann equation, estimate the Lindblad parameters $\alpha$ by performing further measurements on each particle's state. It should be noted that prior to making the non-demolition measurements, we do not have access to the system state, only after performing the measurements $Y(.)$, we get the Belavkin estimate of the one particle state. It should be noted that by the non-demolition property of the Belavkin measurements $Y(.)$, the future evolution of observables in the HPS dynamics is not affected. We can also using this Belavkin equation, design algorithms for nearly cancelling out the effects of Lindblad noise by applying quantum control along the lines indicated in the thesis of Lec-Bouten.

## 0.10 Lec-Bouten's method of Lindblad noise cancellation

Lec-Bouten's method of noise cancellation: In the simplest case of a single particle, the Belavkin state estimate at time $t+dt$ is given in terms of its controlled estimate at time $t$ by the equation

$$\rho_B(t+dt) = \rho_c(t) + dt.\theta(\rho_c(t)) + F_1(t,\rho_c(t)).(dY(t) - F_2(t,\rho_c(t))dt) - - - (113)$$

where

$$\theta(\rho_B(t)) = -i[H,\rho_B] - (1/2)\sum_k (L_k L_k^* \rho_B + \rho_B.L_k L_k^* - 2L_k^* \rho_B L_k) - - - (114)$$

We design the control operator (not necessarily unitary since $Z$ is not restricted to be Hermitian)

$$U_c(t,t+dt) = exp(-iZdY(t)) - - - (115)$$

over the time interval $[t,t+dt]$ where $Z$ is a Hermitian system operator like $H$. Note that $L_k$ is also a system operator but not necessarily Hermitian. Applying this control operation to $\rho_B(t+dt)$ gives us the controlled state at time $t+dt$ as (We are assuming that the non-demolition measurement is quadrature, ie, the input measurement process is a quantum Brownian motion and does not contain any counting/Poisson/Conservation process component. Therefore, we have by quantum Ito's formula, $(dY)^2 = dt$)

$$\rho_c(t+dt) = U_c(t,t+dt)\rho_B(t+dt)U_c(t,t+dt)^* =$$

$$(1-iZdY-Z^2dt/2).(\rho_c(t)+dt.\theta(\rho_c(t))+F_1(t,\rho_c(t)).(dY(t)-F_2(t,\rho_c(t))dt))(1+iZdY-Z^2dt/2)$$

$$= \rho_c(t) + dt.\theta(\rho_c(t)) + F_1(t,\rho_c(t)).(dY(t) - F_2(t,\rho_c(t))dt)$$

$$-i[Z,\rho_c(t)]dY(t) - (dt/2)(Z^2\rho_c(t) + \rho_c(t)Z^2 - 2Z\rho_c(t)Z) - i[Z,F_1(t,\rho_c(t))]dt - - - (116)$$

The idea then is to choose the system operator $Z$, so that

$$\| \theta(\rho_c) - (1/2)(Z^2\rho_c + \rho_c Z^2 - 2Z\rho_c Z) - i[Z,F_1(t,\rho_c)] \|^2 - - - (117)$$

is minimized. To see how this works, suppose that $L$ is a skew-Hermitian system operator. Then $L^* = -L$ and if we assume that

$$\theta(\rho_c) = (-1/2)(LL^*\rho_c + \rho_c LL^* - 2L^*\rho_c L) =$$

$$(1/2)(L^2\rho_c + \rho_c L^2 - 2L\rho_c L) - - - (118)$$

and further taking $M = L$,

$$F_1(t,\rho_c) = M\rho_c + \rho_c M^* - Tr(\rho_c(M+M^*))\rho_c = L\rho_c - \rho_c L = [L,\rho_c] - - - (119)$$

Then, taking $Z = -iL$ (Note that $Z$ will then be Hermitian), we get

$$\theta(\rho_c) - (1/2)(Z^2\rho_c + \rho_c Z^2 - 2Z\rho_c Z) - i[Z, F_1(t, \rho_c)] =$$

$$= (1/2)(L^2\rho_c + \rho_c L^2 - 2L\rho_c L) + (1/2)(L^2\rho_c + \rho_c L^2 - 2L\rho_c L) - [L, [L, \rho_c]] = 0 - - - (120)$$

More generally, we can conceive of a situation in which the measurement comprises of $p > 1$ non-demolition noise processes $Y_1, ..., Y_p$. In that case the Belavkin filter has the form

$$d\rho_B(t) = -i[H, \rho_B] - (1/2)\sum_k (L_k L_k^* \rho_B + \rho_B L_k L_k^* - 2L_k^* \rho_B L_k)$$

$$+ \sum_k (\rho_B L_k + L_k^* \rho_B - Tr(\rho_B(L_k + L_k^*))\rho_B)(dY_k(t) - Tr(\rho_B(L_k + L_k^*))dt) - - - (121)$$

Again, we try to cancel out a part of the Lindblad process noise by means of the control unitary

$$U_c(t, t+dt) = \Pi_{k=1}^p exp(-iZ_k dY_k) = \Pi_{k=1}^p (1 - iZ_k dY_k - Z_k^2 dt/2) = 1 - i\sum_k Z_k dY_k - (1/2)(\sum_k Z_k^2)dt - - - (122)$$

since $Y_k' s$ are quadratures with $dY_k dY_j = \delta_{kj} dt$.

## 0.11 Frequency aspects of the scattered electromagnetic field in the classical Boltzmann equation

Consider first the simple case when there is just one centre generating a potential $U(r)$ in which a large number $N$ point charges, each of charge $q$ execute motion.

By jointly solving the nonlinear classical Boltzmann equation and Maxwell's equations for the scattered field, we easily deduce the fact that the scattered electromagnetic field is a nonlinear functional of the incident electromagnetic field and the initial equilibrium approximate joint Gaussian density of the phonon positions and velocities. Specifically, if we neglected the quadratic nonlinear interaction terms in the classical Boltzmann equation, then the Boltzmann equation would be linear with a driving term being proportional to the equilibrium phonon density. This equation would be

$$\partial_t f(t, r, v) + (v, \nabla_r) f(t, r, v) - (q/m)(\nabla U(r), \nabla_v) f(t, r, v) + (q/m)(E_c(t, r) + v \times B_c(t, r), \nabla_v) f(t, r, v) = 0 - - - (123$$

Here, we are assuming that the binding potential $U(r)$ for each particle is the same. We write

$$f(t, r, v) = f_0(r, v) + \delta f(t, r, v) - - - (124)$$

with

$$f_0(r, v) = Z^{-1} exp(-\beta(qU(r) + mv^2/2)) - - - (125)$$

and after substituting this and making cancellations, we get our final equation as

$$\partial_t \delta f(t,r,v)+(v,\nabla_r)\delta f(t,r,v)-(q/m)(\nabla U(r),\nabla_v)\delta f(t,r,v)-\beta q(E_c(t,r),v)f_0(r,v)$$

$$+(q/m)(E_c(t,r)+v\times B_c(t,r),\nabla_v)\delta f(t,r,v)=0---(126)$$

This equation is exact provided that we neglect the quadratic nonlinear interaction term

$$Q(f,f)(t,r,v)=(N-1)q(\int f(t,r',v')(r-r')d^3r'd^3v'/|r-r'|^3,\nabla_v)f(t,r,v)+(N-1)q(\int f(t,r',v')v'\times(r-r')d^3r$$

Suppose now that we take into account this quadratic term but retain only those terms in it that are linear in $\delta f_1$. Then, we see that the zeroth order equation for the equilibrium density satisfies the nonlinear integro-differential equation

$$(v,\nabla_r)f_0(r,v)-q(\nabla U(r),\nabla_v)f_0(r,v)+Q(f_0,f_0)(r,v)=0---(128)$$

We can solve this perturbatively by assuming $Q(f_0,f_0)$ to be small and writing

$$f_0(r,v)=f_{00}(r,v)+\delta f_0(r,v)---(129)$$

where

$$f_{00}(r,v)=Z^{-1}exp(-\beta(mv^2/2+qU(r)))---(130)$$

satisfies the unperturbed equilibrium equation

$$(v,\nabla_r)f_{00}(r,v)-q(\nabla U(r),\nabla_v)f_{00}(r,v)=0---(131)$$

and $\delta f_0(r,v)$ will then satisfy the first order perturbed equilibrium equation:

$$(v,\nabla_r)\delta f_0(r,v)-q(\nabla U(r),\nabla_v)\delta f_0(r,v)=-Q(f_{00},f_{00})(r,v)----(132)$$

or in terms of operators, the formal solution is

$$\delta f_0(r,v)=-[(v,\nabla_r)\delta f_0(r,v)-q(\nabla U(r),\nabla_v)]^{-1}Q(f_{00},f_{00})(r,v)---(133)$$

Remark: Suppose we have a density of particles in phase space $f(x)$. The interaction energy of these particles with an external potential $U_1(x)$ is $\int f(x)U_1(x)dx$ and the pairwise interaction energy of the particles has the form $(1/2)\int U_2(x,y)f(x)f(y)dxdy$. According the the maximum entropy principle, the equilibrium density $f_0(x)$ will be obtained by maximizing the entropy $-\int f(x).ln(f(x))dx$ subject to the total energy constraint

$$E=\int f(x)U_1(x)dx+\int f(x)f(y)U_2(x,y)dxdy$$

Using Lagrange multipliers to take care of this constraint as well as the density constraint $\int f(x)dx=1$, the functional to be maximized is thus given by

$$S(f,\mu)=-\int f(x)ln(f(x))dx-\mu(E-\int f(x)U_1(x)dx-(1/2)\int f(x)f(y)U_2(x,y)dxdy)-\lambda(1-\int f(x)dx)$$

and setting the variational derivative $\delta S(f).\delta f(x) = 0$ at $f = f_0$ give us the result that $f_0$ satisfies

$$-ln(f(x)) - 1 + \mu(U_1(x) + \int U_2(x,y)f(y)dy + \lambda = 0$$

Formally, we can express this as the following implicit functional equation

$$f(x) = C.exp(\mu(U_1(x) + (U_2 f)(x))$$

where the constants $C, \mu$ are determined from the energy and particle constraints. It is easy to see that the above nonlinear equation for $f_0$ derived as the stationary Boltzmann equation can also be arrived from this argument by an appropriate choice of the functions $U_1, U_2$, namely $x = (r, v)$, $U_1(r, v) = mv^2/2 + qU(r)$ and $U_2(r, v; r', v')$ defined in terms of the kernel of the quadratic form $Q(f, f)$ and finally, $\mu = -\beta$.

Now, taking the quadratic term $Q$ into account, the exact equation satisfied by $\delta f$ is given by

$$\partial_t \delta f(t,r,v) + (v, \nabla_r)\delta f(t,r,v) - (q/m)(\nabla U(r), \nabla_v)\delta f(t,r,v) + (q/m)(E_c(t,r) + v \times B_c(t,r), \nabla_v) f_0(r,v)$$

$$+(q/m)(E_c(t,r) + v \times B_c(t,r), \nabla_v)\delta f(t,r,v) + 2Q(f_0, \delta f)(t,r,v) + Q(\delta f, \delta f)(t,r,v) = 0 - - - (134)$$

where

$$f_0 = f_{00} + \delta f_0 - - - (135)$$

as above. Note that the polarization identity gives

$$Q(f,g) = (Q(f+g, f+g) - Q(f-g, f-g))/4 = (Q(f+g, f+g) - Q(f,f) - Q(g,g))/2 - - - (136)$$

The above equation is exact provided that we assume that the solution to the equilibrium equation for $f_0$ is exact. Note that this becomes a linear integro-partial differential equation for $\delta f(t, r, v)$ provided that we neglect the last term $Q(\delta f, \delta f)$. Writing therefore,

$$\delta f = \delta f_1 + \delta f_2, f_0 = f_{00} + \delta f_0 - - - (137)$$

where $\delta f_1$ is of the first order of smallness and $\delta f_2$ is of the second order of smallness while $f_{00}$ is of zeroth order and $\delta f_0$ is of the first order of smallness, we find that the respective equations satisfied by $\delta f_1$ and $\delta f_2$ are

$$\partial_t \delta f_1(t,r,v) + (v, \nabla_r)\delta f_1(t,r,v) - (q/m)(\nabla U(r), \nabla_v)\delta f_1(t,r,v) - \beta q(E_c(t,r), v) f_{00}(r,v) = 0 - - - (138)$$

if we assume that the incident field $E_c, B_c$ is of the first order of smallness and the kernel of $Q$ is also of the first order of smallness, so that $Q(f_0, \delta f_1)$ becomes of the second order of smallness, and

$$\partial_t \delta f_2(t,r,v) + (v, \nabla_r)\delta f_2(t,r,v) - (q/m)(\nabla U(r), \nabla_v)\delta f_2(t,r,v)$$

$+(q/m)(E_c(t,r)+v\times B_c(t,r),\nabla_v)(\delta f_0(r,v)+\delta f_1(t,r,v))+2Q(f_0,\delta f_1)(t,r,v)=0---(139)$

Remark: Note that $(v\times B_c(t,r),\nabla_v)f_{00}(r,v)=0$ because $f_{00}$ is Gibbsian with energy $mv^2/2+qU(r)$. The story will be complete once we comppute the scattered fields in the non-relativistic approximation as

$$E_s(t,r)=Nq\int(\delta f_0(r',v')+\delta f_1(t,r',v')+\delta f_2(t,r',v'))(r-r')d^3r'd^3v'/|r-r'|^3---(140)$$

$$B_s(t,r)=Nq\int(\delta f_0(r',v')+\delta f_1(t,r',v')+\delta f_2(t,r',v'))v'\times(r-r')d^3r'd^3v'/|r-r'|^3---(141)$$

It is clear from the above equations that $\delta f_1(t,r,v)$ is a linear function of the incident electric field $E_c$ while $\delta f_2(t,r,v)$ is a linear-quadratic function of the incident electric and magnetic fields $E_c,B_c$. The linear term in the equation for $\delta f_2$ comes from $(q/m)(E_c(t,r)+v\times B_c(t,r),\nabla_v)\delta f_0(r,v)$ while the quadratic term comes from $(q/m)(E_c(t,r)+v\times B_c(t,r),\nabla_v)\delta f_1(t,r,v)$. Note that $\delta f_1$ is independent of the incident magnetic field $B_c$. It follows that if $E_c,B_c$ contain frequencies only in a band $I$, then $\delta f_1$ will contain frequencies only in the band $I$ while $\delta f_2$ will contain frequencies only in the band $I\cup(I+I)$. Note that $\delta f_0$ time independent and hence contributes only to the dc term in the scattered electric and magnetic fields which are not of much interest to us.

## 0.12 Frequency spectrum of scattered field from the quantum Boltzmann equation

The two and three particle density operators can be expressed in the form

$$\rho_{12}=\rho_1\otimes\rho_1+g_{12},\rho_{123}=\rho_1\otimes\rho_1\otimes\rho_1+\rho_1\otimes g_{23}+\rho_2\otimes g_{13}+\rho_3\otimes g_{12}+g_{123}---(142)$$

where for consistency, $Tr_2g_{12}=0,Tr_3g_{123}=0$ with $g_{12},g_{123}$ being invariant under particle permutations. These operators satisfy the following differential equations derived from the original $N$-particle Schrodinger equations by forming partial traces over $N-1$ and $N-2$ particles respectively.

$$\partial_t\rho_1=-i[H_1,\rho_1]-i(N-1)Tr_2[V_{12},\rho_{12}]---(143)$$

$$\partial_t\rho_{12}=-i[H_1+H_2+V_{12},\rho_{12}]-i(N-2)Tr_3[V_{13}+V_{23},\rho_{123}---(144)$$

These are exact equations. Substituting the above expressions for $\rho_{12}$ and $\rho_{123}$ and making appropriate cancellations, we derive the following exact equations:

$$\partial_t\rho_1=-i[H_1,\rho_1]-i(N-1)Tr_2[V_{12},\rho_1\otimes\rho_1+g_{12}]---(145)$$

$$\partial_t g_{12}=-i[H_1+H_2+V_{12},g_{12}]-i[V_{12},\rho_1\otimes\rho_1\otimes\rho_1]$$
$$+i.Tr_2[V_{12},\rho_1\otimes\rho_1]\otimes\rho_1+i\rho_1\otimes Tr_2[V_{12},\rho_1\otimes\rho_1]$$

$$+i.\rho_1 \otimes Tr_2[V_{12}, g_{12}] - i(N-2).Tr_3[V_{13}, \rho_1 \otimes g_{23} + \rho_3 \otimes g_{12}]$$

$$-i(N-2).Tr_3[V_{23}, \rho_2 \otimes g_{13} + \rho_3 \otimes g_{12}] - i(N-2).Tr_3[V_{13} + V_{23}, g_{123}] - -- (146)$$

Likewise we can derive exact equations for $g_{123..r}, r = 3, 4, ..., N$ in terms of $\rho_1, g_{12...k}, k = 2, 3, ..., r+1$. The quantum Boltzmann equation upto the $r^{th}$ order involves neglecting $g_{12...r+1}$, so that the system of equations closes on itself at the $r^{th}$ stage. For example, if we require the qbe upto order two, then we must neglect $g_{123}$. These equations are then given by

$$\partial_t \rho_1 = -i[H_1, \rho_1] - i(N-1)Tr_2[V_{12}, \rho_1 \otimes \rho_1 + g_{12}] - -- (147)$$

$$\partial_t g_{12} = -i[H_1 + H_2 + V_{12}, g_{12}] - i[V_{12}, \rho_1 \otimes \rho_1 \otimes \rho_1]$$

$$+i.Tr_2[V_{12}, \rho_1 \otimes \rho_1] \otimes \rho_1 + i\rho_1 \otimes Tr_2[V_{12}, \rho_1 \otimes \rho_1]$$

$$+i.\rho_1 \otimes Tr_2[V_{12}, g_{12}] - i(N-2).Tr_3[V_{13}, \rho_1 \otimes g_{23} + \rho_3 \otimes g_{12}]$$

$$-i(N-2).Tr_3[V_{23}, \rho_2 \otimes g_{13} + \rho_3 \otimes g_{12}] - -- (148)$$

Note that $g_{12}, g_{23}, g_{13}$ are identical copies of each other but acting in the Hilbert spaces $\mathcal{H}_i \otimes \mathcal{H}_j with(i,j) = (1,2), (2,3), (1,3)$ respectively with the $\mathcal{H}_i, i = 1, 2, ..., N$ being identical copies of each other. Likewise, $\rho_i, i = 1, 2, 3$ are identical copies of each other but acting in the Hilbert spaces $\mathcal{H}_i, i = 1, 2, 3$ respectively.

To simplify matters further, we consider the qbe upto first order obtained by setting $g_{12} = 0$: Setting $q = -e$, we get

$$\partial_t \rho_1 = -i[H_1, \rho_1] - i(N-1)Tr_2[V_{12}, \rho_1 \otimes \rho_1] - -- (149)$$

With

$$H_1 = H_0 + \delta H_1(t) - -- (150)$$

where

$H_0 = p_1^2/2m + V_0(r_1), p_1 = -i\nabla_{r_1}, \delta H_1(t) = (-ie/m)(A(t, r_1), \nabla_{r_1}) - eV(t, r_1) + e^2 A(t, r_1)^2/2m = \delta H_{11}(t) + \delta H_1$

where $\delta H_{11}(t)$ is $O(e)$ while $\delta H_{12}(t)$ is $O(e^2)$, and with

$$\rho_1 = \rho_0 + \delta\rho_1 + \delta\rho_2, \rho_0 = Z(\beta)^{-1} exp(-\beta H_0) - -- (152)$$

where $\delta\rho_1$ is $O(e)$ and $\delta\rho_2$ is $O(e^2)$, and we assume that the $V_{12}$ is $O(e^2)$ because it is the electrostatic interaction potential energy between to charged particles of charge $-e$. Thus upto $O(e^2)$, we obtain the equations

$$\partial_t \delta\rho_1 = -i[H_0, \delta\rho_1] - i[\delta H_{11}, \rho_0] - -- (153)$$

$$\partial_t \delta\rho_2 = -i[H_0, \delta\rho_2] - i[\delta H_{11}, \delta\rho_1] - i[\delta H_{12}, \rho_0] - i(N-1)Tr_2[V_{12}, \rho_0 \otimes \rho_0] - -- (154)$$

These have respective solutions with $U_0(t) = exp(-itH_0)$:

$$\delta\rho_1(t) = -i \int_0^t Ad(U_0(t-s))([\delta H_{11}(s), \rho_0]) ds - -- (155)$$

$$\delta\rho_2(t) = -i\int_0^t Ad(U_0(t-s))([\delta H_{11}(s), \delta\rho_1(s)] + [\delta H_{12}(s), \rho_0(s)])ds$$

$$-i(N-1)(\int_0^t Ad(U_0(s))ds)(Tr_2[V_{12}, \rho_0 \otimes \rho_0])$$

$$= -\int_{0<s'<s<t} Ad(U_0(t-s))ad(\delta H_{11}(s))Ad(U_0(s-s'))ad(\delta H_{11}(s'))(\rho_0)dsds'$$

$$-i\int_0^t Ad(U_0(t-s))([\delta H_{12}(s), \rho_0])ds - i(N-1)(\int_0^t Ad(U_0(s))ds)(Tr_2[V_{12}, \rho_0\otimes\rho_0]) \quad ---(156)$$

Noting that $\delta H_{11}$ is linear in the electromagnetic potentials while $\delta H_{12}$ is quadratic in the same, it follows that $\delta\rho_1$ is linear in the electromagnetic field while $\delta\rho_2$ contains a quadratic term in the electromagnetic field plus another term that does not involve the electromagnetic field and that varies with time as $\int_0^t exp(-iad(H_0)s) = (1 - exp(-itad(H_0))/iad(H_0)$. This second term contains frequencies of the form $E_n - E_m$ where $E_n$ is the $n^{th}$ excited stationary energy level of the unperturbed Hamiltonian $H_0$. Combining these results, we deduce that upto second order, the one particle state $\rho_1(t)$ will contain a dc term and terms that are linear-quadratic in the incident electromagnetic field multiplied by terms that are linear in $Ad(U_0(t))$ and in addition, terms that are independent of the electromagnetic field but linear in $Ad(U_0(t))$. Noting that $Ad(U_0(t))$ consists of frequencies corresponding to all energy differences (divided by Planck's constant which we are setting equal to one) of the stationary levels, we see that the frequencies in the one particle state $\rho_1(t)$, apart from the dc term are $E_n - E_m, E_n - E_m \pm \omega, E_n - E_m + \omega \pm \omega', E_n - E_m + E_r - E_s + \omega \pm \omega'$ where $n, m, r, s$ vary over all positive integers that index the stationary one particle energy levels while $\omega, \omega'$ vary over all the frequencies in the incident electromagnetic field. Carrying out the perturbation expansion of the one particle density operator to higher and higher orders, we find that it will contain all the frequencies that can be expressed as sums of $E_n - E_m$ and integer linear combinations of the frequencies present in the incident electromagnetic field. Of course these higher order terms will be diminished in amplitude by powers of the electronic charge. However, the presence of such a wide spectrum of frequencies as compared to that present in the incident field indicates that the field scattered by the charged particles will be nearly white noise w.r.t time.

## 0.13 Complete derivation of the quantum Boltzmann equation for the one and two particle density estimates in quantum filtering theory

Consider the HPS qsde

$$dU(t) = (-(iH+P)dt + \sum_k (L_k dA_k(t) - L_k^* dA_k(t)^*))U(t) --- (157)$$

where $A_k, k = 1,2,...,p$ are annihilation processes and $A_k^*, k = 1,2,...,p$ are the corresponding creation processes satisfying the quantum Ito formula

$$dA_k dA_j^* = \delta_{kj} dt --- (158)$$

and

$$P = (1/2)\sum_k L_k L_k^* --- (159)$$

provides the quantum Ito correction term that guarantees unitarity of $U(t)$. System operators $X$ evolve according to noisy Heisenberg dynamics:

$$j_t(X) = U(t)^* X U(t) --- (160)$$

A simple calculation using quantum Ito's formula shows that

$$dj_t(X) = j_t(\theta_0(X))dt + \sum_k (j_t(\theta_{1k}(X))dA_k + j_t(\theta_{2k}(X))dA_k^*) --- (161)$$

where

$$\theta_0(X) = i[H,X] - (PX+XP) + \sum_l L_k X L_k^*$$

$$= i[H,X] - (1/2)\sum_k (L_k L_k^* X + X L_k L_k^* - 2L_k X L_k^*) --- (162)$$

$$\theta_{1k}(X) = [X, L_k], \theta_{2k}(X) = [L_k^*, X] --- (163)$$

We make several non-demolition measurements (V.P.Belavkin)

$$Y_{ok}(t) = U(t)^* Y_{ik}(t) U(t), Y_{ik}(t) = c(k)A_k(t) + \bar{c}(k)A_k(t)^*, k = 1,2,...,p --- (164)$$

A simple calculation shows that

$$dY_{ok}(t) = c(k)dA_k + \bar{c}(k)dA_k^* - j_t(\bar{c}(k)L_k + c(k)L_k^*)dt, k = 1,2,...,p --- (165)$$

In other words, these measurements correspond to measuring the signals $-(c(k)L_k + \bar{c}(k)L_k^*)$ after unitary evolution plus noise. The output measurement Abelian algebra is given by

$$\eta_o(t) = \sigma(Y_{ok}(s) : s \le t, k = 1,2,...,p) --- (166)$$

The quantum filter is

$$\pi_t(X) = \mathbb{E}(j_t(X)|\eta_o(t)) - - - (167)$$

We define the Abelian process $C(t)$ via the qsde

$$dC(t) = \sum_k f_k(t)C(t)dY_{ok}(t), C(0) = 1 - - - (168)$$

Then, the orthogonality principle for conditional expectations yields

$$\mathbb{E}((j_t(X) - \pi_t(X))C(t)) = 0 - - - (169)$$

Taking differentials of this equation and using the arbitrariness of the functions $f_k(t)$ gives us

$$\mathbb{E}((dj_t(X) - d\pi_t(X))|\eta_o(t)) = 0 - - - (170)$$

$$\mathbb{E}((dj_t(X)-d\pi_t(X))dY_{ok}(t)|\eta_o(t))+\mathbb{E}((j_t(X)-\pi_t(X))dY_{ok}(t)|\eta_o(t)) = 0---(171)$$

Here, conditional expectations are taken with the bath in the coherent state

$$|\phi(u) >= exp(-(1/2) \parallel u \parallel^2)|e(u) > - - -(172)$$

where

$$u = (u_k(t) : k = 1, 2, ..., p, t \in \mathbb{R}_+) - - - (173)$$

We may assume

$$d\pi_t(X) = F_t(X)dt + \sum_k G_{kt}(X)dY_{ok}(t) - - - (174)$$

since $dY_{ok}(t)^2 = |c(k)|^2dt$. The orthogonality equations give us

$$\pi_t(\theta_0(X))+\sum_k(u_k(t)\pi_t(\theta_{1k}(X))+\bar{u}_k(t)\pi_t(\theta_{2k}(X))-F_t(X)-\sum_k(c(k)u_k(t)+\bar{c}(k)\bar{u}_k(t)-\pi_t(\bar{c}(k)L_k+c(k)L_k^*))G_{kt}($$

$$\pi_t(X)\pi_t(\bar{c}(k)L_k+c(k)L_k^*)-\pi_t(X(\bar{c}(k)L_k+c(k)L_k^*))+\bar{c}(k)\pi_t(\theta_{1k}(X))-|c(k)|^2G_{kt}(X) = 0, k = 1, 2, ..., p---(176$$

The second equation gives us

$$|c(k)|^2G_{kt}(X) = \pi_t(X)\pi_t(M_k+M_k^*)-\pi_t(M_kX+XM_k^*), M_k = \bar{c}(k)L_k---(177)$$

Defining the following time varying system operators

$$N_k(t) = -M_k + c(k)u_k(t) = -\bar{c}(k)L_k + c(k)u_k(t), k = 1, 2, ..., p - - - (178)$$

we can finally express the Belavkin quantum filter in the form

$$d\pi_t(X) = \pi_t(\theta_0(X) + \sum_k(u_k(t)\theta_{1k}(X) + \bar{u}_k(t)\theta_{2k}(X)))dt$$

$$+\sum_k |c(k)|^{-2}[\pi_t(N_k(t)X+XN_k(t)^*)-\pi_t(X)\pi_t(N_k(t)+N_k(t)^*)][dY_{ok}(t)-\pi_t(N_k(t)+N_k(t)^*)dt]---(179)$$

Taking the dual of this equation by defining

$$\pi_t(X) = Tr(\rho_B(t)X) --- (180)$$

we get the following Belavkin equation for the filtered state $\rho_B(t)$ after choosing $|c(k)| = 1$(without any loss of generality):

$$d\rho_B(t) = \theta_0^*(\rho_B(t)) + \sum_k (u_k(t)\theta_{1k}^*(\rho_B(t)) + \bar{u}_k(t)\theta_{2k}^*(\rho_B(t)))$$

$$+\sum_k [\rho_B(t)N_k(t)+N_k(t)^*\rho_B(t)-Tr(\rho_B(t)(N_k(t)+N_k(t)^*))\rho_B(t)].[dY_{ok}(t)-Tr(\rho_B(t)(N_k(t)+N_k(t)^*))dt]---(1$$

In order to derive the quantum Boltzmann equation from this, we shall assume that

$$N_k(t) = \sum_j N_{kj}(t)^{\otimes N} --- (182)$$

where $N$ is the number of particles in the system. This amounts to saying that the system Hilbert space is $\mathcal{H}_1^{\otimes N}$ where $\mathcal{H}_1$ is the one particle Hilbert space. Henceforth, we shall denote the $N$-particle Belavkin filtered state $\rho_B(t)$ by simply $\rho(t)$ and $\rho_{12...r}(t)$ is the $r$-particle filtered state:

$$\rho_{12...r}(t) = Tr_{r+1...N}\rho(t) --- (183)$$

We can write the quantum Boltzmann decomposition of $\rho(t)$ as

$$\rho(t) = \rho_1(t)^{\otimes N} + \sum_{r=2}^{N} \sum \rho_1(t)^{\otimes N-r} \otimes g_{12...r}(t) --- (184)$$

where the inner sum here is over all particle permutations, ie, over all the $\binom{N}{r}$ ways of partitioning $\{1, 2, ..., N\}$ into two subsets, the first containing $N - r$ elements and the second containing $r$ elements. More precisely, this inner sum is to be interpreted as

$$\sum \rho_1(t)^{\otimes N-r} \otimes g_{12...r}(t)$$

$$= \sum_{1\leq i_1<i_2<...<i_r\leq N} g_{i_1 i_2 ... i_r}(t) \otimes (\rho_{i_{r+1}}(t) \otimes ... \otimes \rho_{i_N}(t)) --- (185)$$

where for any given $r = 2, ..., N$, $g_{i_1 i_2 ... i_N}(t)$ is a copy of $g_{12...r}(t)$, but acting in the Hilbert space $\otimes_{j=1}^r \mathcal{H}_{i_j}$ and of course for any $i = 1, 2, ..., N$, $\rho_i$ is a copy of $\rho_1$ but acts in $\mathcal{H}_i$. Note that the above assumption on the structure of $N_k(t)$ is equivalent to saying that the bath acts in an indistinguishable way on all the particles. This will be true if

$$H = \sum_k H_k + \sum_{1\leq k<j\leq N} V_{kj} --- (186)$$

$$L_k = \sum_j L_{kj}^{\otimes N}, u(t) = \bigoplus_{k=1}^{N} u_k(t) - - - (187)$$

The last assumption implies that the coherent state $|e(u)>$ is given by

$$|e(u)>=|e(\oplus u_k)>=\bigotimes_{k=1}^{N} |e(u_k)> - - -(188)$$

where the $u_k's$ are the same for all $k$.

Note that for consistency in the form of $\rho$, ie, the $(r+1)^{th}$ marginal $\rho_{12...r+1}$ should induce the $r^{th}$ marginal $\rho_{12...r}$ on partial tracing out over $\mathcal{H}_{r+1}$, we require that

$$Tr_{r+1}g_{12...r+1} = 0, r = 1, 2, ..., N-1 - - - (189)$$

It is clear that

$$\rho_{12...r} = Tr_{r+1...N}\rho$$

$$= \rho_1^{\otimes r} + Tr_{r+1...N} \sum_{2\leq k\leq N} \sum_{(i_1,...,i_k)\subset(1,2,...,N)} g_{i_1...i_k} \otimes \rho_{i_{k+1}} \otimes ... \otimes \rho_{i_N}$$

$$= \rho_1^{\otimes r} + Tr_{r+1...N} \sum_{2\leq k\leq N-r} g_{i_1...i_k} \otimes \rho_{i_{k+1}} \otimes ... \otimes \rho_{i_N} - - - (190)$$

because if $k > N - r$, then

$$Tr_{r+1...N}g_{i_1...i_k} \otimes \rho_{i_{k+1}} \otimes ...\rho_{i_N} = 0 - - - (191)$$

because this partial $(N-r)^{th}$ order partial trace will involve at least one partial trace of $g_{i_1...i_k}$ since $k > N - r$, and any order partial trace of $g_{i_1...i_k}$ is zero.

We shall now make the calculations upto second order, ie, assume that $g_{12...r} = 0, 3 \leq r \leq N$. This amounts to saying that

$$\rho \approx \rho_1^{\otimes N} + \sum \rho_1^{\otimes N-2} \otimes g_{N-1,N} - - - (192)$$

where the sum is over all two element sets obtained by $(N-1, N)$ by general two point subsets $(i_1, i_2)i_1 < i_2$. The relevant first and second marginal qbe's are obtained by subsitituting this expression for the joint state into the Belavkin filter

$$d\rho(t) = \theta_0^*(\rho(t)) + \sum_k (u_k(t)\theta_{1k}^*(\rho(t)) + \bar{u}_k(t)\theta_{2k}^*(\rho(t)))$$

$$+\sum_k [\rho(t)N_k(t)+N_k(t)^*\rho_B(t)-Tr(\rho(t)(N_k(t)+N_k(t)^*))\rho(t)].[dY_{ok}(t)-Tr(\rho(t)(N_k(t)+N_k(t)^*))dt]---(193)$$

and taking $(N-1)^{th}$ and $N^{th}$ order partial traces of the resulting equation. We evaluate the different terms:

$$Tr_{23...N}d(\rho) = Tr_{23...N} \sum_{k=0}^{N-1} \rho_1^{\otimes k} \otimes d\rho_1 \otimes \rho^{N-k-1} = d\rho_1 - - - (194)$$

since $Trd\rho_1 = 0$. Next,

$$Tr_{34...N}d\rho = d\rho_{12} = d(\rho_1^{\otimes 2}) + g_{12}) = d\rho_1 \otimes \rho_1 + \rho_1 \otimes d\rho_1 + dg_{12} - - - (195)$$

Next,

$$Tr_{23....N}\theta_0^*(\rho) = (-1/2)Tr_{23..N}\sum_k (L_k L_k^* \rho + \rho L_k L_k^* - 2L_k^* \rho L_k) - - - (196)$$

Now,

$$Tr_{23...N}(L_k L_k^* \rho) = Tr_{23...N}\sum_{j,m}(L_{kj}L_{km}^*)^{\otimes N}(\rho_1^{\otimes N} + \rho_1^{\otimes N-2} \otimes g_{12} + ...)$$

$$= \sum_{jm}(Tr(L_{kj}L_{km}^*\rho_1))^{N-1})L_{kj}L_{km}^*\rho_1 + (N-1)\sum_{jm}(Tr(L_{kj}L_{km}^*\rho_1))^{N-2}Tr_2((L_{kj}L_{km}^*)^{\otimes 2}g_{12}) - - - (197)$$

Likewise,

$$Tr_{23...N}(\rho L_k L_k^*)$$

$$= \sum_{jm}(Tr(\rho_1(L_{kj}L_{km}^*))^{N-1})\rho_1 L_{kj}L_{km}^* + (N-1)\sum_{jm}(Tr(\rho_1 L_{kj}L_{km}^*)^{N-2}Tr_2(g_{12}(L_{kj}L_{km}^*)^{\otimes 2}) - - - (198)$$

and

$$Tr_{23...N}(L_k^* \rho L_k^*)$$

$$= \sum_{jm}(Tr(L_{km}^*\rho_1 L_{kj}))^{N-1})L_{km}^*\rho_1 L_{kj} + (N-1)\sum_{jm}(Tr(L_{km}^*\rho_1 L_{kj})^{N-2}Tr_2((L_{km}^*)^{\otimes 2}g_{12}(L_{kj})^{\otimes 2}) - - - (199)$$

Combining these three formulas, we get

$$Tr_{23...N}(\theta_0^*(\rho)) = (-1/2)Tr_{23...N}\sum_k (L_k L_k^* \rho + \rho L_k L_k^* - 2L_k^* \rho L_k) =$$

$$-(1/2)\sum_{kjm}(Tr(L_{kj}L_{km}^*\rho_1))^{N-1})(L_{kj}L_{km}^*\rho_1 + \rho_1 L_{kj}L_{km}^* - 2L_{km}^*\rho_1 L_{kj})$$

$$-((N-1)/2)\sum_{kjm}(Tr(L_{kj}L_{km}^*\rho_1))^{N-2}(Tr_2((L_{kj}L_{km}^*)^{\otimes 2}g_{12}) + Tr_2(g_{12}(L_{kj}L_{km}^*)^{\otimes 2}) - 2.Tr_2((L_{km}^*)^{\otimes 2}g_{12}(L_{kj})^{\otimes 2}))$$

Again, we find that

$$Tr_{34...N}(L_k L_k^* \rho) = \sum_{jm}(Tr(L_{kj}L_{km}^*\rho_1)^{N-2}((L_{kj}L_{km}^*)^{\otimes 2})\rho_1^{\otimes 2})$$

$$+(N-2)(Tr(L_{kj}L_{km}^*\rho_1))^{N-3}Tr_3((L_{kj}L_{km}^*)^{\otimes 3}(\rho_1 \otimes g_{23} + \rho_2 \otimes g_{13} + \rho_3 \otimes g_{12})) - - - (201)$$

and hence by an obvious extension, we get

$$Tr_{34...N}\theta_0^*(\rho) =$$

$$-(1/2)\sum_{kjm}Tr(L_{kj}L_{km}^*\rho_1)^{N-2}((L_{kj}L_{km}^*)^{\otimes 2}\rho_1^{\otimes 2} + \rho_1^{\otimes 2}(L_{kj}L_{km}^*)^{\otimes 2}$$

$$-2(L_{km}^*)^{\otimes 2}\rho_1^{\otimes 2}.L_{kj}^{\otimes 2})$$

$$-((N-2)/2)\sum_{kjm}(Tr(L_{kj}L_{km}^*\rho_1))^{N-3}.Tr_3((L_{kj}L_{km}^*)^{\otimes 3}(\rho_1\otimes g_{23}+\rho_2\otimes g_{13}+\rho_3\otimes g_{12})$$

$$+(\rho_1\otimes g_{23}+\rho_2\otimes g_{13}+\rho_3\otimes g_{12})(L_{kj}L_{km}^*)^{\otimes 3}$$

$$-2.(L_{km}^*)^{\otimes 3}(\rho_1\otimes g_{23}+\rho_2\otimes g_{13}+\rho_3\otimes g_{12})(L_{kj})^{\otimes 3})---(202)$$

Proceeding along these lines, it becomes clear how to evaluate the partial traces of the original Belavkin filter equation with the second order approximation. The calculations are lengthy but straightforward if we proceed along the lines indicated.

## 0.14 Conclusions

We have discussed several aspects of the classical and quantum Boltzmann equations in the context of identical electromagnetically interacting charges and have applied it to the problems of computing the scattered electromagnetic field by the charged particles when a field in incident upon the particles in a permutation invariant way. We have also discussed the quantum Boltzmann equation for the Belavkin quantum filter based on non-demolition measurements again when the noisy bath acts symmetrically on all the system particles. We have compared the quantum Boltzmann equation with the classical Boltzmann equation by replacing the quantum density operator in the position representation by its Wigner transform in the particle phase space. Corrections in powers of Planck's constant appear in the quantum Boltzmann equation that are not present in the classical Boltzmann equation.

## 0.15 Acknowledgements

I would like to thank Professor Anand Srivastava, Vice Chancellor, NSUT, for suggesting to me the problem of analyzing the frequency content of the scattered field by a phonon lattice using quantum field theory.