

Research Article

Non-Well-Founded Hierarchies of Nested Systems

Igor Strozzi¹

1. Department of Applied Mathematics, Institute of Mathematics, Universidade Federal do Rio de Janeiro, Brazil

We develop a set-theoretic framework for systems organised across scales of description. A system is a set, its carrier, partitioned into elements and a passive medium, together with valued relations among its elements. Dependence between systems is formalised as *generation* over a set of admissible configurations: one system is generated by another when its relational data factor through the other's. We show that generation is characterised by inclusion of equivalence kernels and that, evaluated at a single configuration, it is vacuous; functional dependence between levels is a property of variables, not of values. On this basis we define subsystems, enclosure (parthood together with membership), superlevels as aggregations whose data are generated from the level below, and hierarchies as level-graded families of systems with partial level transitions. We prove that level transitions are strict, that enclosure is anti-transitive, and that an up-then-down round trip recovers a system up to interderivability if and only if its aggregation is faithful on the admissible configurations. Non-well-founded descent, by enclosure or by refinement of levels, is consistent with ordinary set theory but forces infinite carriers; for finite carriers we give sharp bounds on the length of descent. A dyadic hierarchy shows that bi-infinite hierarchies exist.

Corresponding author: Igor Strozzi, istrozzi@matematica.ufrj.br

1. Introduction

As our knowledge about reality evolves, we tend to perceive intricacy where we once saw simplicity. An improved comprehension often leads to subtler questions about the phenomena under study, and, hopefully, to more profound answers. Of particular interest is the now widely known fact that non-trivial interrelations between even simple entities can make their global behaviour hard to understand, let alone predict, sometimes to the point that this behaviour seems to be of an entirely new nature. The problem of

consciousness is the classical instance: behaviour that may ultimately arise from interacting matter is so unexpected that it has historically motivated explanations appealing to something beyond matter.

The recognition that such problems are ubiquitous led to the development of systems and complexity sciences, which ask, among other things, how an ensemble of entities can give rise to something qualitatively different from its constituents. Some of the obstacles these sciences face seem fundamental; one of them is that complex systems are often hard to treat with the conventional tools used for closely related problems, such as those of dynamical systems theory. That inadequacy, together with the growth of computational power, made simulation one of the main instruments for studying complex systems. It also makes a general theory of complex systems, within the bounds of reason, increasingly necessary: simulation without a theory of what is being simulated cannot tell us which of its observations are structural.

1.1. Motivation and scope

Distributed systems are increasingly present in contemporary society, and their growing complexity is one of the motivations behind frameworks such as *holonic multiagent systems* ^[1]. Our motivation is different: we want tools for understanding complex systems in general, as a contribution to a general and formal theory. More specifically, we are oriented towards the idea of *emergence*, and towards a formalism in which the dependence of coarser descriptions on finer ones can be stated, and its failure detected, precisely. Since simulation is a core instrument of complexity science, we want the formalism to be as close as possible to computational models: the same objects used to analyse a property of a system should be usable, as directly as possible, to implement a model exhibiting that property. Artificial life systems, which provide a virtually unlimited zoology of examples, are a guiding application.

1.2. Approach

Our initial goal was to study nested, hierarchical systems in which subsystems are constrained to be ‘similar’, in a sense to be made precise elsewhere, to their supersystems, and to use that constraint as a parameter both for analysis and for modelling. Laying the foundations required for that study proved longer than anticipated, and this paper is that groundwork. The original goal still motivates several choices below, notably the insistence that the relation between a system and its parts be expressed through maps between their data rather than through inheritance of relations. We are also biased

towards dynamical systems theory, which is well developed and close to complexity theory; the present paper is static, and dynamics is deferred.

1.3. Guiding picture: spatial systems

The definitions are abstracted from physical systems. Every system *occupies* a region, its carrier, and its elements are subregions; a system is not, however, identified with its region, since distinct systems may share a carrier (they may differ in their elements, relations or values). The scale parameter is most naturally read as spatial resolution, and passing between levels as zooming in or out, which explains some of the terminology. The carrier is a bare set: topology, measure or geometry, when a model needs them, enter as relations and valuations. We therefore expect the framework to apply without change to systems that do not live in physical space, although we do not pursue such applications here.

1.4. Contributions

- Systems with elements and a medium, carrying valued relations (Section 3).
- *Generation* over a set Ω of admissible configurations, characterised by kernel inclusion (Lemma 1); on a single configuration every system is generated by every other (Remark 2).
- Subsystems as a structural preorder, enclosure as an anti-transitive one-level parthood (Lemma 3), and superlevels as aggregations generated from below, whose parts are enclosed by construction (Lemma 4).
- Hierarchies with partial level transitions; strictness of transitions, conservativity over relational structures and finite multilevel chains, and a round-trip theorem characterising faithful aggregation (Proposition 4).
- Non-well-founded descent in ZFC; sharp bounds for finite carriers (Corollary 2, Proposition 5); passage from level descent to enclosure descent (Proposition 6); a bi-infinite dyadic hierarchy (Example 1).

2. Preliminaries

2.1. Notation

If S is a set whose elements are sets, $\bigcup S = \{e \mid e \in x \text{ for some } x \in S\}$. A family $\{F_i\}_{i \in I}$ may be written $\{F_i\}$ when I is clear. $\mathcal{P}(X)$ is the power set of X ; $f : X \rightharpoonup Y$ denotes a partial map. For a map f with domain X , its *kernel* is the equivalence relation

$$\ker f = \{(x, x') \in X \times X \mid f(x) = f(x')\}.$$

2.2. Model-theoretic background

Following ^[2], a signature σ consists of constant, function and relation symbols, each function and relation symbol with an arity; a σ -structure with universe E interprets constants as elements of E , m -ary function symbols as maps $E^m \rightarrow E$, and m -ary relation symbols as subsets of E^m . A *relational structure* is one whose signature contains only relation symbols, possibly infinitely many. Our notion of system departs from this in three respects: relational tuples carry values (valuations), the objects of a system are parts of a carrier (blocks of a partition) rather than points, with a remainder that is not an object (the medium), and relations are not tied to a fixed linguistic signature. The model-theoretic approach privileges the interpretation of a language; ours privileges the compositional organisation of wholes, which is what inter-level dependence acts on.

3. Systems

We shall use very simple examples, typically systems of particles. Their simplicity makes the passage between levels transparent: a system of particles is made of particles, and each particle may in turn be regarded as made of further particles. The entities need not have rich behaviour of their own for this nesting to be non-trivial.

3.1. Valued relations

We consider a system, in the established sense, as ‘a set of interrelated objects’ ^[3], but formalise it differently. In two-valued first-order logic a relation either holds or fails; letting relations carry arbitrary values at the level of the language would require logics whose truth values are as varied as the values of interactions, and in principle a different logic for each system. We keep ordinary logic and attach values one level up, to relations taken in the usual mathematical sense.

Definition 1 (Valued relation). Let $n \in \mathbb{N}$, let $S_1, \dots, S_n$ be sets and $R \subseteq \prod_{i=1}^n S_i$ (for $n = 0$, $R \subseteq \{()\}$). Let V be a set, the *set of values*. A map $v : R \rightarrow V$ is a *valuation* of R ; the triple $R_v = (R, V, v)$ is an n -ary *valued relation*, and R is its *underlying relation*. We write $v(R) = \{v(r) \mid r \in R\}$.

An n -ary valued relation is thus an $(n + 1)$ -ary relation functional in its last coordinate, whose last coordinate is typically of a different nature from the others. The underlying relation records *which* tuples

interact; the valuation records *how*: strength, sign, delay, probability, or a qualitative label. A weighted digraph is the basic example.

3.2. The carrier: a mereological commitment

A system is built on a set, its *carrier*, whose points are not the objects of the system. The objects, or elements, are disjoint subsets of the carrier. In the spatial picture the carrier is a region and the elements are subregions with properties, the system being these elements together with their properties and relations. The nature of the points is of little importance; what matters is what is built on them. This choice emphasises the mereological aspects of systems ^[4], and it lets hierarchies be defined through inclusion rather than membership, which is what makes measure-theoretic and topological tools available when needed and, as we shall see, what makes infinite descent harmless (Lemma 6).

3.3. Formal definition

Definition 2 (System). Let S be a set and $\mathfrak{P}$ a partition of S with at least two blocks, and let $\Gamma_{\mathfrak{P}} \subseteq \mathfrak{P}$ be non-empty. Fix a finite set $K \subset \mathbb{N}$ of arities. For each $k \in K$ let $\mathbf{R}_k = (R_{k,i})_{i \in I_k}$ be a finite indexed family of k -ary relations on $\Gamma_{\mathfrak{P}}$, i.e. $R_{k,i} \subseteq \Gamma_{\mathfrak{P}}^k$, and put $\mathbf{R} = (\mathbf{R}_k)_{k \in K}$. For each label (k, i) let $\mathbf{V}_{k,i}$ be a (possibly empty) indexed family of valuations $v : R_{k,i} \rightarrow V_v$, and put $\mathbf{V} = (\mathbf{V}_{k,i})_{k,i}$. A *system* is a quadruple

$$\mathcal{S} = (S, \Gamma_{\mathfrak{P}}, \mathbf{R}, \mathbf{V}),$$

the subscript recording the ambient partition. We write $u(\mathcal{S}) = S$ for the *carrier*, $e(\mathcal{S}) = \Gamma_{\mathfrak{P}}$ for the set of *elements*, $m(\mathcal{S}) = \mathfrak{P} \setminus \Gamma_{\mathfrak{P}}$ for the *medium*, $r(\mathcal{S}) = \mathbf{R}$ and $v(\mathcal{S}) = \mathbf{V}$. Thus

$$S = \left(\bigcup e(\mathcal{S}) \right) \sqcup \left(\bigcup m(\mathcal{S}) \right).$$

The *data* of $\mathcal{S}$ is the labelled tuple $\mathfrak{d}(\mathcal{S}) = (\mathbf{R}, \mathbf{V})$ of all its relations and valuations. A set s is an element of $\mathcal{S}$, written $s \in \mathcal{S}$, iff $s \in \Gamma_{\mathfrak{P}}$.

Relations and valuations are labelled (indexed) so that two distinct labels may carry equal sets; for instance, the empty relation may occur with several arities. The medium carries no data: it is the part of the carrier that the partition resolves but that the system does not treat as an object.

Definition 3 (Restriction). Let $\mathcal{S}$ be a system and $E \subseteq e(\mathcal{S})$ with $|E| \geq 2$. The *restriction* $\mathcal{S}|_E$ is the system with carrier $\bigcup E$, partition and element set E (no medium), relations $R_{k,i} \cap E^k$ under the same labels, and valuations restricted accordingly.

3.4. Remarks

Why valued relations. A property of an object is naturally a value assigned to it; in our definition it is a valuation of a unary relation. Since a valuation of a unary relation on $\Gamma_{\mathfrak{P}}$ is just a partial function on $\Gamma_{\mathfrak{P}}$, this costs nothing, and the same holds in every arity. A 0-ary relation $\{()\}$ with a valuation is a global parameter of the system. What is gained is the ability to speak of relations with values while keeping two separate sets, one of incidences and one of values. One could reduce everything to relations (a function is a functional relation) or to functions (a relation is its characteristic function); keeping incidence and value apart is a matter of convenience, which is also what justifies structures in model theory. It has a computational counterpart: two valued relations with the same incidence share their domain, and a data structure that stores the incidence once and the valuations separately avoids redundant storage.

Remark 1 (On the cardinality constraint). We require at least two blocks, not merely $|S| \geq 2$: a two-point set with the trivial partition is a one-block system. One of the two blocks may be medium, so a single element in an environment is a system (this is used in Section 5.6). The requirement reflects the view that systems are relational: a whole with no internal distinction is, at best, a degenerate case. One may object that an isolated particle is a legitimate system; we would answer that its internal structure (spin, charge, extension), once accounted for, supplies the required distinctions. A consequence used repeatedly below is that *every element of a system is a proper subset of its carrier*.

4. Generation, subsystems and enclosure

4.1. Beyond inherited relations

It is common to take a subsystem of $\mathcal{S}$ to be a system on a subset of the carrier whose relations are inherited from, or compatible with, those of $\mathcal{S}$ [5]. This is inadequate for our purposes, since it excludes the richer relations between levels that emergence requires. A coarser description may carry rescaled versions of finer properties, or fewer properties, or entirely different ones. Consider regions of a gas: rather than the kinetic properties of molecules, one may keep only the temperature of each region. The relevant properties at the macroscopic level (regions) differ from those at the mesoscopic level (intermolecular dynamics), which in turn differ from those at the microscopic level (intramolecular dynamics). The temperature of a region is a function of the velocities of the molecules within it; in that sense the velocities suffice. But we want to account explicitly for this passage, i.e. to carry the map from velocities to temperature as part of the description, and we want to relate all three levels within one

structure. The passage between descriptions will therefore be expressed through maps between data. The first task is to make ‘the data of one system are a function of the data of another’ a non-trivial statement.

4.2. Configurations and generation

Definition 4 (Systems over Ω). Fix a non-empty set Ω of *admissible configurations*: states, scenarios, parameter values, or any range over which a model’s data are allowed to vary. A *system over Ω* is a family $\mathcal{S} = (\mathcal{S}_\omega)_{\omega \in \Omega}$ of systems sharing carrier, partition, element set and labels; only the incidence of relations and the values of valuations may depend on ω . Its *data map* is

$$\mathfrak{d}_{\mathcal{S}} : \Omega \rightarrow \mathfrak{D}_{\mathcal{S}}, \quad \omega \mapsto \mathfrak{d}(\mathcal{S}_\omega),$$

where $\mathfrak{D}_{\mathcal{S}}$ is the set of labelled tuples of the appropriate sorts. Structural notions (u, e, m, Ξ , restriction) are those of any $\mathcal{S}_\omega$. For a family $\mathfrak{A} = \{\mathcal{A}_j\}_{j \in J}$ of systems over Ω , the joint data map is $\mathfrak{d}_{\mathfrak{A}}(\omega) = (\mathfrak{d}_{\mathcal{A}_j}(\omega))_{j \in J}$. A single system is a system over a singleton.

From now on, unless stated otherwise, all systems are systems over a fixed Ω .

Definition 5 (Generation). Let $\mathcal{A}$ (a system or a family of systems) and $\mathcal{B}$ be over Ω . We say $\mathcal{B}$ is *generated by $\mathcal{A}$* , written $\mathcal{A} \rightsquigarrow \mathcal{B}$, if there is a map $\Phi : \mathfrak{d}_{\mathcal{A}}(\Omega) \rightarrow \mathfrak{D}_{\mathcal{B}}$, the *generator*, with

$$\mathfrak{d}_{\mathcal{B}} = \Phi \circ \mathfrak{d}_{\mathcal{A}} \quad \text{on } \Omega.$$

The generator acts coordinatewise: each labelled relation and each labelled valuation of $\mathcal{B}$ is a coordinate of Φ , and may depend on any sub-tuple of the data of $\mathcal{A}$, including global parameters (0-ary data) and relations of any arity. In particular the incidence of a relation and its valuations may be generated from different data: two global coupling constants share their (unary or 0-ary) incidence but may depend on entirely different features of $\mathcal{A}$. No restriction is placed on Φ beyond its existence; a model may impose one (linearity, locality, a class of admissible maps closed under composition) without changing anything below.

Lemma 1 (Kernel criterion). $\mathcal{A} \rightsquigarrow \mathcal{B}$ if and only if $\ker \mathfrak{d}_{\mathcal{A}} \subseteq \ker \mathfrak{d}_{\mathcal{B}}$. In that case the generator is unique, and it is injective if and only if $\ker \mathfrak{d}_{\mathcal{A}} = \ker \mathfrak{d}_{\mathcal{B}}$.

Proof. If $\mathfrak{d}_{\mathcal{B}} = \Phi \circ \mathfrak{d}_{\mathcal{A}}$ and $\mathfrak{d}_{\mathcal{A}}(\omega) = \mathfrak{d}_{\mathcal{A}}(\omega')$, then $\mathfrak{d}_{\mathcal{B}}(\omega) = \mathfrak{d}_{\mathcal{B}}(\omega')$. Conversely, under kernel inclusion $\Phi(\mathfrak{d}_{\mathcal{A}}(\omega)) := \mathfrak{d}_{\mathcal{B}}(\omega)$ is well defined on $\mathfrak{d}_{\mathcal{A}}(\Omega)$. Any generator is forced to take these values, so it is

unique. If the kernels are equal and $\Phi(\mathfrak{d}_{\mathcal{A}}(\omega)) = \Phi(\mathfrak{d}_{\mathcal{A}}(\omega'))$, then $(\omega, \omega') \in \ker \mathfrak{d}_{\mathcal{B}} = \ker \mathfrak{d}_{\mathcal{A}}$, so Φ is injective; if Φ is injective, $\mathfrak{d}_{\mathcal{B}}(\omega) = \mathfrak{d}_{\mathcal{B}}(\omega')$ gives $\mathfrak{d}_{\mathcal{A}}(\omega) = \mathfrak{d}_{\mathcal{A}}(\omega')$.

Corollary 1. *Generation is a preorder on systems over Ω , and $\mathcal{A} \rightsquigarrow \mathcal{B} \rightsquigarrow \mathcal{A}$ iff $\ker \mathfrak{d}_{\mathcal{A}} = \ker \mathfrak{d}_{\mathcal{B}}$, iff the generator is a bijection $\mathfrak{d}_{\mathcal{A}}(\Omega) \rightarrow \mathfrak{d}_{\mathcal{B}}(\Omega)$. We then call $\mathcal{A}$ and $\mathcal{B}$ inter-derivable.*

Remark 2 (Dependence is a property of variables, not of values). If $|\Omega| = 1$ every kernel is $\Omega \times \Omega$, so every system is generated by every other, through a constant map. More generally, data that are constant on Ω are generated by anything. A definition requiring only that the relations of one system ‘be functions of’ those of another, evaluated at one configuration, is therefore vacuous: any relation is the value of a constant function. A functional dependence is a claim about how data co-vary, and Ω makes explicit the range over which it is claimed. Its choice is a modelling decision (all states reachable under a dynamics, all parameter values of a study, all initial conditions of an ensemble), and every statement below is relative to it. When Ω carries σ -algebras and generators are required to be measurable, Lemma 1 becomes the Doob–Dynkin factorisation lemma.

4.3. Subsystems and supersystems

Definition 6 (Subsystem). $\mathcal{S}'$ is a subsystem of $\mathcal{S}$, written $\mathcal{S}' \sqsubseteq \mathcal{S}$, if $u(\mathcal{S}') \subseteq u(\mathcal{S})$ and every element of $\mathcal{S}'$ is contained in an element of $\mathcal{S}$. It is a strict subsystem, $\mathcal{S}' \sqsubset \mathcal{S}$, if moreover $\mathcal{S} \not\sqsubseteq \mathcal{S}'$.

In the spatial picture, a subsystem occupies a subregion and its objects are pieces of objects of the supersystem. The relation is purely structural, hence independent of ω .

Lemma 2 (Structure of $\sqsubseteq$). $\sqsubseteq$ is a preorder; $\mathcal{S} \sqsubseteq \mathcal{S}' \sqsubseteq \mathcal{S}$ iff $\mathcal{S}$ and $\mathcal{S}'$ have the same carrier and the same elements; and $\sqsubset$ is a strict partial order.

Proof. Reflexivity and transitivity are immediate. Suppose $\mathcal{S} \sqsubseteq \mathcal{S}' \sqsubseteq \mathcal{S}$. The carriers coincide. If $b \in \mathfrak{e}(\mathcal{S})$, then $b \subseteq b' \subseteq b''$ with $b' \in \mathfrak{e}(\mathcal{S}')$ and $b'' \in \mathfrak{e}(\mathcal{S})$; since b, b'' are intersecting blocks of one partition, $b = b''$, hence $b = b' \in \mathfrak{e}(\mathcal{S}')$. By symmetry the element sets coincide; the converse is clear. The strict part of a preorder is a strict partial order.

Two systems that differ only in their relations, valuations or medium are thus $\sqsubseteq$ -equivalent, and neither is a strict subsystem of the other. (Requiring merely $\mathcal{S}' \neq \mathcal{S}$ in the definition of a strict subsystem would not give a transitive relation: such a pair would satisfy $\mathcal{S} \sqsubset \mathcal{S}' \sqsubset \mathcal{S}$.)

Remark 3 (Why dependence is not part of the subsystem relation). One may additionally ask that a subsystem’s data be generated by its supersystem’s, $\mathcal{S} \rightsquigarrow \mathcal{S}'$, for instance when parts respond to a

global field. We call such a subsystem *downward determined* and keep this as a separate, optional relation. Building it into $\sqsubseteq$ would make parthood incompatible with lossy aggregation: in the example of Section 5.6, the parts of a system are not generated by the system they compose, because the whole forgets how a total is split among them. The dependence that the passage between levels requires runs the other way, from the finer level to the coarser one, and it is carried by the level transitions of Section 5.

Definition 7 (Supersystem). Let $\mathfrak{S} = \{\mathcal{S}_i\}_{i \in I}$ be a family of systems. A system $\mathcal{S}$ is a *supersystem* of $\mathfrak{S}$ if $\mathcal{S}_i \sqsubseteq \mathcal{S}$ and $\mathfrak{e}(\mathcal{S}_i) \subseteq \mathfrak{e}(\mathcal{S})$ for every i , and $\mathfrak{S} \rightsquigarrow \mathcal{S}$.

A supersystem keeps the objects of its subsystems and is generated by their joint data; it describes them at the same grain, together. Given a subsystem, its supersystem is far from unique: the carrier may be enlarged by arbitrary medium, and data constant on Ω may be added freely, so the supersystems of a family form a proper class. This non-uniqueness is intrinsic: when only a part is known, the whole is underdetermined.

4.4. Enclosure and nested systems

Following ^[6], a nested system is one that ‘is enclosed by, and is enclosing, other systems’. We retain only this structural content; in particular, we do not build into nestedness that enclosing systems emerge from the activity of enclosed ones, although such systems are our main interest. Enclosure requires, besides inclusion, that the enclosed system be *one of the objects* of the enclosing one.

Definition 8 (Enclosure). $\mathcal{S}$ *encloses* $\mathcal{S}'$, written $\mathcal{S}' \sqsubseteq \mathcal{S}$, if $\mathcal{S}' \sqsubseteq \mathcal{S}$ and $\mathfrak{u}(\mathcal{S}') \in \mathcal{S}$. We then call $\mathcal{S}'$ a *part* of $\mathcal{S}$.

A system that is merely a subsystem of $\mathcal{S}$ may play no role in the mesoscale relations of $\mathcal{S}$: it occupies a subregion without being an object. An enclosed system is itself, through its carrier, an object that the relations of $\mathcal{S}$ can address. This captures the intuition that the subsystems at one level become the objects of the next.

Lemma 3 (Enclosure). (a) If $\mathcal{S}' \sqsubseteq \mathcal{S}$, then $\mathfrak{u}(\mathcal{S}') \subsetneq \mathfrak{u}(\mathcal{S})$ and $\mathcal{S}' \sqsubset \mathcal{S}$. (b) Enclosure is irreflexive and anti-transitive: if $\mathcal{S}'' \sqsubseteq \mathcal{S}' \sqsubseteq \mathcal{S}$, then $\mathcal{S}'' \not\sqsubseteq \mathcal{S}$. Its transitive closure is contained in $\sqsubset$.

Proof. (a) $\mathfrak{u}(\mathcal{S}')$ is a block of a partition of $\mathfrak{u}(\mathcal{S})$ with at least two blocks, hence a proper subset (Remark 1); so $\mathfrak{u}(\mathcal{S}) \not\subseteq \mathfrak{u}(\mathcal{S}')$ and $\mathcal{S} \not\sqsubseteq \mathcal{S}'$. (b) Irreflexivity follows from (a). If $\mathcal{S}'' \sqsubseteq \mathcal{S}' \sqsubseteq \mathcal{S}$, then $\emptyset \neq \mathfrak{u}(\mathcal{S}'') \subsetneq \mathfrak{u}(\mathcal{S}') \in \mathfrak{e}(\mathcal{S})$, and a non-empty proper subset of a block is not a block of the same partition, so $\mathfrak{u}(\mathcal{S}'') \notin \mathfrak{e}(\mathcal{S})$. The last claim follows from (a) and transitivity of $\sqsubset$.

Enclosure is thus the immediate, one-level parthood, and its transitive closure the generic one. That parthood in this leveled sense fails to be transitive agrees with mereologies designed for emergent parts ^{[7][8]}; the generic relation $\sqsubseteq$ remains transitive.

Definition 9 (Nested system). A system $\mathcal{S}$ is *nested* if there are systems $\mathcal{R}$ and $\mathcal{T}$ with $\mathcal{R} \sqsubseteq \mathcal{S} \sqsubseteq \mathcal{T}$.

One could call a system nested whenever it has a subsystem and a supersystem. That falls short of the intent of ^[6]: the relations of a system are defined with respect to its elements, not to arbitrary subsystems, so nestedness should involve being an object of the larger system and having objects that are themselves systems. This is what ties the entities of one level to their counterparts above and below.

Remark 4 (Relation to holons). A holon, in the sense of Koestler ^[9], is simultaneously a whole and a part ^[11]. A nested system has exactly this dual role: it is composed of parts ($\mathcal{R} \sqsubseteq \mathcal{S}$) and is itself a part ($\mathcal{S} \sqsubseteq \mathcal{T}$). Iterating the pattern gives chains of enclosure that may continue in both directions; Section 6 studies when downward chains can be infinite.

5. Levels and hierarchies

5.1. Levels of description

By levels we mean levels of description, in essentially the sense of multi-strata systems ^[3], each level being a system in its own right ^[10]. A cell in a multicellular organism interacts, at the cellular level, with other cells and the extracellular space. Depending on the question one may need finer or coarser structures: the cell within its tissue tells one story, its organelles another, its molecules another still; and one can continue downwards to atoms and subatomic structure, or upwards to organs, organisms and beyond. Structures at distant scales need not be independent: the molecular make-up of a cell depends, faintly, on the chemical history of the region in which its planetary system formed.

Which levels are relevant is rarely clear in advance. As a heuristic, the distribution of discernible structures across scales can guide where to cut an infinite description: if the number of distinct structures, as a function of scale, has minima, these are natural places for a cut. This is one reason to consider a system across a wide range of scales from the outset.

5.2. Superlevels and sublevels

The passage to a superlevel aggregates elements into coarser elements and generates the coarser data from the finer ones.

Definition 10 (Aggregation; superlevel; sublevel). Let $\mathcal{S}$ and $\mathcal{S}^+$ be systems over Ω . An *aggregation* from $\mathcal{S}$ to $\mathcal{S}^+$ is a map $a : \mathfrak{e}(\mathcal{S}) \rightarrow \mathfrak{e}(\mathcal{S}^+)$ such that

- i. $e \subseteq a(e)$ for every $e \in \mathfrak{e}(\mathcal{S})$;
- ii. (tiling) $b = \bigcup a^{-1}(b)$ for every b in the image of a ;
- iii. a is not injective.

$\mathcal{S}^+$ is a *superlevel* of $\mathcal{S}$ if $u(\mathcal{S}) \subseteq u(\mathcal{S}^+)$, there is an aggregation from $\mathcal{S}$ to $\mathcal{S}^+$, and $\mathcal{S} \rightsquigarrow \mathcal{S}^+$. We write $\text{Sup}(\mathcal{S})$ for the class of superlevels of $\mathcal{S}$. Dually, $\mathcal{S}^-$ is a *sublevel* of $\mathcal{S}$ if $\mathcal{S} \in \text{Sup}(\mathcal{S}^-)$, and $\text{Sub}(\mathcal{S})$ is the class of sublevels. A superlevel is *exhaustive* if a is surjective. For b in the image of a with $|a^{-1}(b)| \geq 2$, the restriction $\mathcal{S}|_{a^{-1}(b)}$ is the *part* of $\mathcal{S}^+$ at b .

Since coarse elements are disjoint, condition (i) forces $a(e)$ to be the unique coarse element containing e : an aggregation, when it exists, is determined by the two element sets, and is a property rather than additional structure. Coarse elements outside the image of a , and the part of $u(\mathcal{S}^+)$ outside $u(\mathcal{S})$, are *added context*: the superlevel may have elements whose regions are not even contained in the carrier of the level below. Generation from $\mathcal{S}$ means that the coarse data are functions of the fine data, including the mesoscale relations among fine elements, and not only of the data internal to each part. The internal structure of an element of $\mathcal{S}$ reaches $\mathcal{S}^+$ only through the data of $\mathcal{S}$, typically through unary valuations (the temperature of a region); a model in which a superlevel reads finer data directly generates it from a lower level, and transitivity of $\rightsquigarrow$ handles the composite. Two levels may coincide in carrier and differ in their partition: the gas regions and the molecules occupy the same container.

Lemma 4 (Level transitions). Let $\mathcal{S}^+ \in \text{Sup}(\mathcal{S})$ with aggregation a . Then:

- a. $\mathcal{S} \sqsubset \mathcal{S}^+$ and $\mathcal{S} \rightsquigarrow \mathcal{S}^+$;
- b. at least one coarse element has a part, and every part is a system with carrier b satisfying $\mathcal{S}|_{a^{-1}(b)} \sqsubseteq \mathcal{S}^+$.

Proof. (a) $u(\mathcal{S}) \subseteq u(\mathcal{S}^+)$ and $e \subseteq a(e)$ give $\mathcal{S} \sqsubseteq \mathcal{S}^+$. Since a is not injective, there are $e_1 \neq e_2$ with $a(e_1) = a(e_2) = b$. If $b \subseteq c$ for some $c \in \mathfrak{e}(\mathcal{S})$, then $e_1, e_2 \subseteq c$, and since e_1, e_2, c are blocks of one partition, $e_1 = c = e_2$, a contradiction; hence $\mathcal{S}^+ \not\sqsubseteq \mathcal{S}$. Generation holds by definition. (b) Such a b exists by non-injectivity. The restriction to $E = a^{-1}(b)$, $|E| \geq 2$, is a system (Definition 3) with carrier

$\bigcup E = b$ by tiling. Its carrier is contained in $u(\mathcal{S}^+)$ and each of its elements e lies in $a(e) = b \in \mathfrak{e}(\mathcal{S}^+)$, so it is a subsystem of $\mathcal{S}^+$, and its carrier b is an element of $\mathcal{S}^+$.

Lemma 4(b) makes precise the slogan that the subsystems of one level are the objects of the next. It does not assert that the coarse data at b are generated by the part at b alone: they may also read relations between the part and the rest of $\mathcal{S}$, and in general they do.

Remark 5 (Non-uniqueness). A superlevel, and dually a sublevel, of a system is not unique: different aggregations, different added context and different generators all yield superlevels. This reflects the underdetermination of coarse descriptions by fine ones, and, in the other direction, the multiplicity of microscopic realisations of a macroscopic description. Sup and Sub are therefore class-valued correspondences, not operations.

5.3. Hierarchies

An immediate set-theoretic model of a hierarchy is a partially ordered set, or a join-semilattice if one wants an operation. A poset, however, has no notion of moving *one level* up, and the join of a semilattice is idempotent, whereas changing scale is not: zooming out twice is not zooming out once. We therefore separate an order from the level transitions and attach a grading to the latter. Baas's hyperstructures ^[11] iterate observation and interaction across levels in a related spirit; the present construction is a set-theoretic instance in which the inter-level dependence is explicit.

Definition 11 (Hierarchy). A hierarchy over Ω is a quintuple $\mathcal{H} = (\mathcal{L}, \preceq, \downarrow, \uparrow, \ell)$ where:

- i. $\mathcal{L}$ is a set of systems over Ω ;
- ii. $\preceq$ is the relation on $\mathcal{L}$ given by $\mathcal{S}_1 \preceq \mathcal{S}_2$ iff $\mathcal{S}_1 \sqsubseteq \mathcal{S}_2$ and $\mathcal{S}_1 \rightsquigarrow \mathcal{S}_2$;
- iii. $\uparrow : \mathcal{L} \rightarrow \mathcal{L}$ is a partial map with $\uparrow \mathcal{S} \in \text{Sup}(\mathcal{S})$ wherever defined;
- iv. $\downarrow : \mathcal{L} \rightarrow \mathcal{L}$ is a partial map with $\downarrow \mathcal{S} \in \text{Sub}(\mathcal{S})$ wherever defined;
- v. $\ell : \mathcal{L} \rightarrow \mathbb{Z}$ satisfies $\ell(\uparrow \mathcal{S}) = \ell(\mathcal{S}) + 1$ and $\ell(\downarrow \mathcal{S}) = \ell(\mathcal{S}) - 1$ wherever defined.

$\mathcal{H}$ is *unbounded below* (resp. *above*) if $\downarrow$ (resp. $\uparrow$) is total, and *bi-infinite* if both are.

The maps $\uparrow, \downarrow$ are *selections* from the correspondences Sup, Sub; the non-uniqueness remark describes the multiplicity of admissible selections. They are partial on purpose: requiring them to be total would force every member of every hierarchy to have infinitely many points (Proposition 5), excluding every finite model. The level function is defined up to an additive constant, and $\mathbb{Z}$ rather than $\mathbb{N}$ is chosen

so that no level is designated as fundamental. Members of $\mathcal{L}$ are descriptions at a scale; parts, being restrictions, need not be members.

Lemma 5 ($\preceq$). $\preceq$ is a preorder. $\mathcal{S}_1 \preceq \mathcal{S}_2 \preceq \mathcal{S}_1$ iff $\mathcal{S}_1, \mathcal{S}_2$ have the same carrier and elements and are inter-derivable; for example, a relation rescaled by a change of unit. Hence $\preceq$ is a partial order on the quotient $\mathcal{L}/\sim$ by this equivalence. Level transitions are strict: $\mathcal{S} \prec \uparrow \mathcal{S}$ and $\downarrow \mathcal{S} \prec \mathcal{S}$ wherever defined. If ℓ is order-preserving, it is constant on $\sim$ -classes.

Proof. $\preceq$ is the intersection of two preorders (Lemma 2, Corollary 1); the characterisation of $\sim$ combines the two. Strictness follows from Lemma 4(a), since $\uparrow \mathcal{S} \not\sqsubseteq \mathcal{S}$. The last claim is immediate.

Proposition 1 (Local finiteness from finite slices). Suppose ℓ is order-preserving and that, for all $\mathcal{A} \preceq \mathcal{B}$ and every $j \in [\ell(\mathcal{A}), \ell(\mathcal{B})] \cap \mathbb{Z}$, the slice $[\mathcal{A}, \mathcal{B}]_j = \{\mathcal{X} : \mathcal{A} \preceq \mathcal{X} \preceq \mathcal{B}, \ell(\mathcal{X}) = j\}$ is finite. Then every interval $[\mathcal{A}, \mathcal{B}] = \{\mathcal{X} : \mathcal{A} \preceq \mathcal{X} \preceq \mathcal{B}\}$ is finite.

Proof. If $\mathcal{A} \preceq \mathcal{X} \preceq \mathcal{B}$ then $\ell(\mathcal{X}) \in [\ell(\mathcal{A}), \ell(\mathcal{B})] \cap \mathbb{Z}$, a finite set; hence $[\mathcal{A}, \mathcal{B}] = \bigcup_j [\mathcal{A}, \mathcal{B}]_j$ is a finite union of finite sets.

The level function alone does not give local finiteness, since infinitely many systems may share a level; the finite-slice hypothesis excludes exactly that. Note also that local finiteness is a condition on intervals, not on chains: an infinite chain below a member is compatible with it (Example 1). Joins and meets of families of members need not exist, given the non-uniqueness of super- and sublevels, and we do not assume them.

Remark 6 (Near-decomposability). The definition is compatible with, but does not impose, Simon's near-decomposability [12]. Once a dynamics is added, one expects in many systems that intra-level interactions at $\mathcal{S}_1 \prec \mathcal{S}_2$ are fast relative to the interactions that $\mathcal{S}_2$ resolves, so that the levels decouple approximately. Such separation is an idealisation that fails at criticality or under resonant coupling across scales; the framework provides the objects on which the degree of coupling can be measured, not a guarantee of it.

5.4. Conservativity

The apparatus generalises the standard relational and multistrata formalisms.

Proposition 2 (Relational structures are systems). Every relational structure $\mathcal{A} = (E, (r_\lambda)_{\lambda \in \Lambda})$ with finitely many relations $r_\lambda \subseteq E^{k_\lambda}$ and $|E| \geq 2$ is represented by a system with empty medium and no valuations, via an injective, relation-preserving map.

Proof. Let $S = E \times \{0\}$ and $\mathfrak{P} = \Gamma_{\mathfrak{P}} = \{\{(x, 0)\} : x \in E\}$, which has at least two blocks. The bijection $x \mapsto \{(x, 0)\}$ carries each r_λ to $\bar{r}_\lambda \subseteq \Gamma_{\mathfrak{P}}^{k_\lambda}$; take $K = \{k_\lambda\}$, label the $\bar{r}_\lambda$ by λ , and let all $\mathbf{V}_{k,i}$ be empty. A tuple satisfies r_λ iff its image satisfies $\bar{r}_\lambda$, and distinct structures yield distinct systems.

The finiteness hypothesis matches Definition 2; with arbitrary index sets I_k and arity sets K in that definition, the same proof represents every relational structure of finite arities. Lin's multirelational systems ^[13] also admit relations of infinite arity, which are outside the present scope. The converse fails, since systems additionally admit non-singleton elements, a medium and valuations, so the extension is proper.

Proposition 3 (Finite chains of levels). *Let $\mathcal{S}_0, \dots, \mathcal{S}_N$ be systems over Ω with $\mathcal{S}_{j+1} \in \text{Sup}(\mathcal{S}_j)$ for $j < N$. Then $\mathcal{L} = \{\mathcal{S}_0, \dots, \mathcal{S}_N\}$, with $\hat{\lambda} \mathcal{S}_j = \mathcal{S}_{j+1}$ for $j < N$, $\mathcal{S}_j = \mathcal{S}_{j-1}$ for $j > 0$ and $\ell(\mathcal{S}_j) = j$, is a hierarchy, and ℓ is an order isomorphism from $(\mathcal{L}, \preceq)$ onto $\{0, \dots, N\}$.*

Proof. The axioms hold by construction. By Lemma 5, $\mathcal{S}_j \prec \mathcal{S}_{j+1}$; by transitivity of the strict part, $i < j$ implies $\mathcal{S}_i \prec \mathcal{S}_j$, so the $\mathcal{S}_j$ are distinct, $\mathcal{S}_j \not\preceq \mathcal{S}_i$, and $\mathcal{S}_i \preceq \mathcal{S}_j$ iff $i \leq j$.

A finite Mesarović multistrata description ^[3] is an instance of Proposition 3 exactly when each stratum aggregates the objects of the one below and its data are functions of that stratum's data across the admissible configurations. Strata related only by interpretation, without such a dependence, are not levels in the present sense, and the framework then says so rather than absorbing them.

5.5. Round trips

Coarse-graining may forget, and an up-then-down round trip measures what is forgotten.

Proposition 4 (Round trip). *Let $\mathcal{S}^+ \in \text{Sup}(\mathcal{S})$, with generator Φ , i.e. $\mathfrak{d}_{\mathcal{S}^+} = \Phi \circ \mathfrak{d}_{\mathcal{S}}$. Call $\mathcal{S}'$ a reconstruction of $\mathcal{S}$ from $\mathcal{S}^+$ if $\mathcal{S}' \in \text{Sub}(\mathcal{S}^+)$, $\mathcal{S}'$ has the carrier and elements of $\mathcal{S}$, and $\mathcal{S}^+ \rightsquigarrow \mathcal{S}'$. Then:*

- a. every reconstruction satisfies $\mathcal{S} \preceq \mathcal{S}'$ and $\ker \mathfrak{d}_{\mathcal{S}'} = \ker \mathfrak{d}_{\mathcal{S}^+}$;
- b. some reconstruction is inter-derivable with $\mathcal{S}$ iff Φ is injective on $\mathfrak{d}_{\mathcal{S}}(\Omega)$, in which case $\mathcal{S}$ itself is one;
- c. if Φ is not injective on $\mathfrak{d}_{\mathcal{S}}(\Omega)$, then $\mathcal{S} \prec \mathcal{S}'$ for every reconstruction $\mathcal{S}'$.

Proof. (a) $\mathcal{S}' \in \text{Sub}(\mathcal{S}^+)$ gives $\mathcal{S}' \rightsquigarrow \mathcal{S}^+$, which together with $\mathcal{S}^+ \rightsquigarrow \mathcal{S}'$ gives equal kernels (Corollary 1). From $\mathcal{S} \rightsquigarrow \mathcal{S}^+ \rightsquigarrow \mathcal{S}'$ and equal carriers and elements, $\mathcal{S} \preceq \mathcal{S}'$. (b) Since carriers and elements agree, $\mathcal{S}' \sim \mathcal{S}$ iff $\ker \mathfrak{d}_{\mathcal{S}'} = \ker \mathfrak{d}_{\mathcal{S}}$, iff $\ker \mathfrak{d}_{\mathcal{S}^+} = \ker \mathfrak{d}_{\mathcal{S}}$ by (a), iff Φ is injective on

$\mathfrak{d}_{\mathcal{S}}(\Omega)$ (Lemma 1). In that case $\mathcal{S} \in \text{Sub}(\mathcal{S}^+)$ by definition and $\mathcal{S}^+ \rightsquigarrow \mathcal{S}$ via Φ^{-1} , so $\mathcal{S}$ is a reconstruction. (c) By (a) and (b).

We call $\mathcal{S}^+$ *faithful* when Φ is injective on $\mathfrak{d}_{\mathcal{S}}(\Omega)$. Faithfulness depends on Ω : enlarging the admissible configurations can destroy it, and on a single configuration every superlevel is faithful (Remark 2). The order direction encodes the asymmetry of coarse-graining: a round trip can only sit at or above the original, i.e. carry at most its information. A Galois-connection reading of $(\hat{\lambda}, \hat{\mu})$ would require canonical, extremal selections, which we do not assume. Finally, non-faithfulness is the minimal form of inter-level novelty, and it should not by itself be read as strong emergence: an additive total is non-faithful and entirely collective (Example 1).

5.6. A computed example

Let $\Omega = (0, \infty)^2$, with configurations $\omega = (\alpha, \beta)$, and let $S = \{1, 2, 3, 4\}$, $A = \{1, 2\}$, $B = \{3, 4\}$.

Level -1. $\mathcal{S}^-$ has carrier S , partition and elements $\{\{1\}, \{2\}, \{3\}, \{4\}\}$, and one binary relation

$$R^- = \{(\{1\}, \{2\}), (\{2\}, \{1\}), (\{3\}, \{4\}), (\{4\}, \{3\})\},$$

with valuation v_{ω}^- equal to α on the two pairs within A and to β on the two pairs within B : intra-block couplings of strengths α and β .

Level 0. $\mathcal{S}_0$ has carrier S , elements $\{A, B\}$, no medium, and one binary relation $R_0 = \{(A, B), (B, A)\}$ with valuation $v_{0, \omega} \equiv \alpha + \beta$. The map $\{1\}, \{2\} \mapsto A$, $\{3\}, \{4\} \mapsto B$ is an aggregation (each fine element lies in its image, A and B are tiled, and the map is not injective), and $\mathcal{S}^- \rightsquigarrow \mathcal{S}_0$ with generator $\Phi(R^-, v^-) = (R_0, v^-(\{1\}, \{2\}) + v^-(\{3\}, \{4\}))$. Hence $\mathcal{S}_0 \in \text{Sup}(\mathcal{S}^-)$.

Parts. The part of $\mathcal{S}_0$ at A is $\mathcal{S}_A = \mathcal{S}^-|_{\{\{1\}, \{2\}\}}$, with carrier A and coupling α ; by Lemma 4, $\mathcal{S}_A \sqsubseteq \mathcal{S}_0$, and likewise $\mathcal{S}_B \sqsubseteq \mathcal{S}_0$. The parts are *not* downward determined: the configurations $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{4}, \frac{3}{4})$ give $\mathcal{S}_0$ the same data and $\mathcal{S}_A$ different data, so $\mathcal{S}_0 \not\rightsquigarrow \mathcal{S}_A$ (Remark 3).

Round trip. Φ is not injective on $\mathfrak{d}_{\mathcal{S}^-}(\Omega)$, since $(\alpha, \beta) \mapsto \alpha + \beta$ is not; by Proposition 4, every reconstruction $\mathcal{S}'$ of $\mathcal{S}^-$ from $\mathcal{S}_0$ satisfies $\mathcal{S}^- \prec \mathcal{S}'$. The symmetric reconstruction, with both intra-block couplings equal to $(\alpha + \beta)/2$, agrees with $\mathcal{S}^-$ at $(\frac{1}{2}, \frac{1}{2})$ and disagrees at $(\frac{1}{4}, \frac{3}{4})$. Agreement at a configuration is not recovery; this is Remark 2 in concrete form.

Level 1. Aggregating A and B into $W = S$ would give a one-block partition, so we add an environment point: $\mathcal{S}^+$ has carrier $S \cup \{\star\}$, partition $\{W, \{\star\}\}$, element set $\{W\}$, medium $\{\{\star\}\}$, and a unary relation $\{(W)\}$ with valuation

$$\tau_\omega(W) = \sum_{r \in R_0} v_{0,\omega}(r) = 2(\alpha + \beta),$$

the total coupling. The map $A, B \mapsto W$ is an aggregation and $\mathcal{S}_0 \rightsquigarrow \mathcal{S}^+$, so $\mathcal{S}^+ \in \text{Sup}(\mathcal{S}_0)$; its part at W is $\mathcal{S}_0$ itself, whence $\mathcal{S}_0 \sqsubseteq \mathcal{S}^+$.

Summary. $\mathcal{S}_A \sqsubseteq \mathcal{S}_0 \sqsubseteq \mathcal{S}^+$, so $\mathcal{S}_0$ is nested, while $\mathcal{S}_A \not\sqsubseteq \mathcal{S}^+$ (Lemma 3). By Proposition 3, $\{\mathcal{S}^-, \mathcal{S}_0, \mathcal{S}^+\}$ is a hierarchy with levels $-1, 0, 1$. It is well founded, as it must be: $\mathcal{S}^-$ has no sublevel, since a singleton element cannot be tiled by a non-injective aggregation, and $\mathcal{S}_A$ encloses nothing, since its elements are singletons and a carrier needs two points. Both facts are instances of the bounds in Section 6.

6. Non-well-founded hierarchies

The framework permits, but does not require, descent without end. There are two natural forms.

Definition 12 (Non-well-foundedness). A system $\mathcal{S}_0$ is *enclosure-non-well-founded* if there are systems $\mathcal{S}_{-1}, \mathcal{S}_{-2}, \dots$ with $\mathcal{S}_{-k-1} \sqsubseteq \mathcal{S}_{-k}$ for all $k \geq 0$. A hierarchy is *level-non-well-founded* at $\mathcal{S} \in \mathcal{L}$ if $\bigcup_n \mathcal{S}$ is defined for every n .

By Lemma 3(a), an enclosure chain strictly decreases carriers, so no separate properness condition is needed; in particular, systems on a fixed carrier that differ only in their data, however rich, never form such a chain. Non-well-founded hierarchies question the intuition, inherited from set theory, that every structure is grounded in atomic constituents; in physical terms they express the possibility that structure persists at arbitrarily fine scales. Despite the name, nothing here requires an anti-foundation axiom [14].

Lemma 6 (No anti-foundation needed). *Along an enclosure chain $\dots \sqsubseteq \mathcal{S}_{-1} \sqsubseteq \mathcal{S}_0$, the carriers form a strictly decreasing chain*

$$\dots \subsetneq u(\mathcal{S}_{-2}) \subsetneq u(\mathcal{S}_{-1}) \subsetneq u(\mathcal{S}_0)$$

of non-empty subsets of $u(\mathcal{S}_0)$. The descent is along $\subsetneq$, not along $\in$, and the construction is sound in ZFC. The same holds, with $\subseteq$ in place of $\subsetneq$, for the carriers along $\mathcal{S} \supseteq \bigcup_n \mathcal{S} \supseteq \bigcup_n^2 \mathcal{S} \supseteq \dots$.

Proof. By Lemma 3(a). The only memberships used are single steps $u(\mathcal{S}_{-k-1}) \in \epsilon(\mathcal{S}_{-k})$ into a set of blocks; there is no infinite descending $\in$ -sequence, which is what the Axiom of Foundation forbids. For levels, $\bigcup_n \mathcal{S} \sqsubset \mathcal{S}$ by Lemma 5.

Corollary 2 (Finite carriers: enclosure). *If $|\mathfrak{u}(\mathcal{S}_0)| = n < \infty$, every enclosure chain below $\mathcal{S}_0$ has at most $n - 2$ steps, and the bound is attained for every $n \geq 2$. In particular an enclosure-non-well-founded system has an infinite carrier.*

Proof. Carriers along the chain are strictly decreasing (Lemma 6) and each has at least two points (Remark 1); a strictly decreasing sequence of integers in $[2, n]$ has at most $n - 1$ terms. For attainment, let $\mathcal{T}_m$, $2 \leq m \leq n$, have carrier $\{1, \dots, m\}$, partition and elements $\{\{1, \dots, m-1\}, \{m\}\}$ and no relations. Then $\mathcal{T}_{m-1} \sqsubseteq \mathcal{T}_m$ for $3 \leq m \leq n$, a chain of $n - 2$ steps.

Proposition 5 (Finite carriers: levels). *For a system $\mathcal{S}$ put $\psi(\mathcal{S}) = \sum_{b \in \mathfrak{e}(\mathcal{S})} (|b| - 1) \in \mathbb{N} \cup \{\infty\}$, which equals $|\bigcup \mathfrak{e}(\mathcal{S})| - |\mathfrak{e}(\mathcal{S})|$ when $\bigcup \mathfrak{e}(\mathcal{S})$ is finite. If $\bigcup \mathfrak{e}(\mathcal{S})$ is finite, every sequence $\mathcal{S} = \mathcal{S}_0, \mathcal{S}_1, \dots$ with $\mathcal{S}_{j+1} \in \text{Sub}(\mathcal{S}_j)$ has at most $\psi(\mathcal{S})$ steps, and the bound is attained. Consequently, if a hierarchy is level-non-well-founded at $\mathcal{S}$, then $\bigcup \mathfrak{e}(\bigwedge_n \mathcal{S})$ is infinite for every $n \geq 0$.*

Proof. Let $\mathcal{S}^- \in \text{Sub}(\mathcal{S})$ with aggregation a , image $I \subseteq \mathfrak{e}(\mathcal{S})$ and $X = \bigcup I$. Every fine element lies in a fibre and fibres tile their images, so $\bigcup \mathfrak{e}(\mathcal{S}^-) = X$, which is finite. Since a maps $\mathfrak{e}(\mathcal{S}^-)$ onto I and is not injective, $|\mathfrak{e}(\mathcal{S}^-)| \geq |I| + 1$. Hence

$$\psi(\mathcal{S}^-) = |X| - |\mathfrak{e}(\mathcal{S}^-)| \leq |X| - |I| - 1 = \sum_{b \in I} (|b| - 1) - 1 \leq \psi(\mathcal{S}) - 1,$$

so ψ strictly decreases in $\mathbb{N}$ along the sequence. For attainment, let $\bigcup \mathfrak{e}(\mathcal{S}) = \{1, \dots, n\}$ be a single element with medium $\{\{0\}\}$ and no relations, so $\psi(\mathcal{S}) = n - 1$; let $\mathcal{S}_j$ have elements $\{1, \dots, n - j\}, \{n - j + 1\}, \dots, \{n\}$ and the same medium. Each $\mathcal{S}_j$ is a superlevel of $\mathcal{S}_{j+1}$ (the aggregation merges $\{1, \dots, n - j - 1\}$ and $\{n - j\}$ and fixes the singletons), giving $n - 1$ steps. The last claim follows because $\bigwedge_n \mathcal{S}$ is itself level-non-well-founded.

Requiring $\bigwedge$ to be total, as one might be tempted to do in the definition of hierarchy, would therefore exclude every model with a finite member, including the example of Section 5.6 and every finite simulation. The two forms of descent are linked as follows.

Proposition 6 (From level descent to enclosure descent). *Let a hierarchy be level-non-well-founded at $\mathcal{S}_0$, put $\mathcal{S}_{n+1} = \bigwedge \mathcal{S}_n$, and suppose every aggregation $a_n : \mathfrak{e}(\mathcal{S}_{n+1}) \rightarrow \mathfrak{e}(\mathcal{S}_n)$ is surjective with all fibres of size at least two. Then for every $e_0 \in \mathfrak{e}(\mathcal{S}_0)$ there is an infinite enclosure chain $\dots \sqsubseteq P_1 \sqsubseteq P_0$ with $\mathfrak{u}(P_0) = e_0$; in particular every part of $\mathcal{S}_0$ is enclosure-non-well-founded.*

Proof. By dependent choice pick $e_{n+1} \in a_n^{-1}(e_n)$ for all n , which is possible by surjectivity, and let $P_n = \mathcal{S}_{n+1}|_{a_n^{-1}(e_n)}$, a system with carrier e_n (Lemma 4). Then $\mathfrak{u}(P_{n+1}) = e_{n+1} \in a_n^{-1}(e_n) = \mathfrak{e}(P_n)$;

moreover $e_{n+1} \subseteq e_n$, and every element of P_{n+1} lies in e_{n+1} , which is an element of P_n . Hence $P_{n+1} \sqsubseteq P_n$.

Example 1 (A bi-infinite dyadic hierarchy). Let Ω be the set of finite Borel measures on $\mathbb{R}$. For $n \in \mathbb{Z}$ let $\mathcal{D}_n$ have carrier $\mathbb{R}$, partition and elements the dyadic intervals $I_{n,k} = [k2^n, (k+1)2^n)$, $k \in \mathbb{Z}$, the unary relation $\{(I) : I \in \mathfrak{c}(\mathcal{D}_n)\}$ and the mass valuation $m_{n,\mu}(I) = \mu(I)$. Sending each interval to its dyadic parent is an aggregation from $\mathcal{D}_n$ to $\mathcal{D}_{n+1}$ (surjective, fibres of size two), and $\mathcal{D}_n \rightsquigarrow \mathcal{D}_{n+1}$ since the mass of a parent is the sum of the masses of its children. With $\hat{\lambda} \mathcal{D}_n = \mathcal{D}_{n+1}$, $\hat{\lambda} \mathcal{D}_n = \mathcal{D}_{n-1}$ and $\ell(\mathcal{D}_n) = n$, this is a bi-infinite hierarchy. As in Proposition 3, $\mathcal{D}_m \preceq \mathcal{D}_n$ iff $m \leq n$, so ℓ is an order isomorphism onto $\mathbb{Z}$ and every slice is a singleton (Proposition 1). By Proposition 6, every dyadic interval tops an infinite enclosure chain, namely its successive halves. No superlevel is faithful, since a parent's mass forgets its split; yet nothing here is emergent in any strong sense, because the coarse datum is an additive total.

Whether the hierarchies describing actual systems are well founded is an empirical question, bound up with what physics will say about structure at the smallest scales, and a metaphysical one, debated as the question of whether there is a fundamental level [\[15\]](#). The framework is neutral: it describes either possibility, and Corollary 2 and Proposition 5 state the price of the non-well-founded one, namely infinite carriers.

7. Concluding remarks

We have given a formalism for systems across scales in which every inter-level dependence is a factorisation of data over a declared set of admissible configurations. This single choice removes the vacuity that afflicts definitions of the form ‘the relations of one system are functions of those of another’, fixes the direction of dependence (from finer to coarser), and makes faithfulness of coarse-graining, and its failure, a property one can check. Enclosure is the one-level parthood linking levels; it is anti-transitive, and its transitive closure is the ordinary subsystem order. Infinite descent is available within ZFC at the price of infinite carriers, with sharp bounds when carriers are finite.

Two directions are immediate. The first is dynamics: replacing Ω by the state space of a dynamics and asking when a coarse description is closed under it. The second is the original motivation of this work: constraining subsystems to be similar to their supersystems, and using that constraint as a parameter for both analysis and modelling.

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