

ON THE GUPTA-BLEULER AND DIRAC-FOCK QUANTIZATION SCHEMES REVISITED

ANATOLIY K. PRYKARPATSKI¹⁾ AND NIKOLAI N. BOGOLUBOV(JR.)²⁾

ABSTRACT. The Gupta-Bleuler quantization scheme in quantum electromagnetic field theory, based on the so-called “weakened” Lorenz subsidiary condition and indefinite metric on the reduced Fock space, is revisited within the canonical Dirac-Fock quantization approach. The latter, being applied to the Dirac-Fock-Podolsky Lagrangian, allowed to satisfy the strong Lorenz subsidiary condition on the Fock space, to prove its averaged invariance in time and to state positive energy density definiteness of quantum photon states. Simultaneously, there is demonstrated both the time-noninvariance of the averaged “weakened” Lorenz subsidiary condition, proposed by Gupta and Bleuler, and the resulting energy density nonpositive definiteness. There was also reanalyzed the corresponding N.N. Bogolubov’s (Sr.) variational approach to quantum electromagnetic field theory, which succeeded in construction of a perturbed strong Lorenz type operator condition both within the perturbative Schwinger’s S-matrix approach and within the Fock many-temporal proper time paradigm.

1. INTRODUCTION

It is well known that the usually accepted Dirac type quantization scheme [17, 18, 32] in quantum electrodynamics meets some mathematical problems [5, 6, 15, 36, 43] related with the necessity to satisfy simultaneously the requirement of the local positive energy density, the Lorenz subsidiary condition, imposed on the quantum states, and the condition of transversality. Moreover, the entire theory formulation must be [31, 46] relativistic covariant. As the corresponding photon quanta are massless, attempts to carry out the Dirac type quantization scheme to electrodynamics, being an operator vector field theory, encountered fundamental difficulties, resulting in meaningless expressions of physical observables and singularities. At the early fiftieth of the past century there were undertaken different efforts to overcome them by means of some weakly argued from the physics point of view mathematical tricks. This unsatisfactory state of quantum electromagnetic theory was sharply criticized by Gupta [26] and Bleuler [10] regarding the validity of imposing the strong Lorenz operator condition on the electromagnetic operator vector field and proposed to use a suitably defined “weakened” conditions, based both on the physically not motivated indefinite metric [4, 6, 15, 36] and the corresponding weak Lorenz subsidiary constraint, imposed on the related Fock space. The theoretical and mathematical community have accepted [2, 23, 24, 25, 27, 28] this way of dissolving the quantum electromagnetic field problems, and as a state of art, it characterizes the modern quantum electromagnetic theory. To the regret, the Gupta-Bleuler quantization scheme gave rise to both a nonsymmetric and non-positive definite Hamiltonian operator, as well as to the time noninvariance of the averaged Lorenz subsidiary condition. In the early sixties of the past century the great mathematician and mathematical physicist N.N. Bogolubov (Sr.) had written, yet never published a preprint [11], devoted to the variational approach to quantum electromagnetic field theory, in which he both had tried to dissolve the related not invariance problem of the Gupta-Bleuler quantization scheme and succeeded in derivation of a perturbed Lorenz type operator constraint within the quantum interaction representation of the corresponding perturbative S - matrix expansion.

2020 *Mathematics Subject Classification.* 58J70, 58J72, 34A34, 58J70, 58J72.

Key words and phrases. dark type dynamical systems; evolution flows; conservation laws; Lax-Noether condition; asymptotic solutions; linearization; complete integrability;

In the work under regard we have revisited the canonical Dirac type quantization scheme within the Fock space representation of quantum states and applied it to a modified Fock-Podolsky Lagrangian description of electrodynamics under the Lorenz subsidiary condition. Based on this quantization reformulation, there is stated that the Lorenz subsidiary operator constraint is satisfied strongly on the suitably reduced Fock subspace, it is time-invariant and simultaneously guaranteeing both the positive energy density of photon quanta and the related relativistic covariance. Moreover, we have reanalyzed the N.N. Bogolubov's (Sr.) variational approach to quantum electromagnetic field theory and succeeded in constructing a perturbed Lorenz type operator within the Fock many-temporal proper time paradigm. A short Supplement contains an analysis of the Lorenz condition from the spatial electric potential energy invariance point of view and its respective wave propagation is spacetime, which together gives rise to the complete system of the classical Maxwell equations.

2. THE FOCK-PODOLSKI LAGRANGIAN FORMALISM AND THE DIRAC QUANTIZATION REVISITING

Consider on the Minkowski space-time $M_{\tilde{t}}$, specified by the proper temporal parameter $\tilde{t} \in \mathbb{R}_{\tilde{t}}$, the following Fock-Podolsky [19, 15, 6, 5] Lagrangian function

$$(2.1) \quad \mathcal{L}[\varphi, A] := \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla\varphi + \partial A/\partial\tilde{t}|^2 - |\nabla \times A|^2 - (\partial\varphi/\partial\tilde{t} + \langle\nabla|A\rangle)^2) d^3x$$

for the electromagnetic vector fields $(\varphi, A) \in \mathcal{M} \simeq C^2(M^4; \mathbb{R} \times \mathbb{R}^3)$ in vacuum and calculate the related action functional

$$(2.2) \quad S[\varphi, A] := \int_0^T \mathcal{L}[\varphi, A] d\tilde{t}$$

extremum on a temporal interval $[0, T] \subset \mathbb{R}_{\tilde{t}}$. The variational condition $\delta S[\varphi, A] = 0$ on the functional phase space $\mathcal{M}$ at the constraints $(\delta\varphi, \delta A)|_{\tilde{t}=0} = 0 = (\delta\varphi, \delta A)|_{\tilde{t}=T}$, gives rise to the following set of differential Klein-Gordon-Fock wave relationships:

$$(2.3) \quad \partial^2\varphi/\partial\tilde{t}^2 - \Delta\varphi = 0, \quad \partial^2 A/\partial\tilde{t}^2 - \Delta A = 0$$

in the Minkowski space-time M_4 .

The Lagrangian function (2.1) allows the dual Hamiltonian representation

$$(2.4) \quad \begin{aligned} \partial\varphi/\partial\tilde{t} &= \delta H[\varphi, A]/\delta p_0 = -p_0 - \langle\nabla|A\rangle, \quad \partial A/\partial\tilde{t} = \delta H[\varphi, A]/\delta p = p - \nabla\varphi, \\ \partial p_0/\partial\tilde{t} &= -\delta H[\varphi, A]/\delta\varphi = -\langle\nabla|p\rangle, \quad \partial p/\partial\tilde{t} = \delta H[\varphi, A]/\delta A = \nabla \times (\nabla \times A) - \nabla p_0, \end{aligned}$$

where, by definition,

$$(2.5) \quad p_0 := \delta\mathcal{L}[\varphi, A]/\delta(\partial\varphi/\partial\tilde{t}) = -\langle\nabla|A\rangle - \partial\varphi/\partial\tilde{t}, \quad p := \delta\mathcal{L}[\varphi, A]/\delta(\partial A/\partial\tilde{t}) = \nabla\varphi + \partial A/\partial\tilde{t},$$

and the Hamiltonian function

$$(2.6) \quad \begin{aligned} H_b &= \int_{\mathbb{R}_x^3} H_b[\varphi, A] d^3x, \quad H_b[\varphi, A] := p_0\partial\varphi/\partial\tilde{t} + \langle p|\partial A/\partial\tilde{t}\rangle - \mathcal{L}[\varphi, A] = \\ &= \int_{\mathbb{R}_x^3} \left(\frac{1}{2}|p|^2 + \frac{1}{2}|\nabla \times A|^2 - \frac{1}{2}p_0^2 - p_0\langle\nabla|A\rangle - \langle p|\nabla\varphi\rangle \right) d^3x \end{aligned}$$

is *a priori* time-invariant, that is

$$(2.7) \quad dH_b/d\tilde{t} = 0$$

for all $\tilde{t} \in \mathbb{R}_{\tilde{t}}$.

It is now easy to observe that the Hamiltonian equations (2.4) reduce exactly to the Maxwell equations (6.19) and (6.20) in vacuum, as the Lorenz subsidiary condition

$$(2.8) \quad p_0 = \partial\varphi/\partial\tilde{t} + \langle\nabla|A\rangle \Rightarrow 0,$$

proves to be invariant in time, as

$$(2.9) \quad \partial p_0 / \partial \tilde{t} = -\langle \nabla | p \rangle = \rho = 0,$$

that means the zero electric charge density in vacuum and $p := -E$ denotes the corresponding electric field.

The Hamiltonian system (2.4) possesses [1, 3, 8, 7, 44] the canonical Poisson structure

$$(2.10) \quad \begin{aligned} \{\varphi(x), \varphi(y)\} &= 0 = \{A(x), A(y)\}, \{p_0(x), \varphi(y)\} = \delta(x - y), \\ \{p(x), p(y)\} &= 0 = \{p_0(x), p_0(y)\}, \{p(x), \otimes A(y)\} = I\delta(x - y) \end{aligned}$$

for all $(x, y) \in \mathbb{R}^3$ on the cotangent phase space $T^*(\mathcal{M})$ to the functional manifold $\mathcal{M} \simeq C^\infty(\mathbb{R}^3; \mathbb{E}^1 \times \mathbb{E}^3)$. The Poisson brackets (2.10) allow the standard Dirac quantization [9, 7, 17, 32, 33, 44] of the cotangent phase space $T^*(\mathcal{M}) \xrightarrow{\hbar} \hat{T}^*(\mathcal{M})$ via the quantum operator commutator relationships

$$(2.11) \quad \begin{aligned} [\varphi(x), \varphi(y)] &= 0 = [A(x), A(y)], [p_0(x), \varphi(y)] = \frac{\hbar}{i}\delta(x - y), \\ [p(x), p(y)] &= 0 = [p_0(x), p_0(y)], [p(x), \otimes A(y)] = \frac{\hbar}{i}I\delta(x - y) \end{aligned}$$

where $\hbar$ is the classical Planck constant and the self-adjoint canonical pairs of operators (φ, p_0) and $(A, p) \in \hat{T}^*(\mathcal{M}) \simeq \text{End } \Phi$ act on a suitably constructed [19, 15, 21] Fock space Φ , which makes it possible to describe effectively electromagnetic quantum states - the light particles, known as photons. The canonical operator pairs (φ, p_0) and $(A, p) \in \hat{T}^*(\mathcal{M}) \simeq \text{End } \Phi$ possess the natural spatial Fourier integral splitting in positive and negative wave vector components as

$$(2.12) \quad \begin{aligned} \varphi(x) &:= \varphi^-(x) + \varphi^+(x), \quad \varphi^-(x) = \int_{\mathbb{R}^3} \varphi_k^- e^{i\langle k|x \rangle} d^3k, \quad \varphi^+(x) = \int_{\mathbb{R}^3} \varphi_k^+ e^{-i\langle k|x \rangle} d^3k, \\ p_0(x) &:= p_0^-(x) + p_0^+(x), \quad p_0^-(x) = \int_{\mathbb{R}^3} p_{0,k}^- e^{i\langle k|x \rangle} d^3k, \quad p_0^+(x) = \int_{\mathbb{R}^3} p_{0,k}^+ e^{-i\langle k|x \rangle} d^3k, \\ A(x) &:= A^-(x) + A^+(x), \quad A^-(x) = \int_{\mathbb{R}^3} A_k^- e^{i\langle k|x \rangle} d^3k, \quad A^+(x) = \int_{\mathbb{R}^3} A_k^+ e^{-i\langle k|x \rangle} d^3k, \\ p(x) &:= p^-(x) + p^+(x), \quad p^-(x) = \int_{\mathbb{R}^3} p_k^- e^{i\langle k|x \rangle} d^3k, \quad p^+(x) = \int_{\mathbb{R}^3} p_k^+ e^{-i\langle k|x \rangle} d^3k, \end{aligned}$$

where, by definition, $(\varphi_k^-)^* = \varphi_k^+$, $(A_k^-)^* = A_k^+$, $(p_{0,k}^-)^* = p_{0,k}^+$, $(p_k^-)^* = p_k^+$ for all $k \in \mathbb{R}^3$ and the Fock space Φ is respectively defined as a cyclic Hilbert space, endowed with a scalar product $\langle \cdot | \cdot \rangle$, and generated by its vacuum state $|0\rangle \in \Phi$ determining conditions:

$$(2.13) \quad \Phi = \overline{\text{span}}_{(\lambda_0, \lambda) \in \mathbb{C} \times \mathbb{C}^3} \exp(\lambda_0 \varphi^+ + \langle \lambda | A^+ \rangle_{\mathbb{E}^3}) |0\rangle,$$

where, by definition, $\varphi_0^- |0\rangle = 0 = A^- |0\rangle$, jointly with the following canonical simultaneous operator commutation relationships:

$$(2.14) \quad \begin{aligned} [\varphi^\pm(x), \varphi^\pm(y)] &= 0 = [p_0^\pm(x), p_0^\pm(y)], \\ [p_0^-(x), \varphi^+(y)] &= \frac{\hbar}{2i}\delta(x - y) = [\varphi^-(x), p_0^+(y)], \\ [A^\pm(x), \otimes A^\pm(y)] &= 0 = [p^\pm(x) \otimes, p^\pm(y)], \\ [A^\pm(x), \otimes p^\pm(y)] &= 0, [p^-(x), \otimes A^+(y)] = \frac{\hbar}{2i}I\delta(x - y), \end{aligned}$$

satisfied for all $x, y \in \mathbb{R}^3$. The quantized form of the Hamiltonian operator (2.6) looks as follows:

$$\begin{aligned}
 H_b &= \int_{\mathbb{R}^3} \left(\frac{1}{2} |p^- + p^+|^2 + \frac{1}{2} |\nabla \times A^+ + \nabla \times A^-|^2 - \right. \\
 &\quad \left. - \frac{1}{2} |p_0^+ + p_0^-|^2 - (p_0^+ + p_0^-) \langle \nabla | A^+ + A^- \rangle - \right. \\
 &\quad \left. - \langle p^+ + p^- | \nabla \varphi^+ + \nabla \varphi^- \rangle \right) d^3x = \\
 (2.15) \quad &= \int_{\mathbb{R}^3} \left(\langle p_k^+ + ik\varphi_k^+ | p_k^- - ik\varphi_k^- \rangle - p_{0,k}^+ p_{0,k}^- - |k|^2 \varphi_k^+ \varphi_k^- - \right. \\
 &\quad \left. + \langle k \times A_k^+ | k \times A_k^- \rangle - ip_{0,k}^+ \langle k | A_k^- \rangle + i \langle k | A_k^+ \rangle p_{0,k}^- \right) d^3k = \\
 &= \int_{\mathbb{R}^3} \left(\langle k \times A_k^+ | k \times A_k^- \rangle + \langle p_k^+ + ik\varphi_k^+ | p_k^- - ik\varphi_k^- \rangle - \right. \\
 &\quad \left. - |k|^2 \varphi_k^+ \varphi_k^- - ip_{0,k}^+ \langle k | A_k^- \rangle + ip_{0,k}^- \langle k | A_k^+ \rangle \right) d^3k = \\
 &= \int_{\mathbb{R}^3} \left(2 \langle k \times A_k^+ | k \times A_k^- \rangle - ip_{0,k}^+ (\langle k | A_k^- \rangle + |k| \varphi_k^-) + 2i \langle k | A_k^+ \rangle p_{0,k}^- \right) d^3k,
 \end{aligned}$$

which we need to consider as that defined on the reduced Fock subspace

$$(2.16) \quad \Phi_0 := \overline{\ker p_0} \subset \Phi \simeq \{ |f\rangle \in \Phi : p_0 |f\rangle = 0 \sim p_0^+ |f\rangle = 0 = p_0^- |f\rangle \}.$$

The latter yields the following symmetric and, respectively, positive definite quantum Hamiltonian operator

$$(2.17) \quad H_0 = 2 \int_{\mathbb{R}^3} \langle k \times A_k^+ | k \times A_k^- \rangle d^3k,$$

as considered on the reduced Fock subspace Φ_0 . Moreover, owing to the operator evolution relationship

$$(2.18) \quad dp_0/d\tilde{t} = [H_b, p_0] = - \langle \nabla | p \rangle = \rho = 0,$$

satisfied for quantum electromagnetic field in vacuum, the strong operator condition (2.16), imposed on the Fock subspace Φ_0 , is time invariant. The latter can be reformulated as the following proposition.

Proposition 2.1. *The Dirac quantized electromagnetic field in vacuum represents a correct quantum field theory, compatibly defined on the reduced Fock subspace $\Phi_0 \subset \Phi$, defined by the strong invariant in time Lorenz operator relationship (2.16) in vacuum, equivalent to the strong operator constraint*

$$(2.19) \quad p_0 = \langle \nabla | A \rangle + \partial \varphi / \partial \tilde{t} |_{\Phi_0} = 0,$$

for all times $\tilde{t} \in \mathbb{R}_{\tilde{t}}$.

The obtained above result, respectively, means that we can easily construct the corresponding temporal quantization relationships

$$\begin{aligned}
 (2.20) \quad &[\varphi^\pm(\tilde{t}, x), \varphi^\pm(\tilde{s}, y)] = 0 = [p_0^\pm(\tilde{t}, x), p_0^\pm(\tilde{s}, y)], \\
 &[p_0^-(\tilde{t}, x), \varphi^+(\tilde{s}, y)] = \frac{\hbar I}{2i} D_0(\tilde{t} - \tilde{s}, x - y) = [p_0^+(\tilde{s}, y), \varphi^-(\tilde{t}, x)], \\
 &[A^\pm(\tilde{t}, x), \otimes A^\pm(\tilde{s}, y)] = 0 = [p^\pm(\tilde{t}, x) \otimes, p^\pm(\tilde{s}, y)], \\
 &[A^\pm(\tilde{t}, x), \otimes p^\pm(\tilde{s}, y)] = 0, [p^-(\tilde{t}, x), \otimes A^+(\tilde{s}, y)] = \frac{\hbar I}{2i} D_0(\tilde{t} - \tilde{s}, x - y),
 \end{aligned}$$

where $D_0(\tilde{t} - \tilde{s}, x - y)$ denotes the so called relativistic invariant Pauli-Jordan function [6, 15, 37, 36], determined for arbitrary space-time parameters $\tilde{t}, \tilde{s} \in \mathbb{R}_{\tilde{t}}$ and $x, y \in \mathbb{R}_x^3$, as it easy follows from the temporal Heisenberg type operator evolution relationships. The reduced Fock subspace $\Phi_0 \subset \Phi$, constructed above, satisfies the strong temporal Schrödinger type evolution invariance

$$(2.21) \quad \partial |f\rangle / \partial \tilde{t} - i/\hbar H_0 |f\rangle \in \Phi_0$$

for all $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ and any given $f \in \Phi_0$, where the Hamiltonian operator $H_0 \in \text{End } \Phi_0$ is both symmetric and positive definite on the reduced Fock subspace $\Phi_0 \subset \Phi$. Moreover, the averaged Lorenz subsidiary condition $\bar{p}_0 := \langle f | p_0 | f \rangle$ over $\Phi_0 \subset \Phi$, imitating the

classical one, persists to be invariant in time, that is

$$\begin{aligned}
 (2.22) \quad d\bar{p}_0/d\tilde{t} &= \langle \partial f / \partial \tilde{t} | p_0 | f \rangle + \langle f | \partial p_0 / \partial \tilde{t} | f \rangle + \langle f | p_0 | \partial f / \partial \tilde{t} \rangle = \\
 &= \langle \partial f / \partial \tilde{t} | p_0 | f \rangle + \frac{i}{\hbar} \langle f | [H_0, p_0] | f \rangle + \langle p_0 f | \partial f / \partial \tilde{t} \rangle = \\
 &= \frac{i}{\hbar} \langle f | H_0 p_0 f \rangle - \frac{i}{\hbar} \langle f | p_0 H_0 f \rangle = 0,
 \end{aligned}$$

easily following from the condition (2.16) for all $\tilde{t} \in \mathbb{R}_{\tilde{t}}$.

The result above is in strong contrast to the so called "*weak Lorenz condition*", imposed on the Fock space Φ by Gupta [26] and Bleuler [10]: the respectively constrained Fock subspace

$$(2.23) \quad \Phi_0^- := \overline{\ker p_0^-} \subset \Phi \simeq \{ |f\rangle \in \Phi : p_0^- |f\rangle = 0 \},$$

evolving in time as

$$(2.24) \quad \partial |f\rangle / \partial \tilde{t} - i/\hbar H_0^- |f\rangle \in \Phi_0^-,$$

with respect to the reduced Hamiltonian operator $H_0^- := H_b|_{p_0^-=0} \in \text{End } \Phi_0^-$, and does not assure the average Lorenz subsidiary constraint

$$(2.25) \quad \bar{p}_0^- := \langle f | p_0 f \rangle = \langle f | p_0^+ f + p_0^- f \rangle = \langle p_0^- f | g \rangle + \langle f | p_0^- | f \rangle = 0$$

for $|f\rangle \in \Phi_0^-$ and all times $\tilde{t} \in \mathbb{R}_{\tilde{t}}$, as the above expression (2.25) is not time invariant:

$$\begin{aligned}
 (2.26) \quad d\bar{p}_0^-/d\tilde{t} &= \langle \partial f / \partial \tilde{t} | p_0^- | f \rangle + \langle f | \partial p_0^- / \partial \tilde{t} | f \rangle + \langle f | p_0^- | \partial f / \partial \tilde{t} \rangle = \\
 &= \langle \partial f / \partial \tilde{t} | p_0^- | f \rangle + \frac{i}{\hbar} \langle f | H_0^- p_0^- - p_0^- H_0^- | f \rangle + \langle f | p_0^- | \partial f / \partial \tilde{t} \rangle = \\
 &= \langle \partial f / \partial \tilde{t} | p_0^- | f \rangle + \frac{i}{\hbar} \langle f | H_0^- p_0^- | f \rangle - \frac{i}{\hbar} \langle f | p_0^- H_0^- | f \rangle + \langle f | p_0^- | \partial f / \partial \tilde{t} \rangle = \\
 &= -\frac{i}{\hbar} \langle p_0^+ f | H_0^- | f \rangle + \langle p_0^+ f | \partial f / \partial \tilde{t} \rangle = \langle p_0^+ f | (\partial f / \partial \tilde{t}) - \frac{i}{\hbar} H_0^{-*} | f \rangle \rangle \neq 0,
 \end{aligned}$$

if to take into account that the reduced Hamiltonian operator $H_0^- \in \text{End } \Phi_0^-$ is not both positive definite and symmetric, that is $H_0^- \neq H_0^{-*}$ on the reduced Fock subspace $\Phi_0^- \subset \Phi$. As it had been written in the book [15] (p.120) by N.N. Bogolubov (Sr.) and D.V. Shirkov, "*... these conditions (2.23) guarantee that the Lorenz condition (2.25) is satisfied on the average, which is quite sufficient to establish correspondence with the classical field...*" Yet, as follows from the above stated calculative result (2.26), it is not the case, and jointly with the above stated nonsymmetry and positive notdefiniteness of the suitably reduced Hamiltonian operator $H_0^- \in \text{End } \Phi_0^-$ all that makes it possible to formulate the following proposition.

Proposition 2.2. *The Gupta-Bleuler quantization scheme, as applied to electromagnetic field on a suitably reduced Fock subspace, gives rise to a physically ill-posed quantum field theory, specified by the nonsymmetric and nonpositive definite Hamiltonian. In contrast, the respectively used canonical Dirac-Fock quantization scheme give rise to a well posed quantum theory with both symmetric and positive definite Hamiltonian operator on a suitably reduced Fock subspace.*

Moreover, from the physical point of view, this result means that the constrained Fock subspace $\Phi_0^- \subset \Phi$ is not suitable and does not allow to construct an adequate and correct scattering theory and the related Schwinger's S - matrix theory, whose perturbative mathematical structure has been clarified and in detail developed [13, 15] within the so called Bogolubov-Parasyuk R - operation. Moreover, many years ago the great mathematician and mathematical physicist N.N. Bogolubov (Sr.) had written, yet never published a preprint [11], devoted to the variational approach to quantum field theory, in which he tried to dissolve this not invariance problem of the Gupta-Bleuler reduced Fock subspace (2.23) within the interaction representation of the corresponding perturbative S - matrix expansion.

N.N. Bogolubov (Sr.) then had written: "... Consider now a question about subsidiary conditions, which one needs to get subordinated electromagnetic field under interaction. We need then to proceed from the corresponding condition in the noninteracting field theory. In this case, as a rule, one uses the Lorenz condition

$$(2.27) \quad \sum_{(\alpha)} \frac{\partial A_{\alpha}(x)}{\partial x_{\alpha}} f(x) = 0$$

at any space-time point $x \in M^4$, where a wave vector $f \in \Phi$ and a mapping $(A_{\alpha} : \alpha = \overline{0, 3}) : M^4 \rightarrow \mathbb{R} \times \mathbb{E}^3$ denotes the electromagnetic four-vector operator field on the Minkowski space-time. Yet, during past years in his works Gupta and Bleuler criticized the reasonableness of imposing condition (2.27) on the electromagnetic operator vector field and proposed to use a "weakened" condition of the type

$$(2.28) \quad \sum_{(\alpha)} \frac{\partial A_{\alpha}^{-}(x)}{\partial x_{\alpha}} f(x) = 0,$$

where $(A_{\alpha}^{-} : \alpha = \overline{0, 3}) : M^4 \rightarrow \mathbb{R} \times \mathbb{E}^3$ - the positive-frequency part of the electromagnetic field operators. Now we demonstrate, how the conditions (2.27) and (2.28) will change under interaction of the electromagnetic field with ambient charge distribution. Since the S - matrix operator $S(g)$, where $g : M^4 \rightarrow \mathbb{R}$ denotes a seed interaction parameter, establishes a connection between the wave vector for free fields and the wave vector in the presence of interaction, we see that the influence of the operator constraint

$$(2.29) \quad B(x)f(x) = 0,$$

where either $B(x) := \sum_{(\alpha)} \frac{\partial A_{\alpha}(x)}{\partial x_{\alpha}}$ or $\sum_{(\alpha)} \frac{\partial A_{\alpha}^{-}(x)}{\partial x_{\alpha}}$, $x \in M^4$, on the free particle wave vector $f \in \Phi$ can be written as a condition, imposed on the interacting wave vector $f(g) \in \Phi$:

$$(2.30) \quad S(g)B(x)S^{+}(g)f(g) = 0.$$

Let us construct the expansion of (2.30) in the interaction seed parameter $g : M^4 \rightarrow \mathbb{R}$ of the operator expression $S(g)B(x)S^{+}(g) : \Phi \rightarrow \Phi$, $x \in M^4$. Taking into account that

$$(2.31) \quad SBS^{+} = B + (SB - BS)S^{+}$$

and

$$(2.32) \quad \begin{aligned} SB(x) - B(x)S = \sum_{n \in \mathbb{N}} \frac{1}{n!} \int \underbrace{M^4 \times \dots \times M^4}_{n \text{ times}} \{ S_n(x_1, x_2, \dots, x_n) B(x) - \\ - B(x) S_n(x_1, x_2, \dots, x_n) \} \times g(x_1) g(x_2) \dots g(x_n) dx_1 dx_2 \dots dx_n, \end{aligned}$$

as well as the electromagnetic field commutation conditions

$$(2.33) \quad [A_{\mu}(x), A_{\nu}(y)] = \delta_{\mu\nu} D_0(x - y),$$

for arbitrary space-time points $x, y \in M^4$ and $\mu, \nu = \overline{0, 3}$, where D_0 - the already mentioned above invariant Pauli-Jordan function. Since for arbitrary $\nu = \overline{0, 3}$

$$(2.34) \quad \left[\sum_{(\mu)} \frac{\partial A_{\mu}(x)}{\partial x_{\mu}}, A_{\nu}(y) \right] = i \frac{\partial D_0(x - y)}{\partial x_{\nu}}$$

in case (2.27) one obtains that

$$(2.35) \quad [B(x), A_{\nu}(y)] = i \frac{\partial D_0(x - y)}{\partial x_{\nu}},$$

and in case (2.28) one obtains

$$(2.36) \quad [B(x), A_{\nu}(y)] = i \frac{\partial D_0^{-}(x - y)}{\partial x_{\nu}},$$

these expressions give rise to the following commutator relationships:

$$(2.37) \quad \begin{aligned} S_n(x_1, x_2, \dots, x_n)B(x) - B(x)S_n(x_1, x_2, \dots, x_n) = \\ = -i \sum_{(k=\overline{1,n}, \nu)} \frac{\partial D_0(x-y)}{\partial x_\nu} \frac{\partial S_n(x_1, x_2, \dots, x_n)}{\partial A_\nu(x_k)} \end{aligned}$$

in the case (2.27) and similarly to the commutator relationship

$$(2.38) \quad \begin{aligned} S_n(x_1, x_2, \dots, x_n)B(x) - B(x)S_n(x_1, x_2, \dots, x_n) = \\ = -i \sum_{(k=\overline{1,n}, \nu)} \frac{\partial D_0^-(x-y)}{\partial x_\nu} \frac{\partial S_n(x_1, x_2, \dots, x_n)}{\partial A_\nu(x_k)} \end{aligned}$$

in the case (2.28). Thus, the commutator (2.32) becomes

$$(2.39) \quad \begin{aligned} SB(x) - B(x)S = - \sum_{n \in \mathbb{N}} \frac{i}{(n-1)!} \int \underbrace{M^4 \times \dots \times M^4}_{n \text{ times}} \frac{\partial D_0(x-x_1)}{\partial x_\nu} \times \\ \times \frac{\partial S_n(x_1, x_2, \dots, x_n)}{\partial A_\nu(x_1)} g(x_1)g(x_2)\dots g(x_n)dx_1dx_2\dots dx_n = \\ = - \sum_{n \in \mathbb{N}} \frac{i}{n!} \int \underbrace{M^4 \times \dots \times M^4}_{n+1 \text{ times}} \frac{\partial D_0(x-x_1)}{\partial x_\nu} \frac{\partial S_{n+1}(x_1, x_2, \dots, x_n, x_{n+1})}{\partial A_\nu(x_1)} \times \\ \times g(x_1)g(x_2)\dots g(x_n)dx_1dx_2\dots dx_ndx_{n+1} \end{aligned}$$

giving rise to the following result:

$$(2.40) \quad \begin{aligned} (SB(x) - B(x)S) S^+ = - \sum_{(n \in \mathbb{N}, \nu)} \frac{i}{n!} \int \underbrace{M^4 \times \dots \times M^4}_{n \text{ times}} \frac{\partial D_0(x-y)}{\partial x_\nu} \left\{ K \left(\frac{x_1 - x_k}{x_{k+1} - x_n} \right) \frac{\partial S_{k+1}(y, x_1, \dots, x_k)}{\partial A_\nu(y)} \times \right. \\ \times S_{n-k}^+(x_{k+1}, x_{k+2}, \dots, x_n) \left. \right\} g(x_1)g(x_2)\dots g(x_n)g(y)dydx_1dx_2\dots dx_ndx_n = \\ = - \sum_{(n \in \mathbb{N}, \nu)} \frac{i}{\hbar n!} \int \underbrace{M^4 \times \dots \times M^4}_{n \text{ times}} \frac{\partial D_0(x-y)}{\partial x_\nu} \frac{\delta H(y, x_1, \dots, x_n)}{\delta A_\nu(y)} \times \\ \times g(x_1)g(x_2)\dots g(x_n)g(y)dydx_1dx_2\dots dx_ndx_n = \\ = - \sum_{(\nu)} \frac{1}{\hbar} \int_{M^4} \frac{\partial D_0(x-y)}{\partial x_\nu} \frac{\delta H(y, g)}{\delta A_\nu(y)} g(y)dy = - \sum_{(\nu)} \frac{1}{\hbar} \int_{M^4} \frac{\partial D_0(x-y)}{\partial x_\nu} J_\nu(y; g)g(y)dy = \\ = \sum_{(\nu)} \frac{1}{\hbar} \int_{M^4} \frac{\partial D_0(x-y)}{\partial x_\nu} \frac{\partial (J_\nu(y; g)g(y))}{\partial y_\nu} dy, \end{aligned}$$

where $K \left(\frac{x_1 - x_k}{x_{k+1} - x_n} \right)$ denotes combinations from $n \in \mathbb{N}$ marked elements into subsets of k - and $(n - k)$ - element groups, and $H(y; g)$, $y \in M^4$, represents the Hamiltonian operator of the interacting electromagnetic field, as well as when deriving (2.40), there was used the classical variational relationship $\frac{\delta H(y, g)}{\delta A_\nu(y)} = J_\nu(y; g)$, $\nu = \overline{0, 3}$, holding at all points $y \in M^4$. If now to take into account the charge conservation law $\sum_{(\nu)} \frac{\partial J_\nu(y; g)}{\partial x_\nu} = 0$ at all $y \in M^4$, we obtain

$$(2.41) \quad (SB(x) - B(x)S) S^+ = - \sum_{(\nu)} \frac{1}{\hbar} \int_{M^4} \frac{\partial D_0(x-y)}{\partial x_\nu} J_\nu(y; g) \frac{\partial g(y)}{\partial y_\nu} dy,$$

which jointly with the operator expression (2.31) reduces to the following operator relationship:

$$(2.42) \quad \left\{ \sum_{(\mu)} \frac{\partial A_\mu(x)}{\partial x_\mu} - \sum_{(\mu)} \frac{1}{\hbar} \int_{M^4} \frac{\partial D_0(x-y)}{\partial x_\nu} J_\nu(y; g) \frac{\partial g(y)}{\partial y_\nu} dy \right\} f(g) = 0,$$

or similarly for the case (2.28)

$$(2.43) \quad \left\{ \sum_{(\mu)} \frac{\partial A_\mu^-(x)}{\partial x_\mu} - \sum_{(\mu)} \frac{1}{\hbar} \int_{M^4} \frac{\partial D_0^-(x-y)}{\partial x_\nu} J_\nu(y; g) \frac{\partial g(y)}{\partial y_\nu} dy \right\} f(g) = 0.$$

To the regret, these N.N. Bogolubov's (Sr.) notes on the Gupta and Bleuler quantization scheme of electromagnetic field had been not finished with no further both suggestions and inferences concerning the real mathematical physics meaning of the carried out analysis of the problem. Nonetheless, as a natural consequence of these N.N. Bogolubov's (Sr.) calculations and the way of reasonings, presented above, one

can clearly state that the Lorenz subsidiary operator condition (2.27) for the interacting electromagnetic operator field should be deformed by some electric current operator components.

To refine the analytical form of this subsidiary Lorenz operator deformation, predicted by N.N. Bogolubov (Sr.), in the next section we will follow the approach devised in [19] and modified in [39, 40].

3. THE INTERACTING ELECTROMAGNETIC FIELD AND QUANTUM LORENZ TYPE CONDITION DEFORMATION

There are considered the fermionic electron creation and annihilation operator fields $\psi, \psi^+ : \Phi \rightarrow \Phi^4$ together with the related bosonic electromagnetic potential operator field $(\varphi, A) : \Phi \rightarrow \Phi^4$, acting on the related Fock space Φ , and respectively, commuting to each other, which we will treat within the Fock many-temporal description approach [19, 22] subject to the common reference frame $M_{\tilde{t}}^4$, specified by the temporal parameter $\tilde{t} \in \mathbb{R}_{\tilde{t}}$. Based on the classical spin symmetry properties [16, 34, 35, 41, 42] of electron, "minimum interaction" principle [14, 7, 39, 40] and the Lagrangian functional (2.1), there is the corresponding Dirac evolution equation

$$(3.1) \quad \partial\psi/\partial t = [\langle\alpha|\nabla\rangle + \frac{i}{\hbar}\beta m]\psi + \frac{ie}{\hbar}(\varphi - \langle\alpha|A\rangle)\psi$$

on the operator spinor field $\psi : \Phi \rightarrow \Phi^4$, where $\beta, \alpha \in su(2) \times su(2)$ denotes the usual four-vector of Dirac matrices and $m \in \mathbb{R}_+$ is the electron mass, we can write down the related Hamiltonian operator expression

$$(3.2) \quad H_f^{(int)} = \int_{\mathbb{R}^3} d^3x \psi^+ [\langle\alpha|\frac{\hbar}{i}\nabla\rangle + \beta m]\psi + \int_{\mathbb{R}^3} d^3x (e\psi^+\psi\varphi - e\psi^+\langle\alpha|A\rangle\psi),$$

for the electron operator field interacting with the electromagnetic operator field, where e denotes the elementary charge value. Thus, the evolution of the entire quantum field system, consisting of the fermionic and electromagnetic operator fields, is described by means of the joint Hamiltonian operator

$$(3.3) \quad H_{f-b} := H_f^{(int)} + H_b$$

where the bosonic operator $H_b : \Phi \rightarrow \Phi$ is given by the expression (2.6), via the Heisenberg commutator equations

$$(3.4) \quad \partial\psi/\partial t = \frac{i}{\hbar}[H_{f-b}, \psi], \quad \partial\psi^+/\partial t = \frac{i}{\hbar}[H_{f-b}, \psi^+]$$

and

$$(3.5) \quad \partial A/\partial t = \frac{i}{\hbar}[H_{f-b}, A], \quad \partial\varphi/\partial t = \frac{i}{\hbar}[H_{f-b}, \varphi]$$

with respect to the common temporal parameter $t \in \mathbb{R}_t$, as in this case there is assumed that the corresponding proper temporal bosonic $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ and proper temporal fermionic $\bar{t} \in \mathbb{R}_{\bar{t}}$ parameters, respectively, coincide, that is $\tilde{t} = \bar{t} = t \in \mathbb{R}_t$, and where, by definition, the operators $\psi(t, x) := \psi(\tilde{t}, \tilde{t}; x)|_{\tilde{t}=\bar{t}=t}$, $\varphi(t, x) := \varphi(\tilde{t}, x)|_{\tilde{t}=\bar{t}=t}$, $A(t, x) := A(\tilde{t}, x)|_{\tilde{t}=t}$ at any point $t \in \mathbb{R}_t$.

Simultaneously, there should be, evidently, satisfied the before derived both the electromagnetic field evolution equations (2.4) with respect to the own reference frame $M_{\tilde{t}}$ and the modified fermionic charged particle field equations

$$(3.6) \quad \partial\psi/\partial\bar{t} = \frac{i}{\hbar}[H_f^{(int)}, \psi], \quad \partial\psi^+/\partial\bar{t} = \frac{i}{\hbar}[H_f^{(int)}, \psi^+]$$

with respect to the own reference frame $M_{\tilde{t}}$.

Being mostly interested in the evolution of the quantum particle fermionic field $\psi : \Phi \rightarrow \Phi$, we can get rid of the bosonic Hamiltonian impact into the evolution (3.6), having applied to the Fock space Φ the unitary canonical transformation

$$(3.7) \quad \Phi \rightarrow \tilde{\Phi} := U(t) \Phi,$$

where we denoted by $U(t) : \Phi \rightarrow \tilde{\Phi}$ the unitary operator, satisfying the determining equation

$$(3.8) \quad dU(t)/dt = \frac{i}{\hbar} H_b U(t)$$

generated by the bosonic Hamiltonian operator (2.6) with respect to the common temporal parameter $t \in \mathbb{R}$. As a consequence of the transformation (3.7) we obtain the effective fermionic particle field interaction Hamiltonian operator

$$(3.9) \quad \begin{aligned} \tilde{H}_f^{(int)} &:= U(t) H_f^{(int)} U^*(t) = \int_{\mathbb{R}^3} d^3x \psi^+ \langle \alpha | \frac{\hbar}{i} \nabla | \psi + \\ &+ \int_{\mathbb{R}^3} d^3x (e\psi^+ \psi \tilde{\varphi} - e\psi^+ \langle \alpha | \tilde{A} | \psi), \end{aligned}$$

where we put, by definition,

$$(3.10) \quad \tilde{A} := U(t) A U^*(t), \quad \tilde{\varphi} := U(t) \varphi U^*(t),$$

$$\tilde{p}_0 := U(t) p_0 U^*(t), \quad \tilde{p} := U(t) p U^*(t)$$

and took into account that $\tilde{\psi} := U(t) \psi U^*(t) = \psi$, $\tilde{\psi}^+ := U(t) \psi^+ U^*(t) = \psi^+$, subject to which the evolution in the transformed Fock space $\tilde{\Phi}$, governed by the Hamiltonian operator (2.6) in the form

$$(3.11) \quad \tilde{H}_b = \frac{1}{2} \int_{\mathbb{R}_x^3} (|\tilde{p}|^2 + |\nabla \times \tilde{A}|^2 - \tilde{p}_0^2 - \tilde{p}_0 \langle \nabla | \tilde{A} \rangle - \langle \tilde{p} | \nabla \tilde{\varphi} \rangle) d^3x$$

became completely eliminated. The electromagnetic operator field variables and their canonically adjoint ones (3.10) satisfy, respectively, the following evolution equations:

$$(3.12) \quad \partial \tilde{A} / \partial \tilde{t} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{A}], \quad \partial \tilde{\varphi} / \partial \tilde{t} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{\varphi}]$$

and

$$(3.13) \quad \partial \tilde{p}_0 / \partial \tilde{t} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{p}_0], \quad \partial \tilde{p} / \partial \tilde{t} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{p}]$$

with respect to the electromagnetic vector field proper time $\tilde{t} \in \mathbb{R}_{\tilde{t}}$. Really, from equations (3.5) one obtains

$$(3.14) \quad \begin{aligned} \partial \tilde{A} / \partial t &= \frac{i}{\hbar} [\tilde{H}_{f-b}, \tilde{A}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} = \frac{i}{\hbar} [\tilde{H}_f^{(int)}, \tilde{A}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} + \frac{i}{\hbar} [\tilde{H}_b, \tilde{A}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}}. \\ \partial \tilde{\varphi} / \partial t &= \frac{i}{\hbar} [\tilde{H}_{f-b}, \tilde{\varphi}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} = \frac{i}{\hbar} [\tilde{H}_f^{(int)}, \tilde{\varphi}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} + \frac{i}{\hbar} [\tilde{H}_b, \tilde{\varphi}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}}. \end{aligned}$$

At the same time, one can write down equivalently that

$$(3.15) \quad \partial \tilde{A} / \partial t = \partial \tilde{A} / \partial \tilde{t} \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} + \frac{i}{\hbar} [\tilde{H}_f^{(int)}, \tilde{A}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}}, \quad \partial \tilde{\varphi} / \partial t = \partial \tilde{\varphi} / \partial \tilde{t} \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} + \frac{i}{\hbar} [\tilde{H}_f^{(int)}, \tilde{\varphi}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}}.$$

From derived above relationships (3.14) and (3.15) one easily obtains that

$$(3.16) \quad \partial \tilde{A} / \partial \tilde{t} \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{A}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}}, \quad \partial \tilde{\varphi} / \partial \tilde{t} \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{\varphi}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}},$$

coinciding with the above written system of equations (3.12). Similarly one obtains the system of equations

$$(3.17) \quad \frac{\partial \tilde{p}_0}{\partial \tilde{t}} \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{p}_0] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}}, \quad \frac{\partial \tilde{p}}{\partial \tilde{t}} \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{p}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}},$$

coinciding with the above written system of equations (3.13).

Concerning the Hamiltonian operator (3.11), we need to mention that the related Dirac quantized commutation relationships for the operators $(\tilde{\varphi}, \tilde{A}) : \tilde{\Phi} \rightarrow \tilde{\Phi}^4$ and their adjoint ones $(\tilde{p}_0, \tilde{p}) : \tilde{\Phi} \rightarrow \tilde{\Phi}^4$ remain the same, that is

$$(3.18) \quad \begin{aligned} [\tilde{\varphi}(x), \tilde{\varphi}(y)] &= 0 = [\tilde{A}(x), \tilde{A}(y)], [\tilde{p}_0(x), \tilde{p}_0(y)] = 0, [\tilde{p}(x), \tilde{p}(y)] = 0, \\ [\tilde{p}_0(x), \tilde{\varphi}(y)] &= \frac{\hbar}{i} \delta(x - y), [\tilde{p}(x), \otimes \tilde{A}(y)] = \frac{\hbar}{i} I \delta(x - y), \end{aligned}$$

for all $x, y \in \mathbb{R}^3$. Now, concerning the Hamiltonian operators (3.9) and (3.11), the following Heisenberg evolution operator equations

$$(3.19) \quad \partial\psi/\partial\bar{t} = \frac{i}{\hbar} [\tilde{H}_f^{(int)}, \psi], \quad \partial\psi^+/\partial\bar{t} = \frac{i}{\hbar} [\tilde{H}_f^{(int)}, \psi^+]$$

with respect to the proper reference frame $M_{\bar{t}}$ together with the Heisenberg evolution operator equations (3.12) and (3.13) with respect to the proper reference frame $M_{\bar{t}}$ hold.

Being further interested in the evolution equations (3.4) and (3.5), suitably rewritten on the transformed Fock space $\tilde{\Phi}$ with respect to the common temporal parameter $t \in \mathbb{R}$, we need to take into account [22, 39, 40] that the following functional relationships

$$(3.20) \quad \psi(t) := \psi(\bar{t}, \tilde{t})|_{\bar{t}=\tilde{t}=t}, \quad \psi^+(t) := \psi^+(\bar{t}, \tilde{t})|_{\bar{t}=\tilde{t}=t}, \quad \tilde{\varphi}(t) := \tilde{\varphi}(\bar{t}, \tilde{t})|_{\bar{t}=\tilde{t}=t},$$

$$\tilde{A}(t) := \tilde{A}(\bar{t}, \tilde{t})|_{\bar{t}=\tilde{t}=t}, \quad \tilde{p}_0(t) := \tilde{p}_0(\bar{t}, \tilde{t})|_{\bar{t}=\tilde{t}=t}, \quad \tilde{p}(t) := \tilde{p}(\bar{t}, \tilde{t})|_{\bar{t}=\tilde{t}=t},$$

hold. In particular, from (3.20) the evolution expressions follow

$$(3.21) \quad \partial\psi(t)/\partial t = \partial\psi(\bar{t}, \tilde{t})/\partial\bar{t}|_{\bar{t}=\tilde{t}=t} + \partial\psi(\bar{t}, \tilde{t})/\partial\tilde{t}|_{\bar{t}=\tilde{t}=t},$$

$$\partial\tilde{\varphi}(t)/\partial t = \partial\tilde{\varphi}(\bar{t}, \tilde{t})/\partial\bar{t}|_{\bar{t}=\tilde{t}=t} + \partial\tilde{\varphi}(\bar{t}, \tilde{t})/\partial\tilde{t}|_{\bar{t}=\tilde{t}=t},$$

$$\partial\tilde{A}(t)/\partial t = \partial\tilde{A}(\bar{t}, \tilde{t})/\partial\bar{t}|_{\bar{t}=\tilde{t}=t} + \partial\tilde{A}(\bar{t}, \tilde{t})/\partial\tilde{t}|_{\bar{t}=\tilde{t}=t},$$

$$\partial\tilde{p}_0(t)/\partial t = \partial\tilde{p}_0(\bar{t}, \tilde{t})/\partial\bar{t}|_{\bar{t}=\tilde{t}=t} + \partial\tilde{p}_0(\bar{t}, \tilde{t})/\partial\tilde{t}|_{\bar{t}=\tilde{t}=t},$$

$$\partial\tilde{p}(t)/\partial t = \partial\tilde{p}(\bar{t}, \tilde{t})/\partial\bar{t}|_{\bar{t}=\tilde{t}=t} + \partial\tilde{p}(\bar{t}, \tilde{t})/\partial\tilde{t}|_{\bar{t}=\tilde{t}=t},$$

for all $t \in \mathbb{R}$. The calculative expressions (3.21) prove to be useful when deriving the resulting quantum Maxwell electromagnetic equations. Before doing this, we need to check that the weak operator Lorenz constraints (2.19), rewritten in the case of the transformed Fock space $\tilde{\Phi}$, is compatible with the evolution equations (3.19) on the reduced Fock subspace

$$(3.22) \quad \tilde{\Phi}_0 := \{\tilde{f} \in \tilde{\Phi} : (\tilde{p}_0\tilde{f} = (\partial\tilde{\varphi}/\partial\tilde{t} + \langle\nabla|\tilde{A}\rangle)\tilde{f} = 0)\},$$

that is

$$(3.23) \quad [\tilde{p}_0^-(k), \tilde{H}_b]\tilde{f} = 0 = [\tilde{p}_0^+(k), \tilde{H}_b]\tilde{f}$$

for all $k \in \mathbb{R}^3$ and $\tilde{f} \in \tilde{\Phi}_0$, yet they fail to be compatible with respect to the evolution equations (3.18), that is

$$(3.24) \quad [\tilde{p}_0^-(k), \tilde{H}_f^{(int)}]\tilde{f} \neq 0 \neq [\tilde{p}_0^+(k), \tilde{H}_f^{(int)}]\tilde{f}$$

for all $\tilde{f} \in \tilde{\Phi}_0$. This means that we can not impose on the transformed Fock space $\tilde{\Phi}_0$ the constraints (3.22), as

$$(3.25) \quad \tilde{p}_0\tilde{f} := (\partial\tilde{\varphi}/\partial\tilde{t} + \langle\nabla|\tilde{A}\rangle)\tilde{f} \neq 0$$

for arbitrary $\tilde{f} \in \tilde{\Phi}_0$ and all proper parameters $\bar{t} \in \mathbb{R}_{\bar{t}}$ and $\tilde{t} \in \mathbb{R}_{\tilde{t}}$. Yet, it is easy enough to check [14, 16, 19, 42] that the following perturbed Lorenz type operator

$$(3.26) \quad \hat{p}_0(x; \bar{t}, \tilde{t}) = \tilde{p}_0(x; \bar{t}, \tilde{t}) - \frac{e}{2\pi} \int_{\mathbb{R}^3} d^3y \Theta(\bar{t} - \tilde{t}, |x - y|) \psi^+(y) \psi(y),$$

where the generalized function

$$(3.27) \quad \Theta(\bar{t} - \tilde{t}, |x - y|) := \frac{\delta(\bar{t} - \tilde{t} + |x - y|) - \delta(\bar{t} - \tilde{t} - |x - y|)}{2|x - y|}$$

satisfies the relativistic equation (2.3) and solves the N.N. Bogolubov's (Sr.) equation (2.42). It is worth to mention here that the function (3.27) is dual to the well known [45, 13, 15, 36] generalized function

$$(3.28) \quad \delta(|x - y|^2 - (\bar{t} - \tilde{t})^2) = \frac{\delta(\bar{t} - \tilde{t} + |x - y|) + \delta(\bar{t} - \tilde{t} - |x - y|)}{2|x - y|},$$

similarly satisfying the relativistic wave equation (2.3). The perturbed Lorenz type operator (3.26) strongly commutes both to each other and with temporal evolutions, respectively generated by Hamiltonian operators (3.9) and (3.11), that is

$$(3.29) \quad \begin{aligned} [\hat{p}_0(x; \bar{t}, \tilde{t}), \hat{p}_0(y; \bar{t}, \tilde{t})] &= 0, [\hat{p}_0(x; \bar{t}, \tilde{t}), i\hbar\partial/\partial\bar{t} - \tilde{H}_f^{(int)}]\tilde{f} = 0, \\ [\hat{p}_0(x; \bar{t}, \tilde{t}), i\hbar\partial/\partial\tilde{t} - \tilde{H}_b]\tilde{f} &= 0 \end{aligned}$$

for arbitrary proper parameters $\bar{t} \in \mathbb{R}_{\bar{t}}$, $\tilde{t} \in \mathbb{R}_{\tilde{t}}$, all $x, y \in \mathbb{R}^3$ and any $\tilde{f} \in \tilde{\Phi}_0^{(m)}$ from the modified reduced Fock subspace

$$(3.30) \quad \tilde{\Phi}_0^{(m)} := \{\tilde{f} \in \tilde{\Phi} : \hat{p}_0(x; \bar{t}, \tilde{t})\tilde{f} = 0, x \in \mathbb{R}^3\}.$$

Thus, the related evolution flows (3.12), (3.13) and (3.19) on the transformed Fock space $\tilde{\Phi}$ should be correctly considered under the modified reduced Fock subspace (3.30), invariant with respect to the proper temporal parameters $\bar{t} \in \mathbb{R}_{\bar{t}}$ and $\tilde{t} \in \mathbb{R}_{\tilde{t}}$.

Remark 3.1. It is here worthy to mention that the above defined operator $\hat{p}_0(x; \bar{t}, \tilde{t}) : \tilde{\Phi} \rightarrow \tilde{\Phi}$, $x \in \mathbb{R}^3$, depending on the proper temporal parameters $\bar{t} \in \mathbb{R}_{\bar{t}}$ and $\tilde{t} \in \mathbb{R}_{\tilde{t}}$, satisfies the relativistic wave equations

$$(3.31) \quad \partial^2 \hat{p}_0(\bar{t}, \tilde{t}) / \partial \bar{t}^2 - \Delta \hat{p}_0(\bar{t}, \tilde{t}) = 0, \quad \partial^2 \hat{p}_0(\bar{t}, \tilde{t}) / \partial \tilde{t}^2 - \Delta \hat{p}_0(\bar{t}, \tilde{t}) = 0,$$

that can be easily checked, if to make use of the rewritten in the Fock space wave equations (2.3):

$$(3.32) \quad \partial^2 \tilde{A} / \partial \tilde{t}^2 - \Delta \tilde{A} = 0, \quad \partial^2 \tilde{\varphi} / \partial \tilde{t}^2 - \Delta \tilde{\varphi} = 0.$$

Moreover, as it can be shown by means of direct calculations, the transformed bosonic

Hamiltonian operator (3.11) on the reduced Fock subspace $\tilde{\Phi}_0^{(m)} \subset \tilde{\Phi}$ by means of the modified Lorenz type constraints (3.30) persists to be, as before, positive definite.

Now we can proceed to deriving the corresponding quantum Maxwell equations starting from the operator equations (3.32) on the suitably transformed Fock subspace $\tilde{\Phi}_0^{(m)} \subset \tilde{\Phi}$ and the electromagnetic tension operator fields definitions

$$(3.33) \quad \tilde{E} = -\partial \tilde{A} / \partial \tilde{t} - \nabla \tilde{\varphi}, \quad \tilde{B} = \nabla \times \tilde{A}.$$

Applying the differential operator $\nabla \times$ and, respectively, ∇ to operator expressions (3.33), one easily obtains that

$$(3.34) \quad (\nabla \times \tilde{B} - \partial \tilde{E} / \partial \tilde{t}) \tilde{\Phi}_0^{(m)} = \frac{e}{2\pi} \nabla \int_{\mathbb{R}^3} d^3y \Theta(\bar{t} - \tilde{t}, |x - y|) \psi^+(y) \psi(y) \tilde{\Phi}_0^{(m)}$$

and

$$(3.35) \quad \langle \nabla | \tilde{E} \rangle \tilde{\Phi}_0^{(m)} = -\frac{e}{2\pi} \frac{\partial}{\partial \tilde{t}} \int_{\mathbb{R}^3} d^3y \Theta(\bar{t} - \tilde{t}, |x - y|) \psi^+(y) \psi(y) \tilde{\Phi}_0^{(m)}, \quad \langle \nabla | \tilde{B} \rangle \tilde{\Phi}_0^{(m)} = 0,$$

which are considered in the strong operator sense on the reduced Fock subspace $\tilde{\Phi}_0^{(m)} \subset \tilde{\Phi}$ with respect to the proper time reference frame M_t^4 of the electromagnetic vector field. It is also easy to calculate the strong operator relationship

$$(3.36) \quad \tilde{p}_0 \hat{\Phi}_0 := \left(\partial \tilde{\varphi} / \partial t + \langle \nabla | \tilde{A} \rangle \right) \hat{\Phi}_0 = 0,$$

with respect to the limiting common reference frame M_t^4 , where we took into account that $\lim_{\tilde{t} \rightarrow t, \tilde{t} \rightarrow t} \tilde{\Phi}_0^{(m)} = \hat{\Phi}_0$ at $t \in \mathbb{R}_t$.

Now we proceed to calculation of the strong Maxwell type operator relationships (3.34) and (3.35) with respect to the common reference frame M_t^4 :

$$(3.37) \quad \left(\nabla \times \tilde{B} - \partial \tilde{E} / \partial \tilde{t} \right) \tilde{\Phi}_0^{(m)}|_{\tilde{t}=\tilde{t} \rightarrow t} = \frac{e}{2\pi} \nabla \int_{\mathbb{R}^3} d^3 y \Theta(\tilde{t}-\tilde{t}, |x-y|) \psi^+(y) \psi(y) \tilde{\Phi}_0^{(m)}|_{\tilde{t}=\tilde{t} \rightarrow t} = 0$$

and

$$(3.38) \quad \langle \nabla | \tilde{E} \rangle \tilde{\Phi}_0 = -\frac{e}{2\pi} \frac{\partial}{\partial \tilde{t}} \int_{\mathbb{R}^3} d^3 y \Theta(\tilde{t}-\tilde{t}, |x-y|) \psi^+(y) \psi(y) \tilde{\Phi}_0^{(m)}|_{\tilde{t}=\tilde{t} \rightarrow t} = e\psi^+ \psi \hat{\Phi}_0,$$

where there was used the known [15, 45] generalized function relationship $\frac{\partial}{\partial s} \Theta(s, |z|)|_{s=0} = -2\pi\delta(z)$ for any $z \in \mathbb{R}^3$. To calculate further the expression (3.37) with respect to the common reference frame M_t^4 , we need to make use of the strong operator relationships (3.21) and find that

$$(3.39) \quad \frac{\partial \tilde{E}}{\partial \tilde{t}}|_{\tilde{t}=\tilde{t} \rightarrow t} = \frac{\partial \tilde{E}}{\partial t} - \frac{\partial \tilde{E}}{\partial \tilde{t}}|_{\tilde{t}=\tilde{t} \rightarrow t} = \frac{\partial \tilde{E}}{\partial t} - \frac{i}{\hbar} [\tilde{H}_f^{(int)}, \tilde{E}] = \frac{\partial \tilde{E}}{\partial t} + \xi \psi^+ \alpha \psi.$$

Thus, from (3.39) and (3.37) one obtains that

$$(3.40) \quad \left(\nabla \times \tilde{B} - \partial \tilde{E} / \partial t \right) \hat{\Phi}_0 = e\psi^+ \alpha \psi \hat{\Phi}_0$$

with respect to the common reference frame M_t^4 . The combined together strong operator relationships (3.38) and (3.40)

$$(3.41) \quad \left(\nabla \times \tilde{B} - \partial \tilde{E} / \partial t \right) \hat{\Phi}_0 = e\psi^+ \alpha \psi \hat{\Phi}_0, \quad \langle \nabla | \tilde{E} \rangle \hat{\Phi}_0 = e\psi^+ \psi \hat{\Phi}_0$$

on the reduced Fock subspace $\hat{\Phi}_0$ together with with the evident strong operator relationships

$$(3.42) \quad \nabla \times \tilde{E} + \partial \tilde{B} / \partial t = 0, \quad \nabla \times \tilde{B} = 0$$

compile the complete system of quantum Maxwell equations with respect to the common reference frame M_t^4 . The presented above reasonings jointly with the described calculations can be reformulated as the final proposition.

Proposition 3.2. *The electromagnetic vector field, interacting with an ambient electrically charged medium and quantized within the Dirac-Fock-Podolsky Lagrangian formalism, is a well-posed quantum field theory with a positive definite operator energy density, defined on the strongly reduced Fock subspace, which is equal to the closure of the respectively quantized kernel of the auxiliary Lorentz operator condition. Moreover, the quantized electromagnetic field tensions satisfy the related quantum Maxwell equations, defined on this strongly reduced Fock subspace and compatible with the corresponding quantized Dirac operator equations for the ambient charged medium.*

Additionally, from the Heisenberg evolution equations (3.18) it is easy to obtain an operator relationship for the charge flux conservation:

$$(3.43) \quad \partial(e\psi^+ \psi) / \partial t + \langle \nabla | e\psi^+ \alpha \psi \rangle = 0,$$

in which the quantity $\rho := e\psi^+ \psi$ is interpreted as the operator *charge density* and the quantity $J := e\psi^+ \alpha \psi$ is naturally interpreted as the operator *current density* with respect to the common Minkowski reference frame M_t^4 . Whence the strong operator equations (3.41) can be rewritten, taking into account the above charge and current density definitions in the standard Maxwell equations weak form:

$$(3.44) \quad \left(\nabla \times \tilde{B} - \partial \tilde{E} / \partial t \right) \hat{\Phi}_0 = J \hat{\Phi}_0, \quad \langle \nabla | \tilde{E} \rangle \hat{\Phi}_0 = \rho \hat{\Phi}_0,$$

considered on the Fock subspace $\hat{\Phi}_0$, determined by the time invariant Lorenz operator constraint (3.36). Moreover, based on the strong Maxwell operator equations (3.44) and the Lorenz constraint (3.36), one can derive easily the following linear strong operator wave equations

$$(3.45) \quad (\partial^2 \tilde{\varphi} / \partial t^2 - \Delta \tilde{\varphi}) \hat{\Phi}_0 = \rho \hat{\Phi}_0$$

and

$$(3.46) \quad \left(\partial^2 \tilde{A} / \partial t^2 - \Delta \tilde{A} \right) \hat{\Phi}_0 = J \hat{\Phi}_0$$

with respect to the common laboratory reference frame M_t^4 , allowing to calculate the induced by the quantum charged fermionic field the corresponding quantum electromagnetic potentials $(\tilde{\varphi}, \tilde{A}) : \hat{\Phi}_0 \rightarrow \hat{\Phi}_0^4$ in the analytical form:

$$(3.47) \quad \begin{aligned} \tilde{\varphi} &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(t', y) d^3 y}{|x - y|} + \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho_0(t', y) d^3 y}{|x - y|}, \\ \tilde{A} &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{J(t', y) d^3 y}{|x - y|} + \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{J_0(t', y) d^3 y}{|x - y|}, \end{aligned}$$

where $t' := t - |x - y| \in \mathbb{R}$ denotes the "retarded" temporal parameter and neutral density $\rho_0 : \hat{\Phi}_0 \rightarrow \hat{\Phi}_0$ and neutral current $J_0 : \hat{\Phi}_0 \rightarrow \hat{\Phi}_0$ operators are generated by the vacuum $\hat{\Phi}_0$, and satisfy, by definition, the following strong operator constraints:

$$(3.48) \quad \rho_0 \hat{\Phi}_0 = 0 = [\tilde{H}_f^{(int)} - i\hbar \partial / \partial t] \hat{\Phi}_0$$

on the reduced Fock space $\hat{\Phi}_0$. It is interesting to remark here that the neutral charge potential component

$$(3.49) \quad \varphi_0 := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho_0(t', y) d^3 y}{|x - y|}$$

satisfies under the quasi-stationary condition $\partial \rho_0(t, y) \partial t \simeq 0$ the Newton law

$$(3.50) \quad -\Delta \tilde{\varphi}_0 = \rho_0,$$

what together can be interpreted as virtual gravity type interaction source. Owing to the continuity relationship (3.43), the expressions (3.47) satisfy exactly the strong operator Lorenz subsidiary constraint

$$(3.51) \quad \left(\partial \tilde{\varphi} / \partial t + \langle \nabla | \tilde{A} \rangle \right) \hat{\Phi}_0 = 0,$$

coinciding with that of (3.36), regarding the common reference frame M_t^4 .

In conclusion we present an interesting observation from the reasonings, presented in Supplement and related to the characteristic strong wave operator relationship (3.45): if considered classically for the averaged potential $\varphi := \langle \hat{\Phi}_0 | \tilde{\varphi} | \hat{\Phi}_0 \rangle$ and charge density $\rho := \langle \hat{\Phi}_0 | \tilde{\rho} | \hat{\Phi}_0 \rangle$ quantities, it makes it possible to retrieve the complete system of the classical Maxwell equations.

4. CONCLUSION

There is demonstrated within the Dirac and Fock multi-time proper temporal paradigms that the quantum electromagnetic field both in vacuum and interacting with an ambient charged quantum medium, considered on the time invariant strongly reduced Fock subspace, presents a well posed relativistic quantum field theory with a nonnegative definite energy density. As a natural consequence from reasonings and presented calculations there were rederived quantum Maxwell equations, strongly defined on the mentioned above reduced Fock subspace. Moreover, there was reanalyzed the N.N. Bogolubov's (Sr.) prediction concerning a deformation of the strong Lorenz subsidiary constraint, imposed on quantum states of the related Fock space. There was also shown that the "weakened" Lorenz subsidiary condition, proposed by Gupta and Bleuler and widely accepted in the modern manuals on quantum electrodynamics is not time-invariant, as well as the respectively reduced Hamiltonian operator is not both symmetric and positive definite, thus makes appealing to the poor physically

motivated so called Hilbert space with "indefinite" metric. Moreover, the averaged "weakened" Lorenz subsidiary condition proves to be not time invariant, thus not sufficient to establish correspondence with the classical field, as it had been claimed in [15], making appealing to the poor physically motivated so Hilbert space "indefinite" metric also not satisfactory.

5. ACKNOWLEDGEMENTS

Both of authors are cordially thankful to the Abdus Salam International Centre for Theoretical Physics in Trieste, Italy, for the hospitality during their research scholarships. Authors are also cordially appreciated to Martin Land and Tepper Gill, the Organizers of the 14th Biennial Conference on Classical and Quantum Relativistic Dynamics of Particles and Fields (3 - 5 June 2024, at Aalto University of Helsinki, Finland) for a kind invitation to present a report and a very inspiring atmosphere during the event. They also are much appreciated to V.M. Simulik, V.I. Gerasimenko and D.B. Kurylyak for numerous discussions, instrumental comments and valuable suggestions.

6. SUPPLEMENT: A SPATIAL ELECTRIC POTENTIAL ENERGY INVARIANCE AND THE MAXWELL ELECTROMAGNETIC EQUATIONS RETRIEVING

Consider from the very beginning a field theoretical description of the classical electromagnetic vector field, based on a real smooth scalar charge density distribution $\rho \in C^1(M^4; \mathbb{R})$ on the Minkowski space $M^4 \simeq \mathbb{R}_{\tilde{t}} \times \mathbb{R}_x^3$ with coordinates $(\tilde{t}, x) \in M^4$, generating via the classical integral expression the following electric retarded potential distribution

$$(6.1) \quad \varphi(\tilde{t}, x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(\tilde{t} - |x - y|, y) d^3 y}{|x - y|},$$

satisfying the well known relativistic [6, 15, 31, 36, 46] Klein-Gordon-Fock wave propagation equation

$$(6.2) \quad \partial^2 \varphi(\tilde{t}, x) / \partial \tilde{t}^2 - \Delta \varphi(\tilde{t}, x) = \rho(\tilde{t}, x),$$

where we assumed, for brevity, the light velocity $c = 1$. Owing to the charge conservation property

$$(6.3) \quad dQ[\rho; \Omega_{\tilde{t}}] / d\tilde{t} = 0$$

for any temporal parameter $\tilde{t} \in \mathbb{R}_{\tilde{t}}$, where

$$(6.4) \quad Q[\rho; \Omega_{\tilde{t}}] = \int_{\Omega_{\tilde{t}}} \rho(\tilde{t}, x(\tilde{t})) d^3 x(\tilde{t})$$

denotes the charge inside any closed spatial region $\Omega_{\tilde{t}} \subset \mathbb{R}_x^3$ with the smooth boundary $\partial\Omega_{\tilde{t}}$, whose space-time pattern is formed by the evolution of the ambient charge. As it is easy to calculate, the latter means that the following partial differential relationship holds:

$$(6.5) \quad \partial \rho(\tilde{t}, x) / \partial \tilde{t} + \langle \nabla | J(\tilde{t}, x) \rangle = 0,$$

where $J(\tilde{t}, x(\tilde{t})) := \rho(\tilde{t}, x(\tilde{t})) u(\tilde{t}, x(\tilde{t}))$ denotes the corresponding electric current, generated by the charge velocity $u(\tilde{t}, x(\tilde{t})) := dx(\tilde{t}) / d\tilde{t}$ at internal points $(\tilde{t}, x(\tilde{t})) \in \Omega_{\tilde{t}}$ for all $\tilde{t} \in \mathbb{R}_{\tilde{t}}$. Based on the charge conservation equation (6.5) and the wave propagation equation (6.2), one states the following instructive proposition.

Proposition 6.1. *Consider the electric vector potential*

$$(6.6) \quad \varphi(\tilde{t}, x) = \frac{e}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(\tilde{t} - |x - y|, y) d^3 y}{|x - y|}$$

at points $x = x(\tilde{t}, x_0) \in \Omega_{\tilde{t}}$, $x_0 \in \Omega_0$, of an arbitrary closed domain $\Omega_{\tilde{t}} \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega_{\tilde{t}}$ for all $\tilde{t} \in \mathbb{R}_{\tilde{t}}$. Then the following statements hold:

a) the delayed temporal parameter $\tilde{t}' = \tilde{t} - |x - y(\tilde{t}', y_0)|$ satisfies the wave propagation equation

$$(6.7) \quad \partial^2 \tilde{t}' / \partial \tilde{t}^2 - \Delta \tilde{t}' = 0$$

for any spatial evolution $y(\tilde{t}', y_0) \in \mathbb{R}_{x, y_0}^3 \in \mathbb{R}^3$, and all $\tilde{t} \in \mathbb{R}_{\tilde{t}}$;
b) the electric potential energy functional

$$(6.8) \quad I[\varphi; \Omega_{\tilde{t}}] = \int_{\Omega_{\tilde{t}}} \varphi(\tilde{t}, x(\tilde{t})) d^3 x(\tilde{t}).$$

satisfies the temporal invariance condition

$$(6.9) \quad dI[\varphi; \Omega_{\tilde{t}}] / d\tilde{t} = 0$$

for all $\tilde{t} \in \mathbb{R}_{\tilde{t}}$, which is equivalent to the classical Lorenz subsidiary condition

$$(6.10) \quad \partial \varphi(\tilde{t}, x) / \partial \tilde{t} + \langle \nabla | A(\tilde{t}, x) \rangle = 0,$$

where, by definition, $A(\tilde{t}, x) := \varphi(\tilde{t}, x) dx(\tilde{t}, x_0) / d\tilde{t}$, $x(\tilde{t}, x_0) \in \Omega_{\tilde{t}}$, denotes the induced magnetic vector potential [29] for all temporal parameters $\tilde{t} \in \mathbb{R}_{\tilde{t}}$;
c) the classical Lorenz subsidiary condition (6.10) jointly with the differential relationships (6.2) and (6.5) reduces to the complete classical system of the Maxwell equations:

$$(6.11) \quad \begin{aligned} \nabla \times B(\tilde{t}, x) &= J(\tilde{t}, x) + \partial E(\tilde{t}, x) / \partial \tilde{t}, \quad \langle \nabla | E(\tilde{t}, x) \rangle = \rho(\tilde{t}, x), \\ \partial B(\tilde{t}, x) / \partial \tilde{t} &= -\nabla \times E(\tilde{t}, x), \quad \langle \nabla | B(\tilde{t}, x) \rangle = 0, \end{aligned}$$

satisfied at all $(x, \tilde{t}) \in M^4$.

Proof. a) It is easy to observe that the first derivatives of the delayed parameter $\tilde{t}' = \tilde{t} - |x - y(\tilde{t}', y_0)|$ equal to the following expressions:

$$(6.12) \quad \frac{\partial \tilde{t}'}{\partial \tilde{t}} = U(\tilde{t}, x), \quad \frac{\partial \tilde{t}'}{\partial x} = -U(\tilde{t}, x) \frac{x - y(\tilde{t}', y_0)}{|x - y(\tilde{t}', y_0)|},$$

where the scalar multiplier $U(\tilde{t}, x) := (1 - \frac{\langle x - y(\tilde{t}', y_0) | \dot{y}(\tilde{t}', y_0) \rangle}{|x - y(\tilde{t}', y_0)|})^{-1}$, $\dot{y}(\tilde{t}', y_0) := \partial y(\tilde{t}', y_0) / \partial \tilde{t}'$, is well defined for all $(\tilde{t}, x) \in M^4$. Respectively, calculating the second temporal $\partial^2 \tilde{t}' / \partial \tilde{t}^2$ and spatial gradient $\langle \nabla | \nabla \tilde{t}' \rangle$ derivatives, one easily proves their equality, so stating the wave propagation equation (6.7).

b) Taking into account both the charge invariance (6.5) and the electric potential

$$(6.13) \quad \varphi(\tilde{t}, x(t)) = \frac{e}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(\tilde{t} - |x(\tilde{t}, x_0) - y|, y) d^3 y}{|x(\tilde{t}, x_0) - y|}$$

relationships, one can obtain by means of direct calculations the following potential energy invariance:

$$(6.14) \quad \begin{aligned} dI[\varphi; \Omega_{\tilde{t}}] / d\tilde{t} &= \frac{d}{d\tilde{t}} \int_{\Omega_{\tilde{t}}} d^3 x(t, x_0) \frac{e}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(\tilde{t} - |x(t, x_0) - y|, y) d^3 y}{|x(t, x_0) - y|} = \\ &= \frac{e}{4\pi} \int_{\mathbb{R}^3} d^3 z \frac{d}{d\tilde{t}} \int_{\Omega_0} \frac{d^3 x_0}{|z|} \rho(\tilde{t} - |z|, x(\tilde{t}, x_0) - z) \text{Jac}(x(t, x_0)) = \\ &= \frac{e}{4\pi} \int_{\mathbb{R}^3} d^3 z \int_{\Omega_0} \frac{d^3 x_0}{|z|} [\partial \rho(\tilde{t} - |z|, x(\tilde{t}, x_0) - z) / \partial \tilde{t} + \\ &\quad + \langle \nabla | \rho(\tilde{t} - |z|, x(\tilde{t}, x_0) - z) \dot{x}(\tilde{t}, x_0) \rangle \text{Jac}(x(t, x_0))] = \\ &= \frac{e}{4\pi} \int_{\mathbb{R}^3} d^3 z \int_{\Omega_t} \frac{d^3 x(\tilde{t}, x_0)}{|z|} [\partial \rho(\tilde{t} - |z|, x(\tilde{t}, x_0) - z) / \partial \tilde{t} + \\ &\quad + \langle \nabla | J(\tilde{t} - |z|, x(\tilde{t}, x_0) - z) \rangle] = 0, \end{aligned}$$

where we put, by definition, $\text{Jac}(x(\tilde{t}, x_0)) := |\det \partial x(\tilde{t}, x_0) / \partial x_0|$ - the usual Jacobian of the transformation $\Omega_0 \ni x_0 \rightarrow x(\tilde{t}, x_0) \in \Omega_{\tilde{t}}$, for all $x_0 \in \Omega_0$, $\tilde{t} \in \mathbb{R}_{\tilde{t}}$. On the other hand, the expression (6.14) can be respectively recalculated as

$$(6.15) \quad \begin{aligned} dI[\varphi; \Omega_{\tilde{t}}] / d\tilde{t} &= \frac{d}{d\tilde{t}} \int_{\Theta_{\tilde{t}}} \varphi(\tilde{t}, x(\tilde{t}, x_0)) d^3 x(\tilde{t}, x_0) = \\ &= \frac{d}{d\tilde{t}} \int_{\Omega_0} \varphi(\tilde{t}, x(\tilde{t}, x_0)) \text{Jac}(x(\tilde{t}, x_0)) d^3 x_0 = \\ &= \int_{\Omega_0} [\partial \varphi(\tilde{t}, x(\tilde{t}, x_0)) / \partial \tilde{t} + \langle \nabla | u(\tilde{t}, x) \varphi(\tilde{t}, x(\tilde{t}, x_0)) \rangle \text{Jac}(x(\tilde{t}, x_0))] d^3 x_0 = \\ &= \int_{\Omega_{\tilde{t}}} [\partial \varphi(\tilde{t}, x(\tilde{t}, x_0)) / \partial \tilde{t} + \langle \nabla | u(\tilde{t}, x) \varphi(\tilde{t}, x(\tilde{t}, x_0)) \rangle] d^3 x(\tilde{t}, x_0) = 0, \end{aligned}$$

what is exactly equivalent to the condition (6.10) at all points $(x, \tilde{t}) \in M^4$.
 c) Having defined the electric field tension $E(\tilde{t}, x) := -\nabla\varphi(\tilde{t}, x) - \partial A(\tilde{t}, x)/\partial\tilde{t}$ and the magnetic field tension $B(\tilde{t}, x) := \nabla \times A(\tilde{t}, x)$ at all points $(\tilde{t}, x) \in M^4$, upon applying to them curl-operation $\nabla \times$ one easily obtains

$$(6.16) \quad \begin{aligned} \nabla \times E(\tilde{t}, x) &= \nabla \times (-\nabla\varphi(\tilde{t}, x) - \partial A(\tilde{t}, x)/\partial\tilde{t}) = -\partial B(\tilde{t}, x)/\partial\tilde{t}, \\ \nabla \times B(\tilde{t}, x) &= \nabla \times (\nabla \times A(\tilde{t}, x)) = \nabla(\nabla|A(\tilde{t}, x)) - \Delta A. \end{aligned}$$

Taking now into account the Lorenz condition (6.10), one obtains

$$(6.17) \quad \begin{aligned} \nabla \times E(\tilde{t}, x) &= -\partial B(\tilde{t}, x)/\partial\tilde{t}, \quad \nabla \times B(\tilde{t}, x) := \nabla \times (\nabla \times A(\tilde{t}, x)) = \\ &= \partial E(\tilde{t}, x)/\partial\tilde{t} + [\partial^2 A(\tilde{t}, x)/\partial\tilde{t}^2 - \Delta A(\tilde{t}, x)]. \end{aligned}$$

Similarly, taking into account the wave propagation equation (6.2) jointly with the charge continuity equation (6.5), one obtains

$$(6.18) \quad \langle \nabla|\partial^2 A(\tilde{t}, x)/\partial\tilde{t}^2 - \Delta A(\tilde{t}, x) \rangle = \langle \nabla|J(\tilde{t}, x) \rangle,$$

that upon substituting it into (6.17) gives rise to the system

$$(6.19) \quad \begin{aligned} \nabla \times E(\tilde{t}, x) &= -\partial B(\tilde{t}, x)/\partial\tilde{t}, \\ \nabla \times B(\tilde{t}, x) &= \partial E(\tilde{t}, x)/\partial\tilde{t} + J(\tilde{t}, x), \end{aligned}$$

coinciding exactly with the classical Faraday and Bio-Savart laws in the Minkowski space-time M^4 . Similarly, having applied the gradient operation ∇ to the vector magnetic and electric potentials, one simply derives from the wave propagation equation (6.2) and the Lorenz condition (6.10) the corresponding boundary conditions

$$(6.20) \quad \langle \nabla|B(\tilde{t}, x) \rangle = 0, \quad \langle \nabla|E(\tilde{t}, x) \rangle = \rho(\tilde{t}, x)$$

for the above evolution equations (6.19) at all points $(\tilde{t}, x) \in M^4$. Combining the differential relationships (6.19) with those of (6.20), we obtain the classical Maxwell equations, proving the proposition. $\square$

REFERENCES

- [1] Abraham R., Marsden J., Foundations of Mechanics, second edition, Benjamin Cummings, NY, 1978
- [2] Araki H., Indecomposable Representations with Invariant Inner Product: A Theory of the Gupta-Bleuler Triplet. Commun. Math. Phys. 97 (1985), 149-159
- [3] Arnold V.I., Mathematical Methods of Classical Mechanics, Springer, NY, 1978.
- [4] Bedirc S., Kong O.C.W., Ting H.K., Group Theoretical Approach to Pseudo-Hermitian Quantum Mechanics with Lorentz covariance and $c \rightarrow \infty$ limit. arXiv:2009.07499v2 [quant-ph] 28 Dec 2020
- [5] Berestetskii V.B., Lifshitz E.M., Pitaevskii L.P., Relativistic Quantum Theory, vol. 14, Pergamon Press, 1971
- [6] Bjorken J.D., and Drell S.D., Relativistic quantum mechanics, Mc-Grow Hill Company, 1965
- [7] Blackmore D., Prykarpatsky A.K. and Samoylenko V.Hr. Nonlinear dynamical systems of mathematical physics: spectral and differential-geometrical integrability analysis. World Scientific Publ., NJ, USA, 2011
- [8] Błaszak M., Multi-Hamiltonian Theory of Dynamical Systems, Springer, Berlin, 1998.
- [9] Błaszak M., Quantum versus classical mechanics and integrability problems, Springer, 2019
- [10] Bleuler K., Eine neue Methode zur Behandlung der longitudinalen und skalaren Photonen. Helv. Phys. Acta, 23 (1950), 567-586
- [11] Bogolubov N.N., Equations in variations of quantum field theory, Preprint. (unpublished; the N.N. Bogolubov (Jr.) courtesy)
- [12] Bogolubov N.N., Bogolubov N.N. (Jr.), Introduction to Quantum Statistical Mechanics. Gordon and Breach, New York, London, 1994.
- [13] Bogolubov N.N., Logunov A.A. Oksak A.I., Todorov I. T. General Principles of Quantum Field Theory. Kluwer Publ., Netherlands, 1990
- [14] Bogolubov N.N. (Jr.), Prykarpatsky A.K. and Blackmore D., Maxwell-Lorentz Electrodynamics Revisited via the Lagrangian Formalism and Feynman Proper Time Paradigm. Mathematics, 3 (2015), 190-257
- [15] Bogolubov, N.N., Shirkov, D.V., Quantum Fields. Addison-Wesley, 1982
- [16] Prykarpatski A.K., Bogolubov N.N. (Jr.), Some Remarks on the Electron Spin $SU(2) \times SU(2)$ -Symmetry Structure and the Related Ground State Properties. Physics of Particles and Nuclei, 57(1) (2026), 151-166
- [17] Dirac P.A.M., The principles of quantum mechanics. Oxford, Clarendon Press, 1947

- [18] Dirac P.A.M., Spinors in Hilbert space. Springer, 1974
- [19] Dirac P.A.M., Fock V.A., Podolsky B. On quantum electrodynamics. Sow. Physics, 2, 1932, p. 468-479
- [20] Finster F., Strohmaier A., Gupta-Bleuler Quantization of the Maxwell Field in Globally Hyperbolic Space-Times. Ann. Henri Poincaré 16 (2015), 1837-1868
- [21] Fock V. Die Eigenzeit in der klassischen und in der Quantenmechanik. Sow. Phys., 1937, 12, p. 404-425
- [22] Fock V.A., Podolsky B. On the quantization of electromagnetic waves and the interaction of charges in Dirac's theory. Sow. Physics, 1, 1932, p. 801-817.
- [23] Futakachi S., Dynamics for non-symmetric Hamiltonians, and Gupta-Bleuler formalism for Dirac-Maxwell operator. RIMS, (1961), 42-55
- [24] Grundling H., Systems with Outer Constraints. Gupta-Bleuler Electromagnetism as an Algebraic Field Theory. Commun.Math. Phys. 114 (1988), 69-91
- [25] Grundland H., Hurst C.A., Algebraic classification of the curvature of three-dimensional manifolds with indefinite metric. J. Math. Phys. 44, 4374 (2003); 10.1063/1.1592611618
- [26] Gupta S.N., Theory of Longitudinal Photons in Quantum Electrodynamics. Proc. Roy. Soc., 63A (1950),
- [27] Land M. Electrostatic Physics in Stukelberger-Horwitz electrodynamics. Journal of Physics: Conf. Series 437 (2013) 012012
- [28] Land M.C., Shnerb N., Horwitz L.P. Land M. On Feynman approach to the foundation of gauge theory. J. Math. Phys. 36(7) 1995, 3263-3288
- [29] Landau L.D., Lifschitz E.M., The classical theory of fields. Pergamon Press, Oxford, 1971
- [30] Minkowski H., Raum und Zeit. Physik. Z. 10, 1909, p. 104
- [31] Pauli W., Theory of Relativity. Oxford Publ., 1958
- [32] Pauli W., General Principles of Quantum Mechanics, Springer, 1980.
- [33] Pauli W., On Dirac's new method of field quantization. Review of Modern Physics, 15(3) 1943, 175-207
- [34] Pauli, W. Über den Zusammenhang des Abschlusses der Elektronengruppen im Atom mit der Komplexstruktur der Spektren. Zeitschrift für Physik. 31(1) (1925), 765-783
- [35] Pauli W., Zur Quantenmechanik des magnetischen Elektrons. Z. Physik 43 (1927), 601-623
- [36] Peskin M.E., Schroeder D.V., Introduction to quantum field theory. Perseus Book, Massachusetts, USA, 1995
- [37] Petryna D.Y., Quantum field theory. Kyiv, "Vyscha Shkola" Publisher, 1984
- [38] Poincare H. Sur la dynamique de l'électron. C.R. Acad. Sci. (Paris) 140, 1905, p. 1504-1508
- [39] Prykarpatski A.K. On the electron spin and spectrum energy problems within the Fock many temporal and Feynman proper time paradigms. Journal of Physics: Conference Series, 2482 (2023), 012017(18).
- [40] Prykarpatski A.K., The quantum fermionic charged particle self-interaction problem within the Fock multi-time and Feynman proper time paradigms. Ukr. J. Phys. 62(2) (2017), 172-183
- [41] Prykarpatski A.K., On the electron spin origin and its $SU(2)$ -symmetry structure. Journal of Physics: Conference Series 3152 (2025) 012026; doi:10.1088/1742-6596/3152/1/012026
- [42] Prykarpatski A.K., On the electron spin and spectrum energy problems within the Fock many temporal and Feynman proper time paradigms. Journal of Physics: Conference Series, 2482 (2023), 012017(18)
- [43] Rebenko A.I., Theory of Interacting Quantum Fields. De Gruyter, 2010
- [44] Takhtajan L.A. Lectures on Quantum Mechanics. Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651, USA
- [45] Vladimirov V.S. Equations of Mathematical Physics. Marcel and Dekker Ink., 1971
- [46] Weyl H. The Theory of Groups and Quantum Mechanics, Kessinger Publishing, USA, 2010

Conflicts of Interest: The authors declare no conflict of interest.

¹)DEPARTMENT OF COMPUTER SCIENCE AND TELECOMMUNICATION AT THE CRACOW UNIVERSITY OF TECHNOLOGY, KRAKOW, 31-155, POLAND, AND LVIV POLYTECHNIC UNIVERSITY, LVIV, 79000, UKRAINE, ²ICTP, TRIESTE, ITALY

Email address: ¹)pryk.anat@cybergal.com, ²) nikolai.bogolubov@hotmail.com