

# ON THE GUPTA-BLEULER AND DIRAC-FOCK QUANTIZATION SCHEMES REVISITED

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**ABSTRACT.** The classical Gupta-Bleuler quantization scheme in quantum electromagnetic field theory, based on the so-called “weakened” Lorenz subsidiary condition and indefinite metric on the reduced Fock space, is revisited within the canonical Dirac-Fock quantization approach. The latter, as applied to the Fermi-Fock-Podolsky Lagrangian, allowed both to satisfy the strong Lorenz subsidiary condition on the Fock space and to prove its averaged invariance in time, as well as to state positive energy density definiteness of quantum photon states. There is also demonstrated both the time-noninvariance of the averaged “weakened” Lorenz subsidiary condition within the Gupta and Bleuler approach and the resulting energy density positive nondefiniteness. We also reanalyzed the corresponding N.N. Bogolubov’s (Sr.) variational approach to constructing a deformed strong Lorenz type operator condition both within the perturbative Scwinger’s S-matrix approach and within the Fock many-temporal proper time paradigm. A short supplement is devoted to analysis of the classical Lorenz-type subsidiary condition and the related temporal invariance of the electric potential energy, providing a derivation of the complete system of classical Maxwell’s equations.

## 1. INTRODUCTION

It is well known that the commonly accepted Dirac-type quantization scheme [17, 18, 30, 37] in quantum electrodynamics encounters certain mathematical problems [7, 8, 16, 30, 41, 49] related to the need to simultaneously satisfy the requirement of local positive energy density, the Lorenz auxiliary condition imposed on quantum states, and the transversality condition. Furthermore, the entire formulation of the theory must be [36, 52] relativistic. Since the corresponding photon quanta are massless, attempts to transfer the Dirac-type quantization scheme to electrodynamics, which is a vector operator field theory, encountered fundamental difficulties, resulting in nonsensical expressions of physical observables and singularities. In the early 1950s, various attempts were made to overcome these problems by using certain mathematical tricks that were poorly justified from the theoretical and mathematical physics points of view. This unsatisfactory state of quantum electromagnetic theory was severely criticized by Gupta [29] and Bleuler [11] with respect to the validity of imposing the strong Lorenz operator condition on the electromagnetic operator vector field and proposed the use of appropriately defined “weakened” conditions based on both the physically unmotivated indefinite metric [6, 8, 16, 30, 31, 41] and the corresponding weak Lorenz dependent constraint imposed on the associated Fock space. The theoretical and mathematical physics community has accepted [2, 25, 27, 28, 31, 32, 33] this way of solving quantum electromagnetic field problems and it characterizes modern quantum electromagnetic theory as state of the art. Unfortunately, the Gupta-Bleuler quantization scheme, as applied directly to the Fermi Lagrangian, leads both to a nonsymmetric and a nonpositive definite Hamiltonian operator, as well as to the time noninvariance of the averaged Lorenz auxiliary condition, weakly imposed on the Fock space with an indefinite Krein type metric. Taking into account that these physical inconsistencies have been generated by the zero charge density constraint, the latter has been noncanonically replaced by

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means of the corresponding Klein-Gordon equation, satisfied in the strong sense on the Fock space. These important aspects were later strongly reanalyzed within the so called BRST approach [4, 5, 30, 31, 26], which generalized and suitably replaced the Gupta-Bleuler formalism. Namely, the introduced by Gupta-Bleuler so called "weakened" condition on the corresponding Fock space became the abelian precursor to the BRST basic cohomology [30] closure condition, imposed on the Fock space, what allowed to construct the physical Hilbert space rather than an indefinite metric state truncation. The corresponding Hamiltonian of the electromagnetic field in these covariant Lorenz-gauge and BRST schemes, being not positive-definite on the full Krein indefinite metric space, becomes positive definite on physical transverse states, once the physical subspace is properly quotiented or constrained.

In the late fifties of the past century, the great mathematician and mathematical physicist N.N. Bogolubov (Sr.) wrote, but never published, a preprint [12], devoted to a variational approach to the quantum theory of the electromagnetic field. In this work he attempted to resolve the related problem of the invariance of the Gupta-Bleuler quantization scheme and succeeded in deriving a perturbed Lorenz-type operator within the quantum interaction representation of the corresponding  $S$ -matrix expansion. Inspired by this N.N. Bogolubov (Sr.) work, we decided to reanalyze the canonical Dirac-type quantization scheme applied to the suitably constructed Fock space and applied it to a modified description of the Fermi-Fock-Podolsky Lagrangian electrodynamics [20, 21, 19] extended with the Lorenz subsidiary component. Based on the Dirac type quantization [17, 18, 30] with first and secondary type constraints, we succeeded in demonstrating that the Lorenz-type auxiliary operator constraint is strongly satisfied on a suitably reduced Fock subspace and is time-invariant, that simultaneously guarantees both the positive definite energy density of photon quanta and the associated relativistic covariance. Furthermore, we reexamined N.N. Bogolubov's (Sr.) variational approach to quantum electromagnetic field theory and succeeded in constructing a suitably perturbed Lorenz-type operator expression within the classical multi-time Fock paradigm, based on the proper time reference frames, assigned to electromagnetic and electron fields. In a short supplement we analyze the classical Lorenz-type subsidiary condition and the related temporal invariance of the electric potential energy, providing a derivation of the complete system of classical Maxwell's equations.

## 2. THE FOCK-PODOLSKI LAGRANGIAN FORMALISM AND THE DIRAC QUANTIZATION REVISITING

Consider on the Minkowski space-time  $M_{\tilde{t}}^4$ , specified by the proper temporal parameter  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ , the following Fermi-Fock-Podolsky [16, 8, 7, 19, 21] Lagrangian function

$$(2.1) \quad \mathcal{L}[\varphi, A] := \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla \varphi + \partial A / \partial \tilde{t}|^2 - |\nabla \times A|^2 - (\partial \varphi / \partial \tilde{t} + \langle \nabla | A \rangle)^2) d^3 x$$

for the electromagnetic vector fields  $(\varphi, A) \in \mathcal{M} \simeq C^2(M^4; \mathbb{R} \times \mathbb{R}^3)$  in vacuum and calculate the related action functional

$$(2.2) \quad S[\varphi, A] := \int_0^T \mathcal{L}[\varphi, A] d\tilde{t}$$

extremum on a temporal interval  $[0, T) \subset \mathbb{R}_{\tilde{t}}$ . The variational condition  $\delta S[\varphi, A] = 0$  on the functional phase space  $\mathcal{M}$  at the constraints  $(\delta \varphi, \delta A)|_{\tilde{t}=0} = 0 = (\delta \varphi, \delta A)|_{\tilde{t}=T}$ , gives rise to the following set of differential Klein-Gordon-Fock wave relationships:

$$(2.3) \quad \partial^2 \varphi / \partial \tilde{t}^2 - \Delta \varphi = 0, \quad \partial^2 A / \partial \tilde{t}^2 - \Delta A = 0$$

in the Minkowski space-time  $M_{\tilde{t}}^4$ .

The Lagrangian function (2.1) allows the dual Hamiltonian representation

$$(2.4) \quad \begin{aligned} \partial\varphi/\partial\tilde{t} &= \delta H[\varphi, A]/\delta p_0 = -p_0 - \langle \nabla | A \rangle, \quad \partial A/\partial\tilde{t} = \delta H[\varphi, A]/\delta p = p - \nabla\varphi, \\ \partial p_0/\partial\tilde{t} &= -\delta H[\varphi, A]/\delta\varphi = -\langle \nabla | p \rangle, \quad \partial p/\partial\tilde{t} = \delta H[\varphi, A]/\delta A = \nabla \times (\nabla \times A) - \nabla p_0 \end{aligned}$$

with respect to the canonical Poisson brackets

$$(2.5) \quad \begin{aligned} \{\varphi(x), \varphi(y)\} &= 0 = \{A(x), A(y)\}, \quad \{p_0(x), \varphi(y)\} = \delta(x - y), \\ \{\varphi(x), A(y)\} &= 0, \quad \{p(x), \otimes A(y)\} = I\delta(x - y), \\ \{p(x), p(y)\} &= 0 = \{p_0(x), p_0(y)\}, \quad \{p_0(x), p(y)\} = 0, \\ \{p_0(x), A(y)\} &= 0 = \{p(x), \varphi(y)\} = 0 \end{aligned}$$

for all  $x, y \in \mathbb{R}^3$ , where, by definition,

$$(2.6) \quad \begin{aligned} p_0 &:= \delta \mathcal{L}[\varphi, A]/\delta (\partial\varphi/\partial\tilde{t}) = -\langle \nabla | A \rangle - \partial\varphi/\partial\tilde{t}, \\ p &:= \delta \mathcal{L}[\varphi, A]/\delta (\partial A/\partial\tilde{t}) = \nabla\varphi + \partial A/\partial\tilde{t}, \end{aligned}$$

and the Hamiltonian function

$$(2.7) \quad H_b = \int_{\mathbb{R}_x^3} H_b[\varphi, A] d^3x, \quad H_b[\varphi, A] := p_0 \partial\varphi/\partial\tilde{t} + \langle p | \partial A/\partial\tilde{t} \rangle - \mathcal{L}[\varphi, A],$$

that is

$$(2.8) \quad H_b = \int_{\mathbb{R}_x^3} \left[ \frac{1}{2} |p|^2 + \frac{1}{2} |\nabla \times A|^2 - \frac{1}{2} p_0^2 - p_0 \langle \nabla | A \rangle - \langle p | \nabla \varphi \rangle \right] d^3x$$

is *a priori* time-invariant:

$$(2.9) \quad dH_b/d\tilde{t} = 0$$

for all  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ . It is easy to observe that the Hamiltonian equations (2.4) reduce exactly to the Maxwell equations (5.19) and (5.20) in vacuum, if the Lorenz subsidiary condition

$$(2.10) \quad p_0 = -(\partial\varphi/\partial\tilde{t} + \langle \nabla | A \rangle),$$

persists to be invariant in time:

$$(2.11) \quad dp_0/\partial\tilde{t} = -\langle \nabla | p \rangle \Rightarrow 0$$

under the additional constraint

$$(2.12) \quad d\langle \nabla | p \rangle/\partial\tilde{t} = -\langle \nabla | \nabla p_0 \rangle \Rightarrow 0,$$

satisfied for all  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ , regarding the imposed external zero electric charge density condition  $\rho = \langle \nabla | E \rangle = 0$  in vacuum, where the vector momentum  $p := -E$  denotes the corresponding electric field.

Thus, we can claim here that the classical electromagnetic field theory is compatible iff the main determining constraints (2.11) and (2.12) are satisfied for all  $(\tilde{t}, x) \in M_{\tilde{t}}^4$ . As these constraints are of the first class within the Dirac [17, 18, 30] approach to constructing reduced Poisson brackets, the latter imposes the corresponding deformation of the canonical Poisson structures (2.5) to the following reduced form:

$$(2.13) \quad \begin{aligned} \{\varphi(x), \varphi(y)\} &= 0 = \{A(x), A(y)\}, \quad \{p_0(x_0), \varphi(y)\} = 0, \\ \{\varphi(x), A(y)\} &= 0, \quad \{p(x), \otimes A(y)\} = I\delta(x - y), \\ \{p(x), p(y)\} &= 0, \quad \{p_0(x_0), p_0(x_0)\} = 0, \quad \{p_0(x_0), p(y)\} = 0, \\ \{p_0(x_0), A(y)\} &= 0 = \{p(x), \varphi(y)\} = 0 \end{aligned}$$

for all  $x, y \in \mathbb{R}^3$  on the cotangent phase space  $T^*(\mathcal{M})$  to the functional manifold  $\mathcal{M} \simeq C^\infty(\mathbb{R}^3; \mathbb{E}^1 \times \mathbb{E}^3)$ , where we took into account that the Lorenz expression  $p_0 : M_t^4 \rightarrow \mathbb{R}$  is constant on the space time  $M_t^4$ , that is  $p_0(x) = p_0(x_0) \in \mathbb{R}$  for some fixed point  $x_0 \in \mathbb{R}^3$ , which entails the vanishing of the delta function  $\delta(x - y)|_{x=x_0} = 0$

on the whole space  $\mathbb{R}^3$ , as its one point  $x_0 \in \mathbb{R}^3$  support is of zero measure in  $\mathbb{R}^3$ . Moreover, the following temporal evolution

$$(2.14) \quad \begin{aligned} 0 &= \frac{\partial}{\partial \tilde{t}} \{p_0(x), \varphi(y)\} = \left\{ \frac{\partial}{\partial \tilde{t}} p_0(x), \varphi(y) \right\} + \left\{ p_0(x), \frac{\partial}{\partial \tilde{t}} \varphi(y) \right\} = \\ &= - \underbrace{\{\langle \nabla | p(x) \rangle, \varphi(y)\}}_0 + \underbrace{\{p_0(x), p_0(y) - \langle \nabla | A(y) \rangle\}}_0 = 0 \end{aligned}$$

confirms the zero bracket  $\{p_0(x), \varphi(y)\} = 0$  invariance for all  $x, y \in \mathbb{R}^3$  and  $\tilde{t} \in \mathbb{R}$ . These reasonings make it possible to apply the standard Dirac quantization scheme [9, 10, 17, 30, 37, 38, 50] to the cotangent phase space  $T^*(\mathcal{M}) \xrightarrow{\hbar} \hat{T}^*(\mathcal{M})$  via the quantum operator commutator relationships

$$(2.15) \quad \begin{aligned} [\varphi(x), \varphi(y)] &= 0 = [A(x), A(y)], \quad [\varphi(x), A(y)] = 0, \\ [p_0(x), \varphi(y)] &= 0, [p(x), p(y)] = 0, [p_0(x_0), A(y)] = 0 \\ [p_0(x), p_0(y)] &= 0, \quad [p(x), \otimes A(y)] = \frac{\hbar}{i} I \delta(x - y) \end{aligned}$$

for all  $x, y \in \mathbb{R}^3$ , where  $\hbar$  is the classical Planck constant and the adjoint pairs of operators  $(\varphi, p_0)$  and  $(A, p) \in \hat{T}^*(\mathcal{M}) \simeq \text{End } \Phi_0$  act on a suitably constructed [19, 16, 23] Fock space  $\Phi_0$ , which should describe effectively electromagnetic quantum states - the real light particles, known as photons.

To make this quantization procedure physically reasonable, we first take into account that the canonical operator pairs  $(\varphi, p_0)$  and  $(A, p) \in \hat{T}^*(\mathcal{M}) \simeq \text{End } \Phi_0$  possess the natural spatial Fourier integral splitting in positive and negative wave vector components as

$$(2.16) \quad \begin{aligned} p_0(x) &= p_0^+(x) + p_0^-(x), \quad p_0^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} p_{0,k}^\pm e^{\mp i \langle k | x \rangle} d^3 k, \\ \varphi(x) &= \varphi^+(x) + \varphi^-(x), \quad \varphi^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \varphi_k^\pm e^{\mp i \langle k | x \rangle} d^3, \\ p(x) &= p^+(x) + p^-(x), \quad p^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} p_k^\pm e^{\mp i \langle k | x \rangle} d^3, \\ A(x) &= A^+(x) + A^-(x), \quad A^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} A_k^\pm e^{\mp i \langle k | x \rangle} d^3, \end{aligned}$$

where, by definition,  $(\varphi_k^-)^* = \varphi_k^+$ ,  $(A_k^-)^* = A_k^+$ ,  $(p_{0,k}^-)^* = p_{0,k}^+$  and  $(p_k^-)^* = p_k^+$  for all momentums  $k \in \mathbb{R}^3$ . Then the corresponding Fock space  $\Phi_0$  can be, respectively, defined as a cyclic Hilbert space, endowed with a scalar product  $(\cdot | \cdot)$ , and generated by its vacuum vector state  $|0\rangle \in \Phi_0$  determining conditions:

$$(2.17) \quad \begin{aligned} \Phi_0 &= \text{span}_{(\lambda_0, \lambda) \in L_2(\mathbb{R}^3; \mathbb{C})^2, (\mu_0, \mu) \in L_2(\mathbb{C}^3)^2} \exp[\lambda_{0,k} p_{0,k}^+ + \langle \mu_{0,k} | p_k^+ \rangle_{\mathbb{E}^3} + \\ &\quad + \lambda_k \varphi_k^+ + \langle \mu_k | A_k^+ \rangle_{\mathbb{E}^3}] |0\rangle, \end{aligned}$$

where, by definition, the vacuum state  $|0\rangle \in \Phi_0$  satisfies the following constraint:

$$(2.18) \quad \begin{aligned} \text{span}_{(\lambda_0, \lambda) \in L_2(\mathbb{R}^3; \mathbb{C})^2, (\mu_0, \mu) \in L_2(\mathbb{R}^3; \mathbb{C}^3)^2} \{ \lambda_{0,k} p_{0,k}^- + \langle \mu_{0,k} | p_k^- \rangle_{\mathbb{E}^3} + \\ + \lambda_k \varphi_k^- + \langle \mu_k | A_k^- \rangle_{\mathbb{E}^3} \} |0\rangle = 0 \end{aligned}$$

for all  $k \in \mathbb{R}^3$ , jointly with the canonical simultaneous operator commutations:

$$(2.19) \quad \begin{aligned} [\varphi_k^\pm, \varphi_s^\pm] &= 0 = [p_{0,k}^\pm, p_{0,s}^\pm], [\varphi_k^\pm, A_s^\pm] = 0, \\ [p_{0,k}^-, \varphi_s^+] &= \frac{\hbar}{2i} \delta_{ks} = [p_{0,k}^+, \varphi_s^-], [p_{0,k}^-, \varphi_s^-] = 0 = [p_{0,k}^+, \varphi_s^+] \\ [A_k^\pm, \otimes p_s^\pm] &= 0, [p_k^\pm, \otimes A_s^\mp] = \frac{\hbar}{2i} I \delta_{ks}, [A_k^\pm, \otimes A_s^\pm] = 0 = [p_k^\pm, \otimes p_s^\pm], \end{aligned}$$

satisfied for all momentums  $k, s \in \mathbb{R}^3$ . If to take now into account the quantized form of the Hamiltonian operator (2.8):

$$\begin{aligned}
 H_b &= \int_{\mathbb{R}^3} [\frac{1}{2}|p^- + p^+|^2 + \frac{1}{2}|\nabla \times A^+ + \nabla \times A^-|^2 - \\
 &\quad - \frac{1}{2}|p_0^+ + p_0^-|^2 - (p_0^+ + p_0^-)\langle \nabla | A^+ + A^- \rangle - \\
 &\quad - \langle p^+ + p^- | \nabla \varphi^+ + \nabla \varphi^- \rangle] d^3x = \\
 (2.20) \quad &= 2 \int_{\mathbb{R}^3} [\langle p_k^+ | p_k^- \rangle - p_{0,k}^+ p_{0,k}^- + \langle k \times A_k^+ | k \times A_k^- \rangle - \langle p_k^+ | ik \varphi_k^- \rangle + \\
 &\quad + \langle p_k^- | ik \varphi_k^+ \rangle - ip_{0,k}^+ \langle k | A_k^- \rangle + i \langle k | A_k^+ \rangle p_{0,k}^-] d^3k,
 \end{aligned}$$

one can observe that it becomes positive semi-definite on the reduced Fock subspace

$$(2.21) \quad \Phi_0^{(red)} := \{ |f\rangle \in \Phi_0 : p_{0,k}^\pm |f\rangle = 0, \langle k | p_k^\pm |f\rangle = 0, k \in \mathbb{R}^3 \},$$

which is defined by means of imposing on the Fock space  $\Phi_0$  the additional invariant in time constraint  $\langle \nabla | p \rangle \sim \langle k | p_k^\pm \rangle = 0, k \in \mathbb{R}^3$ , as follows from the Heisenberg operator equation

$$(2.22) \quad d\langle k | p_k^\pm \rangle / d\tilde{t} = \frac{i}{\hbar} [H_b, \langle k | p_k^\pm \rangle] = i|k|^2 p_{0,k}^\pm = 0,$$

satisfied on the Fock space  $\Phi_0$  for arbitrary  $k \in \mathbb{R}^3$ . The latter yields the symmetric and, respectively, positive semidefinite reduced quantum Hamiltonian operator

$$(2.23) \quad H_{b,0} := H_b|_{\Phi_0^{(red)}} = 2 \int_{\mathbb{R}^3} [\langle k \times A_k^+ | k \times A_k^- \rangle + \langle p_k^+ | p_k^- \rangle] d^3k,$$

as considered on the reduced Fock subspace  $\Phi_0^{(red)}$ , jointly with the suitably reduced commutator relationships:

$$\begin{aligned}
 (2.24) \quad &[\varphi_k^\pm, \varphi_s^\pm] = 0, \quad [\varphi_k^\pm, A_s^\pm] = 0, \\
 &[A_k^\pm, \otimes A_s^\pm] = 0 = [p_k^\pm \otimes, p_s^\pm], [A_k^\pm, \otimes p_s^\pm] = 0, \\
 &[p_k^-(x), \otimes A_s^+] = \frac{\hbar}{2i} I \delta_{ks} = [p_k^+(x), \otimes A_s^-],
 \end{aligned}$$

where  $(p_{0,k}^-)^* = 0 = p_{0,k}^+|_{\Phi_0^{(red)}}$  and  $\langle k | p_k^\pm |_{\Phi_0^{(red)}} = 0$  for all momentums  $k \in \mathbb{R}^3$  on the reduced Fock subspace  $\Phi_0^{(red)} \subset \Phi_0$ . The reasonings above can be reformulated as the following proposition.

**Proposition 2.1.** *The Dirac-Fock quantized electromagnetic field in vacuum presents a physically correct quantum field theory, compatibly defined on the reduced Fock subspace  $\Phi_0^{(red)} \subset \Phi_0$ , defined by the strong invariant in time quantized Lorenz subsidiary operator condition in vacuum, equivalent to the strong operator constraints*

$$\begin{aligned}
 (2.25) \quad &p_0(\tilde{t}, x_0) = -(\langle \nabla | A(\tilde{t}, x) \rangle + \partial \varphi(\tilde{t}, x) / \partial \tilde{t})|_{\Phi_0^{(red)}} = 0, \\
 &\nabla p_0(\tilde{t}, x)|_{\Phi_0^{(red)}} = 0, \langle \nabla | p(\tilde{t}, x) \rangle|_{\Phi_0^{(red)}} = 0
 \end{aligned}$$

for all spacetime variables  $(\tilde{t}, x) \in M_{\tilde{t}}$  and an arbitrarily chosen point  $x_0 \in \mathbb{R}^3$ .

This result, in particular, means that we can also easily construct the reduced temporal quantization relationships

$$\begin{aligned}
 (2.26) \quad &[\varphi^\pm(\tilde{t}, x), \varphi^\pm(\tilde{s}, y)] = 0 = [p_0^\pm(\tilde{t}, x_0), p_0^\pm(\tilde{s}, x_0)], \\
 &[p_0^-(\tilde{t}, x_0), \varphi^+(\tilde{s}, y)] = 0 = [p_0^+(\tilde{s}, x_0), \varphi^-(\tilde{t}, x)], \\
 &[A^\pm(\tilde{t}, x), \otimes A^\pm(\tilde{s}, y)] = 0 = [p^\pm(\tilde{t}, x) \otimes, p^\pm(\tilde{s}, y)], \\
 &[A^\pm(\tilde{t}, x), \otimes p^\pm(\tilde{s}, y)] = 0, \\
 &[p^-(\tilde{t}, x), \otimes A^+(\tilde{s}, y)] = \frac{\hbar I}{2i} D_0(\tilde{t} - \tilde{s}, x - y),
 \end{aligned}$$

where  $D_0(\tilde{t} - \tilde{s}, x - y)$  denotes the so called relativistic invariant Pauli-Jordan function [8, 16, 42, 41], determined for arbitrary space-time parameters  $\tilde{t}, \tilde{s} \in \mathbb{R}_{\tilde{t}}$  and  $x, y \in \mathbb{R}_x^3$ , as it easy follows from the temporal Heisenberg type operator evolution

relationships. The reduced Fock subspace  $\Phi_0^{(red)}$ , constructed above, satisfies the strong temporal Schrödinger type evolution invariance

$$(2.27) \quad \partial|f\rangle/\partial\tilde{t} - i/\hbar H_{b,0}|f\rangle \in \Phi_0^{(m)}$$

for all  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$  and any given  $|f\rangle \in \Phi_0^{(red)}$ , where the Hamiltonian operator  $H_{b,0} \in \text{End } \Phi_0^{(red)}$  is both symmetric and positive definite on the reduced Fock subspace  $\Phi_0^{(red)} \subset \Phi_0$ . Moreover, the averaged Lorenz subsidiary condition  $\bar{p}_0 := (f|p_0|f)$  over  $\Phi_0^{(red)} \subset \Phi^{(red)}$ , imitating the classical one, persists to be invariant in time, that is

$$(2.28) \quad \begin{aligned} d\bar{p}_0/d\tilde{t} &= (\partial f/\partial\tilde{t}|p_0|f) + (f|\partial p_0/\partial\tilde{t}|f) + (f|p_0|\partial f/\partial\tilde{t}) = \\ &= (\partial f/\partial\tilde{t}|p_0|f) + \frac{i}{\hbar}(f|[H_{b,0}, p_0]f) + (p_0 f|\partial f/\partial\tilde{t}) = \\ &= \frac{i}{\hbar}(f|H_{b,0}p_0 f) - \frac{i}{\hbar}(f|p_0 H_{b,0} f) = 0, \end{aligned}$$

easily following from the condition (2.25) for all  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ .

Reminding now the generally accepted Gupta [29] and Bleuler [11] quantization scheme, the result above is in strong contrast to the so called "*weak Lorenz condition*", imposed by them on the corresponding Fock space  $\Phi$

$$(2.29) \quad \begin{aligned} \Phi = \text{span}_{(\lambda_0, \lambda) \in L_2(\mathbb{R}^3; \mathbb{C})^2} \exp[\lambda_{0,k} p_{0,k}^+ + \langle \mu_{0,k} | p_k^+ \rangle_{\mathbb{E}^3} + \\ + \lambda_k \varphi_k^+ + \langle \mu_k | A_k^+ \rangle_{\mathbb{E}^3}] |0\rangle, \end{aligned}$$

generated by the canonical quantum commutator relationships (2.19), where, by definition, the vacuum state  $|0\rangle \in \Phi$  satisfies the following constraint:

$$(2.30) \quad \begin{aligned} \text{span}_{(\lambda_0, \lambda) \in L_2(\mathbb{R}^3; \mathbb{C})^2} \{ \lambda_{0,k} p_{0,k}^- + \langle \mu_{0,k} | p_k^- \rangle_{\mathbb{E}^3} + \\ + \lambda_k \varphi_k^- + \langle \mu_k | A_k^- \rangle_{\mathbb{E}^3} \} |0\rangle = 0 \end{aligned}$$

for all momentums  $k \in \mathbb{R}^3$ . The respectively reduced Fock subspace

$$(2.31) \quad \Phi^+ := \overline{\ker p_0^+} \subset \Phi \simeq \{ |f\rangle \in \Phi : p_{0,k}^+ |f\rangle = 0, k \in \mathbb{R}^3 \}$$

neglects both the condition  $\nabla p_0|_{\Phi^+}$  and the zero charge density value  $\langle \nabla |p\rangle|_{\Phi^+} = 0$  constraint, which should be imposed on the suitably reduced Fock subspace  $\Phi^+$ . Moreover, with respect to the reduced Hamiltonian operator  $H_b^+ := H_b|_{\Phi^+} \in \text{End } \Phi^+$  the temporal evolution of any state  $|f\rangle \in \Phi^+$  does not assure the corresponding vanishing of the averaged Lorenz subsidiary constraint:

$$(2.32) \quad \bar{p}_0^+ := (f|p_0 f) = (f|p_0^+ f + p_0^- f) = (p_0^+ f|f) + (f|p_0^+ |f) = 0$$

for all times  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ . Really, the above expression (2.32) is not time invariant:

$$(2.33) \quad \begin{aligned} dp_0^+/d\tilde{t} &= (\partial f/\partial\tilde{t}|p_0^+ |f) + (f|\partial p_0^+/\partial\tilde{t}|f) + (f|p_0^+ |\partial f/\partial\tilde{t}) = \\ &= (\partial f/\partial\tilde{t}|p_0^+ |f) + \frac{i}{\hbar}(f|H_b^+ p_0^+ - p_0^+ H_b^+ |f) + (f|p_0^+ |\partial f/\partial\tilde{t}) = \\ &= (\partial f/\partial\tilde{t}|p_0^+ |f) + \frac{i}{\hbar}(f|H_b^+ p_0^+ |f) - \frac{i}{\hbar}(f|p_0^- H_b^+ |f) + (f|p_0^+ |\partial f/\partial\tilde{t}) = \\ &= -\frac{i}{\hbar}(p_0^- f|H_b^+ |f) + (p_0^- f|\partial f/\partial\tilde{t}) = (p_0^- f|(\partial f/\partial\tilde{t}) - \frac{i}{\hbar} H_b^{+*} |f)) \neq 0, \end{aligned}$$

if to take into account that the reduced Hamiltonian operator  $H_b^+ \in \text{End } \Phi_0^+$  is not both positive semidefinite and symmetric, that is  $H_b^+ \neq H_b^{+*}$  on the reduced Fock subspace  $\Phi^+ \subset \Phi$ .

Faced with these inconsistencies, Gupta [29] and Bleuler [11] proposed to consider the Fock space  $\Phi$ , rigged with a suitably constructed indefinite metric jointly with the noncanonical second order in time Klein-Gordon evolution equation to satisfy the zero charge density constraint, discussed above. However, all of this directly addresses the essence of the mathematical and conceptual failure of the Gupta-Bleuler quantization approach. It highlights exactly why physicists were forced to abandon

it for more sophisticated frameworks like the BRST [4, 5, 30, 31, 26] quantization method, as ad-hoc introduction of an indefinite metric Fock space to fix a manifestation of negative energy feels mathematically artificial and much unconvincing. Moreover, forcing a second-order constraint, based on the Lorenz condition and the Klein-Gordon equation, into a first-order canonical framework completely violates the strict mathematical rules laid out in Dirac's canonical quantization of constrained systems. This is why the BRST approach is fundamentally superior and resolves these two fatal flaws, on which we will not dwell here in more detail.

As it had been written in the book [16] (p.120) by N.N. Bogolubov (Sr.) and D.V. Shirkov, "... these conditions (2.31) guarantee that the Lorenz condition (2.32) is satisfied on the average, which is quite sufficient to establish correspondence with the classical field..." Yet, as follows from the above stated calculative result (2.33), it is not the case, that jointly with the above stated nonsymmetry and positive not semi-definiteness of the suitably reduced Hamiltonian operator  $H_{b,0}^+ : \Phi_0^+ \rightarrow \Phi_0^+$  make it possible to suggest that the Gupta-Bleuler quantization scheme, as applied to electromagnetic field on a suitably reduced Fock subspace  $\Phi^+ \subset \Phi$ , gives rise to a physically ill-posed quantum field theory. In contrast, as it was demonstrate above, the respectively applied canonical Dirac-Fock quantization scheme gives rise to a well posed quantum electromagnetic field theory with both symmetric and positive semi-definite Hamiltonian operator  $H_{b,0} : \Phi_0^{(red)} \rightarrow \Phi_0^{(red)}$  on a suitably reduced Fock subspace  $\Phi_0^{(red)} \subset \Phi_0$ .

Moreover, from the physical point of view, this result means that the constrained Fock subspace  $\Phi_0^+ \subset \Phi$  is not suitable for applying the canonical Dirac type quantization scheme and does not allow to construct an adequate and correct scattering theory as well as the related Schwinger's  $S$  - matrix theory, whose perturbative mathematical structure has been clarified and in detail developed [14, 16] within the so called Bogolubov-Parasyuk  $R$  - operation. Many years ago, concerning these quantization aspects, the great mathematician and mathematical physicist N.N. Bogolubov (Sr.) had written, yet never published a preprint [12], devoted to the variational approach to quantum electromagnetic field theory, in which he tried to dissolve this not invariance problem and related inconsistencies of the Gupta-Bleuler reduced Fock subspace (2.31) within the interaction representation of the corresponding perturbative  $S$  - matrix expansion.

N.N. Bogolubov (Sr.) then had written: "... Consider now a question about subsidiary conditions, which one needs to get subordinated electromagnetic field under interaction. We need then to proceed from the corresponding condition in the noninteracting field theory. In this case, as a rule, one uses the Lorenz condition

$$(2.34) \quad \sum_{(\alpha)} \frac{\partial A_\alpha(x)}{\partial x_\alpha} f(x) = 0$$

at any space-time point  $x \in M^4$ , where a wave vector  $f \in \tilde{\Phi}$  and a mapping  $(A_\alpha : \alpha = \overline{0,3}) : M^4 \rightarrow \mathbb{R} \times \mathbb{E}^3$  denotes the electromagnetic four-vector operator field on the Minkowski space-time. Yet, during past years in his works Gupta and Bleuler criticized the reasonableness of imposing condition (2.34) on the electromagnetic operator vector field and proposed to use a "weakened" condition of the type

$$(2.35) \quad \sum_{(\alpha)} \frac{\partial A_\alpha^+(x)}{\partial x_\alpha} f(x) = 0,$$

where  $(A_\alpha^- : \alpha = \overline{0,3}) : M^4 \rightarrow \mathbb{R} \times \mathbb{E}^3$  - the positive-frequency part of the electromagnetic field operators. Now we demonstrate, how the conditions (2.34) and (2.35) will change under interaction of the electromagnetic field with ambient charge distribution. Since the  $S$  - matrix operator  $S(g)$ , where  $g : M^4 \rightarrow \mathbb{R}$  denotes a seed interaction parameter, establishes a connection between the wave vector for free fields and the wave vector in the presence of interaction, we see that the influence of the

operator constraint

$$(2.36) \quad B(x)f(x) = 0,$$

where either  $B(x) := \sum_{(\alpha)} \frac{\partial A_{\alpha}(x)}{\partial x_{\alpha}}$  or  $\sum_{(\alpha)} \frac{\partial A_{\alpha}^{-}(x)}{\partial x_{\alpha}}$ ,  $x \in M^4$ , on the free particle wave vector  $f \in \tilde{\Phi}$  can be written as a condition, imposed on the interacting wave vector  $f(g) \in \tilde{\Phi}$ :

$$(2.37) \quad S(g)B(x)S^{+}(g)f(g) = 0.$$

Let us construct the expansion of (2.37) in the interaction seed parameter  $g : M^4 \rightarrow \mathbb{R}$  of the operator expression  $S(g)B(x)S^{+}(g) : \tilde{\Phi} \rightarrow \tilde{\Phi}$ ,  $x \in M^4$ . Taking into account that

$$(2.38) \quad SBS^{+} = B + (SB - BS)S^{+}$$

and

$$(2.39) \quad \begin{aligned} SB(x) - B(x)S = \sum_{n \in \mathbb{N}} \frac{1}{n!} \int \underbrace{M^4 \times \dots \times M^4}_{n \text{ times}} \{ S_n(x_1, x_2, \dots, x_n) B(x) - \\ - B(x) S_n(x_1, x_2, \dots, x_n) \} \times g(x_1) g(x_2) \dots g(x_n) dx_1 dx_2 \dots dx_n, \end{aligned}$$

as well as the electromagnetic field commutation conditions

$$(2.40) \quad [A_{\mu}(x), A_{\nu}(y)] = \delta_{\mu\nu} D_0(x - y),$$

for arbitrary space-time points  $x, y \in M^4$  and  $\mu, \nu = \overline{0, 3}$ , where  $D_0$  - the already mentioned above invariant Pauli-Jordan function. Since for arbitrary  $\nu = \overline{0, 3}$

$$(2.41) \quad \left[ \sum_{(\mu)} \frac{\partial A_{\mu}(x)}{\partial x_{\mu}}, A_{\nu}(y) \right] = i \frac{\partial D_0(x - y)}{\partial x_{\nu}}$$

in case (2.34) one obtains that

$$(2.42) \quad [B(x), A_{\nu}(y)] = i \frac{\partial D_0(x - y)}{\partial x_{\nu}},$$

and in case (2.35) one obtains

$$(2.43) \quad [B(x), A_{\nu}(y)] = i \frac{\partial D_0^{+}(x - y)}{\partial x_{\nu}},$$

these expressions give rise to the following commutator relationships:

$$(2.44) \quad \begin{aligned} S_n(x_1, x_2, \dots, x_n) B(x) - B(x) S_n(x_1, x_2, \dots, x_n) = \\ = -i \sum_{(k=\overline{1, n, \nu})} \frac{\partial D_0(x - y)}{\partial x_{\nu}} \frac{\partial S_n(x_1, x_2, \dots, x_n)}{\partial A_{\nu}(x_k)} \end{aligned}$$

in the case (2.34) and similarly to the commutator relationship

$$(2.45) \quad \begin{aligned} S_n(x_1, x_2, \dots, x_n) B(x) - B(x) S_n(x_1, x_2, \dots, x_n) = \\ = -i \sum_{(k=\overline{1, n, \nu})} \frac{\partial D_0^{-}(x - y)}{\partial x_{\nu}} \frac{\partial S_n(x_1, x_2, \dots, x_n)}{\partial A_{\nu}(x_k)} \end{aligned}$$

in the case (2.35). Thus, the commutator (2.39) becomes

$$(2.46) \quad \begin{aligned} SB(x) - B(x)S = - \sum_{n \in \mathbb{N}} \frac{i}{(n-1)!} \int \underbrace{M^4 \times \dots \times M^4}_{n \text{ times}} \frac{\partial D_0(x - x_1)}{\partial x_{\nu}} \times \\ \times \frac{\partial S_n(x_1, x_2, \dots, x_n)}{\partial A_{\nu}(x_1)} g(x_1) g(x_2) \dots g(x_n) dx_1 dx_2 \dots dx_n = \\ = - \sum_{n \in \mathbb{N}} \frac{i}{n!} \int \underbrace{M^4 \times \dots \times M^4}_{n+1 \text{ times}} \frac{\partial D_0(x - x_1)}{\partial x_{\nu}} \frac{\partial S_{n+1}(x_1, x_2, \dots, x_{n+1})}{\partial A_{\nu}(x_1)} \times \\ \times g(x_1) g(x_2) \dots g(x_{n+1}) dx_1 dx_2 \dots dx_{n+1} \end{aligned}$$

giving rise to the following result:

$$\begin{aligned}
(2.47) \quad (SB(x) - B(x)S) S^+ &= - \sum_{(n \in \mathbb{N}, \nu)} \frac{i}{n!} \underbrace{\int M^4 \times \dots \times M^4}_{n \text{ times}} \frac{\partial D_0(x-y)}{\partial x_\nu} \left\{ K \left( \frac{x_1 - x_k}{x_{k+1} - x_n} \right) \frac{\partial S_{k+1}(y, x_1, \dots, x_k)}{\partial A_\nu(y)} \times \right. \\
&\quad \times S_{n-k}^+(x_{k+1}, x_{k+2}, \dots, x_n) \left. \right\} g(x_1) g(x_2) \dots g(x_n) g(y) dy dx_1 dx_2 \dots dx_n dx_n = \\
&= - \sum_{(n \in \mathbb{N}, \nu)} \frac{i}{\hbar n!} \underbrace{\int M^4 \times \dots \times M^4}_{n \text{ times}} \frac{\partial D_0(x-y)}{\partial x_\nu} \frac{\delta H(y, x_1, \dots, x_n)}{\delta A_\nu(y)} \times \\
&\quad \times g(x_1) g(x_2) \dots g(x_n) g(y) dy dx_1 dx_2 \dots dx_n dx_n = \\
&= - \sum_{(\nu)} \frac{1}{\hbar} \int_{M^4} \frac{\partial D_0(x-y)}{\partial x_\nu} \frac{\delta H(y, g)}{\delta A_\nu(y)} g(y) dy = - \sum_{(\nu)} \frac{1}{\hbar} \int_{M^4} \frac{\partial D_0(x-y)}{\partial x_\nu} J_\nu(y; g) g(y) dy = \\
&= \sum_{(\nu)} \frac{1}{\hbar} \int_{M^4} \frac{\partial D_0(x-y)}{\partial x_\nu} \frac{\partial (J_\nu(y; g) g(y))}{\partial y_\nu} dy,
\end{aligned}$$

where  $K \left( \frac{x_1 - x_k}{x_{k+1} - x_n} \right)$  denotes combinations from  $n \in \mathbb{N}$  marked elements into subsets of  $k$  - and  $(n - k)$  - element groups, and  $H(y; g)$ ,  $y \in M^4$ , represents the Hamiltonian operator of the interacting electromagnetic field, as well as when deriving (2.47), there was used the classical variational relationship  $\frac{\delta H(y, g)}{\delta A_\nu(y)} = J_\nu(y; g)$ ,  $\nu = \overline{0, 3}$ , holding at all points  $y \in M^4$ . If now to take into account the charge conservation law  $\sum_{(\nu)} \frac{\partial J_\nu(y; g)}{\partial x_\nu} = 0$  at all  $y \in M^4$ , we obtain

$$(2.48) \quad (SB(x) - B(x)S) S^+ = - \sum_{(\nu)} \frac{1}{\hbar} \int_{M^4} \frac{\partial D_0(x-y)}{\partial x_\nu} J_\nu(y; g) \frac{\partial g(y)}{\partial y_\nu} dy,$$

which jointly with the operator expression (2.38) reduces to the following operator relationship:

$$(2.49) \quad \left\{ \sum_{(\mu)} \frac{\partial A_\mu(x)}{\partial x_\mu} - \sum_{(\mu)} \frac{1}{\hbar} \int_{M^4} \frac{\partial D_0(x-y)}{\partial x_\nu} J_\nu(y; g) \frac{\partial g(y)}{\partial y_\nu} dy \right\} f(g) = 0,$$

or similarly for the case (2.35)

$$(2.50) \quad \left\{ \sum_{(\mu)} \frac{\partial A_\mu^+(x)}{\partial x_\mu} - \sum_{(\mu)} \frac{1}{\hbar} \int_{M^4} \frac{\partial D_0^+(x-y)}{\partial x_\nu} J_\nu(y; g) \frac{\partial g(y)}{\partial y_\nu} dy \right\} f(g) = 0.$$

To the regret, these N.N. Bogolubov's (Sr.) notes on the Gupta and Bleuler quantization scheme of electromagnetic field had been not finished with no further both suggestions and inferences concerning the real mathematical physics meaning of the carried out analysis of the problem. Nonetheless, as a natural consequence of these N.N. Bogolubov's (Sr.) calculations and the way of reasonings, presented above, one can clearly state that the Lorenz subsidiary operator condition (2.34) for the interacting electromagnetic operator field should be deformed by some electric current operator components.

To refine the analytical form of this subsidiary Lorenz operator deformation, predicted by N.N. Bogolubov (Sr.), in the next section we will follow the approach devised in [19] and modified in [44, 45].

### 3. THE INTERACTING ELECTROMAGNETIC FIELD AND QUANTUM LORENZ TYPE CONDITION DEFORMATION

There are considered the fermionic electron creation and annihilation operator fields  $\psi, \psi^+ : \Phi \rightarrow \Phi^4$  together with the related bosonic electromagnetic potential operator field  $(\varphi, A) : \Phi \rightarrow \Phi^4$ , acting on the related Fock space  $\Phi$ , and respectively, commuting to each other, which we will treat within the Fock many-temporal description approach [19, 24] subject to the common reference frame  $M_{\vec{t}}^4$ , specified by the temporal parameter  $\vec{t} \in \mathbb{R}_{\vec{t}}$ . Based on the classical spin symmetry properties [48, 39, 40, 46, 47] of electron, "minimum interaction" principle [15, 9, 44, 45] and the Lagrangian functional (2.1), there is the corresponding Dirac evolution equation

$$(3.1) \quad \partial \psi / \partial t = [ \langle \alpha | \nabla \rangle + \frac{i}{\hbar} \beta m ] \psi + \frac{ie}{\hbar} (\varphi - \langle \alpha | A \rangle) \psi$$

on the operator spinor field  $\psi : \Phi \rightarrow \Phi^4$ , where  $\beta, \alpha \in su(2) \times su(2)$  denotes the usual four-vector of Dirac matrices and  $m \in \mathbb{R}_+$  is the electron mass, we can write down the related Hamiltonian operator expression

$$(3.2) \quad H_f^{(int)} = \int_{\mathbb{R}^3} d^3x \psi^+ [\langle \alpha | \frac{\hbar}{i} \nabla \rangle + \beta m] \psi + \int_{\mathbb{R}^3} d^3x (e \psi^+ \psi \varphi - e \psi^+ \langle \alpha | A \rangle \psi),$$

for the electron operator field interacting with the electromagnetic operator field, where  $e$  denotes the elementary charge value. Thus, the evolution of the entire quantum field system, consisting of the fermionic and electromagnetic operator fields, is described by means of the joint Hamiltonian operator

$$(3.3) \quad H_{f-b} := H_f^{(int)} + H_b$$

where the bosonic operator  $H_b : \Phi \rightarrow \Phi$  is given by the expression (2.7), via the Heisenberg commutator equations

$$(3.4) \quad \partial \psi / \partial t = \frac{i}{\hbar} [H_{f-b}, \psi], \quad \partial \psi^+ / \partial t = \frac{i}{\hbar} [H_{f-b}, \psi^+]$$

and

$$(3.5) \quad \partial A / \partial t = \frac{i}{\hbar} [H_{f-b}, A], \quad \partial \varphi / \partial t = \frac{i}{\hbar} [H_{f-b}, \varphi]$$

with respect to the common temporal parameter  $t \in \mathbb{R}_t$ , as in this case there is assumed that the corresponding proper temporal bosonic  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$  and proper temporal fermionic  $\bar{t} \in \mathbb{R}_{\bar{t}}$  parameters, respectively, coincide, that is  $\tilde{t} = \bar{t} = t \in \mathbb{R}_t$ , and where, by definition, the operators  $\psi(t, x) := \psi(\tilde{t}, \tilde{t}; x)|_{\tilde{t}=\bar{t}=t}$ ,  $\varphi(t, x) := \varphi(\tilde{t}, x)|_{\tilde{t}=\bar{t}=t}$ ,  $A(t, x) := A(\tilde{t}, x)|_{\tilde{t}=t}$  at any point  $t \in \mathbb{R}_t$ .

Simultaneously, there should be, evidently, satisfied the before derived both the electromagnetic field evolution equations (2.4) with respect to the own reference frame  $M_{\tilde{t}}$  and the modified fermionic charged particle field equations

$$(3.6) \quad \partial \psi / \partial \bar{t} = \frac{i}{\hbar} [H_f^{(int)}, \psi], \quad \partial \psi^+ / \partial \bar{t} = \frac{i}{\hbar} [H_f^{(int)}, \psi^+]$$

with respect to the own reference frame  $M_{\tilde{t}}$ .

Being mostly interested in the evolution of the quantum particle fermionic field  $\psi : \Phi \rightarrow \Phi$ , we can get rid of the bosonic Hamiltonian impact into the evolution (3.6), having applied to the Fock space  $\Phi$  the unitary canonical transformation

$$(3.7) \quad \Phi \rightarrow \tilde{\Phi} := U(t) \Phi,$$

where we denoted by  $U(t) : \Phi \rightarrow \tilde{\Phi}$  the unitary operator, satisfying the determining equation

$$(3.8) \quad dU(t)/dt = \frac{i}{\hbar} H_b U(t)$$

generated by the bosonic Hamiltonian operator (2.7) with respect to the common temporal parameter  $t \in \mathbb{R}$ . As a consequence of the transformation (3.7) we obtain the effective fermionic particle field interaction Hamiltonian operator

$$(3.9) \quad \tilde{H}_f^{(int)} := U(t) H_f^{(int)} U^*(t) = \int_{\mathbb{R}^3} d^3x \psi^+ \langle \alpha | \frac{\hbar}{i} \nabla \rangle \psi + \\ + \int_{\mathbb{R}^3} d^3x (e \psi^+ \psi \tilde{\varphi} - e \psi^+ \langle \alpha | \tilde{A} \rangle \psi),$$

where we put, by definition,

$$(3.10) \quad \tilde{A} := U(t) A U^*(t), \quad \tilde{\varphi} := U(t) \varphi U^*(t),$$

$$\tilde{p}_0 := U(t) p_0 U^*(t), \quad \tilde{p} := U(t) p U^*(t)$$

and took into account that  $\tilde{\psi} := U(t)\psi U^*(t) = \psi$ ,  $\tilde{\psi}^+ := U(t)\psi^+ U^*(t) = \psi^+$ , subject to which the evolution in the transformed Fock space  $\tilde{\Phi}$ , governed by the Hamiltonian operator (2.7) in the form

$$(3.11) \quad \tilde{H}_b = \frac{1}{2} \int_{\mathbb{R}_x^3} (|\tilde{p}|^2 + |\nabla \times \tilde{A}|^2 - \tilde{p}_0^2 - \tilde{p}_0 \langle \nabla | \tilde{A} \rangle - \langle \tilde{p} | \nabla \tilde{\varphi} \rangle) d^3x$$

became completely eliminated. The electromagnetic operator field variables and their canonically adjoint ones (3.10) satisfy, respectively, the following evolution equations:

$$(3.12) \quad \partial \tilde{A} / \partial \tilde{t} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{A}], \quad \partial \tilde{\varphi} / \partial \tilde{t} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{\varphi}]$$

and

$$(3.13) \quad \partial \tilde{p}_0 / \partial \tilde{t} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{p}_0], \quad \partial \tilde{p} / \partial \tilde{t} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{p}]$$

with respect to the electromagnetic vector field proper time  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ . Really, from equations (3.5) one obtains

$$(3.14) \quad \begin{aligned} \partial \tilde{A} / \partial t &= \frac{i}{\hbar} [\tilde{H}_{f-b}, \tilde{A}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} = \frac{i}{\hbar} [\tilde{H}_f^{(int)}, \tilde{A}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} + \frac{i}{\hbar} [\tilde{H}_b, \tilde{A}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}}, \\ \partial \tilde{\varphi} / \partial t &= \frac{i}{\hbar} [\tilde{H}_{f-b}, \tilde{\varphi}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} = \frac{i}{\hbar} [\tilde{H}_f^{(int)}, \tilde{\varphi}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} + \frac{i}{\hbar} [\tilde{H}_b, \tilde{\varphi}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}}. \end{aligned}$$

At the same time, one can write down equivalently that

$$(3.15) \quad \partial \tilde{A} / \partial t = \partial \tilde{A} / \partial \tilde{t} \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} + \frac{i}{\hbar} [\tilde{H}_f^{(int)}, \tilde{A}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}}, \quad \partial \tilde{\varphi} / \partial t = \partial \tilde{\varphi} / \partial \tilde{t} \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} + \frac{i}{\hbar} [\tilde{H}_f^{(int)}, \tilde{\varphi}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}}.$$

From derived above relationships (3.14) and (3.15) one easily obtains that

$$(3.16) \quad \partial \tilde{A} / \partial \tilde{t} \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{A}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}}, \quad \partial \tilde{\varphi} / \partial \tilde{t} \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{\varphi}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}},$$

coinciding with the above written system of equations (3.12). Similarly one obtains the system of equations

$$(3.17) \quad \frac{\partial \tilde{p}_0}{\partial \tilde{t}} \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{p}_0] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}}, \quad \frac{\partial \tilde{p}}{\partial \tilde{t}} \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}} = \frac{i}{\hbar} [\tilde{H}_b, \tilde{p}] \Big|_{\substack{\tilde{t}=t, \\ \tilde{t}=t}},$$

coinciding with the above written system of equations (3.13).

Concerning the Hamiltonian operator (3.11), we need to mention that the related Dirac quantized commutation relationships for the operators  $(\tilde{\varphi}, \tilde{A}) : \tilde{\Phi} \rightarrow \tilde{\Phi}^4$  and their adjoint ones  $(\tilde{p}_0, \tilde{p}) : \tilde{\Phi} \rightarrow \tilde{\Phi}^4$  remain the same, that is

$$(3.18) \quad \begin{aligned} [\tilde{\varphi}(x), \tilde{\varphi}(y)] &= 0 = [\tilde{A}(x), \tilde{A}(y)], [\tilde{p}_0(x), \tilde{p}_0(y)] = 0, [\tilde{p}(x), \tilde{p}(y)] = 0, \\ [\tilde{p}_0(x), \tilde{\varphi}(y)] &= \frac{\hbar}{i} \delta(x - y), [\tilde{p}(x), \otimes \tilde{A}(y)] = \frac{\hbar}{i} I \delta(x - y), \end{aligned}$$

for all  $x, y \in \mathbb{R}^3$ . Now, concerning the Hamiltonian operators (3.9) and (3.11), the following Heisenberg evolution operator equations

$$(3.19) \quad \partial \psi / \partial \tilde{t} = \frac{i}{\hbar} [\tilde{H}_f^{(int)}, \psi], \quad \partial \psi^+ / \partial \tilde{t} = \frac{i}{\hbar} [\tilde{H}_f^{(int)}, \psi^+]$$

with respect to the proper reference frame  $M_{\tilde{t}}$  together with the Heisenberg evolution operator equations (3.12) and (3.13) with respect to the proper reference frame  $M_{\tilde{t}}$  hold.

Being further interested in the evolution equations (3.4) and (3.5), suitably rewritten on the transformed Fock space  $\tilde{\Phi}$  with respect to the common temporal parameter  $t \in \mathbb{R}$ , we need to take into account [24, 44, 45] that the following functional

relationships

$$(3.20) \quad \psi(t) := \psi(\bar{t}, \tilde{t})|_{\bar{t}=\tilde{t}=t}, \quad \psi^+(t) := \psi^+(\bar{t}, \tilde{t})|_{\bar{t}=\tilde{t}=t}, \quad \tilde{\varphi}(t) := \tilde{\varphi}(\bar{t}, \tilde{t})|_{\bar{t}=\tilde{t}=t},$$

$$\tilde{A}(t) := \tilde{A}(\bar{t}, \tilde{t})|_{\bar{t}=\tilde{t}=t}, \quad \tilde{p}_0(t) := \tilde{p}_0(\bar{t}, \tilde{t})|_{\bar{t}=\tilde{t}=t}, \quad \tilde{p}(t) := \tilde{p}(\bar{t}, \tilde{t})|_{\bar{t}=\tilde{t}=t},$$

hold. In particular, from (3.20) the evolution expressions follow

$$(3.21) \quad \partial\psi(t)/\partial t = \partial\psi(\bar{t}, \tilde{t})/\partial\bar{t}|_{\bar{t}=\tilde{t}=t} + \partial\psi(\bar{t}, \tilde{t})/\partial\tilde{t}|_{\bar{t}=\tilde{t}=t},$$

$$\partial\tilde{\varphi}(t)/\partial t = \partial\tilde{\varphi}(\bar{t}, \tilde{t})/\partial\bar{t}|_{\bar{t}=\tilde{t}=t} + \partial\tilde{\varphi}(\bar{t}, \tilde{t})/\partial\tilde{t}|_{\bar{t}=\tilde{t}=t},$$

$$\partial\tilde{A}(t)/\partial t = \partial\tilde{A}(\bar{t}, \tilde{t})/\partial\bar{t}|_{\bar{t}=\tilde{t}=t} + \partial\tilde{A}(\bar{t}, \tilde{t})/\partial\tilde{t}|_{\bar{t}=\tilde{t}=t},$$

$$\partial\tilde{p}_0(t)/\partial t = \partial\tilde{p}_0(\bar{t}, \tilde{t})/\partial\bar{t}|_{\bar{t}=\tilde{t}=t} + \partial\tilde{p}_0(\bar{t}, \tilde{t})/\partial\tilde{t}|_{\bar{t}=\tilde{t}=t},$$

$$\partial\tilde{p}(t)/\partial t = \partial\tilde{p}(\bar{t}, \tilde{t})/\partial\bar{t}|_{\bar{t}=\tilde{t}=t} + \partial\tilde{p}(\bar{t}, \tilde{t})/\partial\tilde{t}|_{\bar{t}=\tilde{t}=t},$$

for all  $t \in \mathbb{R}$ . The calculative expressions (3.21) prove to be useful when deriving the resulting quantum Maxwell electromagnetic equations. Before doing this, we need to check that the weak operator Lorenz constraints (2.25), rewritten in the case of the transformed Fock space  $\tilde{\Phi}$ , is compatible with the evolution equations (3.19) on the reduced Fock subspace

$$(3.22) \quad \tilde{\Phi}_0 := \{\tilde{f} \in \tilde{\Phi} : \tilde{p}_0(x)\tilde{f} = 0 \sim -(\partial\tilde{\varphi}/\partial\tilde{t} + \langle\nabla|\tilde{A}\rangle)|f\rangle = 0\}$$

for all  $x \in \mathbb{R}^3$  with respect to the proper time  $\tilde{t} \in \mathbb{R}$ :

$$(3.23) \quad [\tilde{p}_{0,k}^-, i\hbar\partial/\partial\tilde{t} - \tilde{H}_{b,0}]\tilde{f} = 0 = [\tilde{p}_{0,k}^+, i\hbar\partial/\partial\tilde{t} - \tilde{H}_{b,0}]\tilde{f}$$

for all momentums  $k \in \mathbb{R}^3$  and  $|\tilde{f}\rangle \in \tilde{\Phi}_0$ , yet they fail to be compatible with respect to the evolution equations (3.19), that is

$$(3.24) \quad [\tilde{p}_{0,k}^-, i\hbar\partial/\partial\tilde{t} - \tilde{H}_f^{(int)}]\tilde{f} \neq 0 \neq [\tilde{p}_{0,k}^+, i\hbar\partial/\partial\tilde{t} - \tilde{H}_f^{(int)}]\tilde{f}$$

for all  $|\tilde{f}\rangle \in \tilde{\Phi}_0$  with respect to the electron proper time  $\bar{t} \in \mathbb{R}$ . This means that we can not impose on the Fock space  $\tilde{\Phi}_0$  the constraint (3.22), as

$$(3.25) \quad \tilde{p}_0(x)|\tilde{f}\rangle := -(\partial\tilde{\varphi}/\partial\tilde{t} + \langle\nabla|\tilde{A}\rangle)|\tilde{f}\rangle \neq 0$$

for arbitrary  $|\tilde{f}\rangle \in \tilde{\Phi}_0$  and all proper time parameters  $\bar{t} \in \mathbb{R}_{\bar{t}}$  and  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ .

Yet, it is easy enough to check [15, 48, 19, 47] that the following deformed Lorenz type operator

$$(3.26) \quad \tilde{p}_0^{(def)}(x) = \tilde{p}_0(x) + \frac{e}{2\pi} \int_{\mathbb{R}^3} d^3y \tilde{\Theta}(\tilde{t}; x-y) \psi^+(y) \psi(y),$$

where the kernel

$$(3.27) \quad \tilde{\Theta}(\tilde{t}; x-y) = \frac{\delta(\tilde{t} + |x-y|) - \delta(\tilde{t} - |x-y|)}{2|x-y|}$$

is the well known Pauli-Jordan [16] generalized function, consisting of the retarded minus advanced Green's functions of the d'Alembert wave equations (2.3), already satisfies the compatibility conditions

$$(3.28) \quad [\tilde{p}_0^{(def)}(x), \tilde{p}_0^{(def)}(y)] = 0, \quad [\tilde{p}_0^{(p)}(x), i\hbar\partial/\partial\bar{t} - \tilde{H}_f^{(int)}]\tilde{f} = 0, \\ [\tilde{p}_0^{(p)}(x), i\hbar\partial/\partial\tilde{t} - \tilde{H}_0]\tilde{f} = 0$$

for arbitrary proper temporal parameters  $\bar{t} \in \mathbb{R}_{\bar{t}}$ ,  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ , all  $x, y \in \mathbb{R}^3$  and any  $|\tilde{f}\rangle \in \tilde{\Phi}_0^{(def)}$  from the deformed reduced Fock subspace

$$(3.29) \quad \tilde{\Phi}_0^{(def)} := \{\tilde{f} \in \tilde{\Phi} : \tilde{p}_0^{(def)}(x)|f\rangle = 0, x \in \mathbb{R}^3\}.$$

Moreover, the operator expression

$$(3.30) \quad \hat{\Theta}(\tilde{t}, \bar{t}; x) = \frac{e}{2\pi\hbar} \int_{\mathbb{R}^3} d^3y \tilde{\Theta}(\tilde{t}; x - y) \psi^+(y) \psi(y)$$

satisfies for all  $x \in \mathbb{R}^3$  and  $\tilde{t}, \bar{t} \in \mathbb{R}$  the determining differential-operator relationship

$$(3.31) \quad \frac{\partial}{\partial \tilde{t}} \hat{\Theta}(\tilde{t}, \bar{t}; x) + \frac{1}{\hbar} [H_f, \hat{\Theta}(\tilde{t}, \bar{t}; x)] = \frac{i}{\hbar} e \psi^+(x) \psi(x)$$

and solves, in part, the N.N. Bogolubov's (Sr.) equation (2.49). It is worth to remark here, that the determining differential-operator equation (3.31) allows another integral operator solution

$$(3.32) \quad \hat{\Theta}(\tilde{t}, \bar{t}; x) = \frac{e}{2\pi\hbar} \int_{\mathbb{R}^3} d^3z \int_{\mathbb{R}^3} d^3y \tilde{\Theta}(x; y, z) \tilde{p}_0(z) \psi^+(y) \psi(z),$$

whose kernel is given by the generalized function

$$(3.33) \quad \tilde{\Theta}(x; y, z) := \frac{\langle (y - x) | \nabla_y \delta^3(y - z) \rangle}{2i |y - x|^3}$$

for all  $x \neq y$  and  $z \in \mathbb{R}^3$ . Thus, the related evolution flows (3.12), (3.13) and (3.19) on the reduced Fock subspace should be correctly considered on the perturbed reduced Fock subspace (3.29), invariant with respect to the proper temporal parameters  $\bar{t} \in \mathbb{R}_{\bar{t}}$  and  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ .

Now we can proceed to deriving the corresponding quantum Maxwell equations starting from the operator equations (3.26) on the suitably transformed Fock space  $\tilde{\Phi}_0^{(def)} \subset \tilde{\Phi}$  and the electromagnetic tension operator fields definitions

$$(3.34) \quad \tilde{E} = -\partial \tilde{A} / \partial \tilde{t} - \nabla \tilde{\varphi}, \quad \tilde{B} = \nabla \times \tilde{A}.$$

Applying the differential operator  $\nabla \times$  and, respectively,  $\nabla$  to operator expressions (3.34), one easily obtains that

$$(3.35) \quad (\nabla \times \tilde{B} - \partial \tilde{E} / \partial \tilde{t})|_{\tilde{\Phi}_0^{(def)}} = \frac{e}{2\pi} \nabla \int_{\mathbb{R}^3} d^3y \Theta(x; y, z) \psi^+(y) \psi(z)|_{\tilde{\Phi}_0^{(def)}}$$

and

$$(3.36) \quad \langle \nabla | \tilde{E} \rangle|_{\tilde{\Phi}_0^{(def)}} = -\frac{e}{2\pi} \frac{\partial}{\partial \tilde{t}} \int_{\mathbb{R}^3} d^3y \Theta(x; y, z) \psi^+(y) \psi(z)|_{\tilde{\Phi}_0^{(def)}}, \quad \langle \nabla | \tilde{B} \rangle|_{\tilde{\Phi}_0^{(def)}} = 0,$$

which are considered in the strong operator sense on the reduced Fock subspace  $\tilde{\Phi}_0^{(def)} \subset \tilde{\Phi}$  with respect to the proper time reference frame  $M_t^4$  of the electromagnetic vector field. It is also easy to calculate the strong operator relationship

$$(3.37) \quad \tilde{p}_0|_{\tilde{\Phi}_0^{(lim)}} := -\left(\partial \tilde{\varphi} / \partial t + \langle \nabla | \tilde{A} \rangle\right)|_{\tilde{\Phi}_0^{(lim)}} = 0,$$

with respect to the limiting common reference frame  $M_t^4$ , where we took into account that the space  $\tilde{\Phi}_0^{(lim)} := \lim_{\tilde{t} \rightarrow t} \tilde{\Phi}_0^{(def)}|_{\tilde{t}=\tilde{t} \rightarrow t}$  at any  $t \in \mathbb{R}_t$ .

Now we proceed to calculation of the strong Maxwell type operator relationships (3.35) and (3.36) with respect to the common reference frame  $M_t^4$ :

$$(3.38) \quad (\nabla \times \tilde{B} - \partial \tilde{E} / \partial \tilde{t})|_{\tilde{\Phi}_0^{(def)}|_{\tilde{t}=\tilde{t} \rightarrow t}} = \frac{e}{2\pi} \nabla \int_{\mathbb{R}^3} d^3y \Theta(x; y, z) \psi^+(y) \psi(z)|_{\tilde{\Phi}_0^{(def)}|_{\tilde{t}=\tilde{t} \rightarrow t}} = 0$$

and

$$(3.39) \quad \begin{aligned} \langle \nabla | \tilde{E} \rangle|_{\tilde{\Phi}_0^{(lim)}} &= \frac{i}{\hbar} [H_f, \int_{\mathbb{R}^3} d^3y \Theta(x; y, z) \psi^+(y) \psi(z)] - \\ &- \frac{e}{2\pi} \frac{\partial}{\partial \tilde{t}} \int_{\mathbb{R}^3} d^3y \Theta(x; y, z) \psi^+(y) \psi(z)|_{\tilde{\Phi}_0^{(p)}|_{\tilde{t}=\tilde{t}=t}} = e \psi^+ \psi|_{\tilde{\Phi}_0^{(lim)}}, \end{aligned}$$

where there was used the corresponding evolution equations [16, 51] for operators  $\psi^+(y), \psi(z) : \tilde{\Phi}_0^{(lim)} \rightarrow \tilde{\Phi}_0^{(lim)}$ ,  $y, z \in \mathbb{R}^3$ . To calculate further the expression (3.38)

with respect to the common reference frame  $M_t^4$ , we need to make use of the strong operator relationships (3.21) and find that

$$(3.40) \quad \frac{\partial \tilde{E}}{\partial t} \Big|_{\tilde{t}=\tilde{t} \rightarrow t} = \frac{\partial \tilde{E}}{\partial t} - \frac{\partial \tilde{E}}{\partial \tilde{t}} \Big|_{\tilde{t}=\tilde{t} \rightarrow t} = \frac{\partial \tilde{E}}{\partial t} - \frac{i}{\hbar} [\tilde{H}_f^{(int)}, \tilde{E}] = \frac{\partial \tilde{E}}{\partial t} + e\psi^+ \alpha \psi.$$

Thus, from (3.40) and (3.38) one obtains that

$$(3.41) \quad \left( \nabla \times \tilde{B} - \partial \tilde{E} / \partial t \right) \Big|_{\tilde{\Phi}_0^{(lim)}} = e\psi^+ \alpha \psi \Big|_{\tilde{\Phi}_0^{(lim)}}$$

with respect to the common reference frame  $M_t^4$ . The combined together strong operator relationships (3.39) and (3.41)

$$(3.42) \quad \left( \nabla \times \tilde{B} - \partial \tilde{E} / \partial t \right) \Big|_{\tilde{\Phi}_0^{(lim)}} = e\psi^+ \alpha \psi \Big|_{\tilde{\Phi}_0^{(lim)}}, \quad \langle \nabla | \tilde{E} \rangle \Big|_{\tilde{\Phi}_0} = e\psi^+ \psi \Big|_{\tilde{\Phi}_0^{(lim)}}$$

on the reduced Fock subspace  $\tilde{\Phi}_0^{(lim)}$  together with the evident strong operator relationships

$$(3.43) \quad \nabla \times \tilde{E} + \partial \tilde{B} / \partial t = 0, \quad \nabla \times \tilde{B} = 0$$

compile the complete system of quantum Maxwell equations with respect to the common reference frame  $M_t^4$ . The presented above reasonings jointly with the described calculations can be reformulated as the final proposition.

**Proposition 3.1.** *The electromagnetic vector field, interacting with an ambient electrically charged medium and quantized within the Dirac-Fock-Podolsky Lagrangian formalism, is a well-posed quantum field theory with a positive definite operator energy density, defined on the strongly reduced Fock subspace, which is equal to the closure of the respectively quantized kernel of the auxiliary Lorentz operator condition. Moreover, the quantized electromagnetic field tensions satisfy the related quantum Maxwell equations, defined on this strongly reduced Fock subspace and compatible with the corresponding quantized Dirac operator equations for the ambient charged medium.*

Additionally, from the Heisenberg evolution equations (3.18) it is easy to obtain an operator relationship for the charge flux conservation:

$$(3.44) \quad \partial(e\psi^+ \psi) / \partial t + \langle \nabla | e\psi^+ \alpha \psi \rangle = 0,$$

in which the quantity  $\rho := e\psi^+ \psi$  is interpreted as the operator *charge density* and the quantity  $J := e\psi^+ \alpha \psi$  is naturally interpreted as the operator *current density* with respect to the common Minkowski reference frame  $M_t^4$ . Whence the strong operator equations (3.42) can be rewritten, taking into account the above charge and current density definitions in the standard Maxwell equations weak form:

$$(3.45) \quad \left( \nabla \times \tilde{B} - \partial \tilde{E} / \partial t \right) \Big|_{\tilde{\Phi}_0^{(lim)}} = J \Big|_{\tilde{\Phi}_0^{(lim)}}, \quad \langle \nabla | \tilde{E} \rangle \Big|_{\tilde{\Phi}_0^{(lim)}} = \rho \Big|_{\tilde{\Phi}_0^{(lim)}},$$

considered on the Fock subspace  $\tilde{\Phi}_0^{(lim)}$ , determined by the time invariant Lorenz operator constraint (3.37). Moreover, based on the strong Maxwell operator equations (3.45) and the Lorenz constraint (3.37), one can derive easily the following linear strong operator wave equations

$$(3.46) \quad \left( \partial^2 \tilde{\varphi} / \partial t^2 - \Delta \tilde{\varphi} \right) \Big|_{\tilde{\Phi}_0^{(lim)}} = \rho \Big|_{\tilde{\Phi}_0^{(lim)}}$$

and

$$(3.47) \quad \left( \partial^2 \tilde{A} / \partial t^2 - \Delta \tilde{A} \right) \Big|_{\tilde{\Phi}_0^{(lim)}} = J \Big|_{\tilde{\Phi}_0^{(lim)}}$$

with respect to the common laboratory reference frame  $M_t^4$ , allowing to calculate the induced by the quantum charged fermionic field the corresponding quantum electromagnetic potentials  $(\tilde{\varphi}, \tilde{A}) : \tilde{\Phi}_0^{(lim)} \rightarrow \left( \tilde{\Phi}_0^{(lim)} \right)^4$  in the analytical form:

$$(3.48) \quad \begin{aligned} \tilde{\varphi} &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(t', y) d^3 y}{|x - y|} + \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho_0(t', y) d^3 y}{|x - y|}, \\ \tilde{A} &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{J(t', y) d^3 y}{|x - y|} + \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{J_0(t', y) d^3 y}{|x - y|}, \end{aligned}$$

where  $t' := t - |x - y| \in \mathbb{R}$  denotes the "*retarded*" temporal parameter and neutral density  $\rho_0 : \tilde{\Phi}_0^{(\text{lim})} \rightarrow \tilde{\Phi}_0^{(\text{lim})}$  and neutral current  $J_0 : \tilde{\Phi}_0^{(\text{lim})} \rightarrow \tilde{\Phi}_0^{(\text{lim})}$  operators are generated by the vacuum  $\tilde{\Phi}_0^{(\text{lim})}$ , and satisfy, by definition, the following strong operator constraints:

$$(3.49) \quad \rho_0|_{\tilde{\Phi}_0^{(\text{lim})}} = 0, \quad [\tilde{H}_f^{(int)} + i\hbar\partial/\partial t, \rho_0]|_{\tilde{\Phi}_0^{(\text{lim})}} = 0$$

on the reduced Fock space  $\tilde{\Phi}_0^{(\text{lim})}$ . It is interesting to remark here that, owing to the conditions (3.49), the neutral charge potential component

$$(3.50) \quad \tilde{\varphi}_0 := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho_0(t', y) d^3 y}{|x - y|}$$

satisfies the quasi-stationary conditions  $\partial\rho_0(t, y)\partial t \simeq 0$ ,  $\langle \nabla | J_0(t, y) \rangle \simeq 0$  for all  $t \in \mathbb{R}$  and  $y \in \mathbb{R}$ , thus giving rise to the related Newton law

$$(3.51) \quad -\Delta\tilde{\varphi}_0 = \rho_0,$$

what together can be interpreted as a virtual gravity type interaction source.

Owing to the continuity relationship (3.44), the expressions (3.48) satisfy exactly the strong operator Lorenz subsidiary constraint

$$(3.52) \quad \left( \partial\tilde{\varphi}/\partial t + \langle \nabla | \tilde{A} \rangle \right) |_{\tilde{\Phi}_0^{(\text{lim})}} = 0,$$

coinciding with that of (3.37), regarding the common reference frame  $M_t^4$ .

In conclusion we present an interesting observation from the reasonings, presented in Supplement and related to the characteristic strong wave operator relationship (3.46): if considered classically for the averaged potential  $\bar{\varphi} := (\tilde{f}_0|\varphi|\tilde{f}_0)$  and charge density  $\bar{\rho} := (\tilde{f}_0|\rho|\tilde{f}_0)$  quantities for  $|\tilde{f}_0\rangle \in \tilde{\Phi}_0^{(\text{lim})}$ , one makes it possible to retrieve the complete system of the classical Maxwell equations.

There is demonstrated within the Dirac and Fock multi-time proper temporal paradigms that the quantum electromagnetic field both in vacuum and interacting with an ambient charged quantum medium, considered on the time invariant strongly reduced Fock subspace, presents a well posed relativistic quantum field theory with a nonnegative definite energy density. As a natural consequence from reasonings and presented calculations there were rederived quantum Maxwell equations, strongly defined on the mentioned above reduced Fock subspace. Moreover, there was reanalyzed the N.N. Bogolubov's (Sr.) prediction concerning a deformation of the strong Lorenz subsidiary constraint, imposed on quantum states of the related Fock space. There was also shown that the "weakened" Lorenz subsidiary condition, proposed by Gupta and Bleuler and widely accepted in the modern manuals on quantum electrodynamics is not time-invariant, as well as the respectively reduced Hamiltonian operator is not both symmetric and positive definite, thus makes appealing to the poor physically motivated so called Hilbert space with "indefinite" metric. Moreover, the averaged "weakened" Lorenz subsidiary condition proves to be not time invariant, thus not sufficient to establish correspondence with the classical field, as it had been claimed in [16], making appealing to the poor physically motivated so Hilbert space "indefinite" metric also not satisfactory.

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5. SUPPLEMENT: A SPATIAL ELECTRIC POTENTIAL ENERGY INVARIANCE AND THE  
MAXWELL ELECTROMAGNETIC EQUATIONS RETRIEVING

Consider from the very beginning a field theoretical description of the classical electromagnetic vector field, based on a real smooth scalar charge density distribution  $\rho \in C^1(M^4; \mathbb{R})$  on the Minkowski space  $M^4 \simeq \mathbb{R}_{\tilde{t}} \times \mathbb{R}_x^3$  with coordinates  $(\tilde{t}, x) \in M^4$ , generating via the classical integral expression the following electric retarded potential distribution

$$(5.1) \quad \varphi(\tilde{t}, x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(\tilde{t} - |x - y|, y) d^3 y}{|x - y|},$$

satisfying the well known relativistic [8, 16, 36, 41, 52] Klein-Gordon-Fock wave propagation equation

$$(5.2) \quad \partial^2 \varphi(\tilde{t}, x) / \partial \tilde{t}^2 - \Delta \varphi(\tilde{t}, x) = \rho(\tilde{t}, x),$$

where we assumed, for brevity, the light velocity  $c = 1$ . Owing to the charge conservation property

$$(5.3) \quad dQ[\rho; \Omega_{\tilde{t}}] / d\tilde{t} = 0$$

for any temporal parameter  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ , where

$$(5.4) \quad Q[\rho; \Omega_{\tilde{t}}] = \int_{\Omega_{\tilde{t}}} \rho(\tilde{t}, x(\tilde{t})) d^3 x(\tilde{t})$$

denotes the charge inside any closed spatial region  $\Omega_{\tilde{t}} \subset \mathbb{R}_x^3$  with the smooth boundary  $\partial\Omega_{\tilde{t}}$ , whose space-time pattern is formed by the evolution of the ambient charge. As it is easy to calculate, the latter means that the following partial differential relationship holds:

$$(5.5) \quad \partial \rho(\tilde{t}, x) / \partial \tilde{t} + \langle \nabla | J(\tilde{t}, x) \rangle = 0,$$

where  $J(\tilde{t}, x(\tilde{t})) := \rho(\tilde{t}, x(\tilde{t})) u(\tilde{t}, x(\tilde{t}))$  denotes the corresponding electric current, generated by the charge velocity  $u(\tilde{t}, x(\tilde{t})) := dx(\tilde{t}) / d\tilde{t}$  at internal points  $(\tilde{t}, x(\tilde{t})) \in \Omega_{\tilde{t}}$  for all  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ . Based on the charge conservation equation (5.5) and the wave propagation equation (5.2), one states the following instructive proposition.

**Proposition 5.1.** *Consider the electric vector potential*

$$(5.6) \quad \varphi(\tilde{t}, x) = \frac{e}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(\tilde{t} - |x - y|, y) d^3 y}{|x - y|}$$

at points  $x = x(\tilde{t}, x_0) \in \Omega_{\tilde{t}}$ ,  $x_0 \in \Omega_0$ , of an arbitrary closed domain  $\Omega_{\tilde{t}} \subset \mathbb{R}^3$  with smooth boundary  $\partial\Omega_{\tilde{t}}$  for all  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ . Then the following statements hold: a) the delayed temporal parameter  $\tilde{t}' = \tilde{t} - |x - y(\tilde{t}', y_0)|$  satisfies the wave propagation equation

$$(5.7) \quad \partial^2 \tilde{t}' / \partial \tilde{t}^2 - \Delta \tilde{t}' = 0$$

for any spatial evolution  $y(\tilde{t}', y_0) \in \mathbb{R}_x^3$ ,  $y_0 \in \mathbb{R}^3$ , and all  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ ; b) the electric potential energy functional

$$(5.8) \quad I[\varphi; \Omega_{\tilde{t}}] = \int_{\Omega_{\tilde{t}}} \varphi(\tilde{t}, x(\tilde{t})) d^3 x(\tilde{t}).$$

satisfies the temporal invariance condition

$$(5.9) \quad dI[\varphi; \Omega_{\tilde{t}}] / d\tilde{t} = 0$$

for all  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ , which is equivalent to the classical Lorenz subsidiary condition

$$(5.10) \quad \partial \varphi(\tilde{t}, x) / \partial \tilde{t} + \langle \nabla | A(\tilde{t}, x) \rangle = 0,$$

where, by definition,  $A(\tilde{t}, x) := \varphi(\tilde{t}, x) dx(\tilde{t}, x_0) / d\tilde{t}$ ,  $x(\tilde{t}, x_0) \in \Omega_{\tilde{t}}$ , denotes the induced magnetic vector potential [34] for all temporal parameters  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ ;

c) the classical Lorenz subsidiary condition (5.10) jointly with the differential relationships (5.2) and (5.5) reduces to the complete classical system of the Maxwell equations:

$$(5.11) \quad \begin{aligned} \nabla \times B(\tilde{t}, x) &= J(\tilde{t}, x) + \partial E(\tilde{t}, x)/\partial \tilde{t}, \quad \langle \nabla | E(\tilde{t}, x) \rangle = \rho(\tilde{t}, x), \\ \partial B(\tilde{t}, x)/\partial \tilde{t} &= -\nabla \times E(\tilde{t}, x), \quad \langle \nabla | B(\tilde{t}, x) \rangle = 0, \end{aligned}$$

satisfied at all  $(x, \tilde{t}) \in M^4$ .

*Proof.* a) It is easy to observe that the first derivatives of the delayed parameter  $\tilde{t}' = \tilde{t} - |x - y(\tilde{t}', y_0)|$  equal to the following expressions:

$$(5.12) \quad \frac{\partial \tilde{t}'}{\partial \tilde{t}} = U(\tilde{t}, x), \quad \frac{\partial \tilde{t}'}{\partial x} = -U(\tilde{t}, x) \frac{x - y(\tilde{t}', y_0)}{|x - y(\tilde{t}', y_0)|},$$

where the scalar multiplier  $U(\tilde{t}, x) := (1 - \frac{\langle x - y(\tilde{t}', y_0) | \dot{y}(\tilde{t}', y_0) \rangle}{|x - y(\tilde{t}', y_0)|})^{-1}$ ,  $\dot{y}(\tilde{t}', y_0) := \partial y(\tilde{t}', y_0)/\partial \tilde{t}'$ , is well defined for all  $(\tilde{t}, x) \in M^4$ . Respectively, calculating the second temporal  $\partial^2 \tilde{t}'/\partial \tilde{t}^2$  and spatial gradient  $\langle \nabla | \nabla \tilde{t}' \rangle$  derivatives, one easily proves their equality, so stating the wave propagation equation (5.7).

b) Taking into account both the charge invariance (5.5) and the electric potential

$$(5.13) \quad \varphi(\tilde{t}, x(t)) = \frac{e}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(\tilde{t} - |x(\tilde{t}, x_0) - y|, y) d^3 y}{|x(\tilde{t}, x_0) - y|}$$

relationships, one can obtain by means of direct calculations the following potential energy invariance:

$$(5.14) \quad \begin{aligned} dI[\varphi; \Omega_{\tilde{t}}]/d\tilde{t} &= \frac{d}{d\tilde{t}} \int_{\Omega_{\tilde{t}}} d^3 x(t, x_0) \frac{e}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(\tilde{t} - |x(t, x_0) - y|, y) d^3 y}{|x(t, x_0) - y|} = \\ &= \frac{e}{4\pi} \int_{\mathbb{R}^3} d^3 z \frac{d}{d\tilde{t}} \int_{\Omega_0} \frac{d^3 x_0}{|z|} \rho(\tilde{t} - |z|, x(\tilde{t}, x_0) - z) \text{Jac}(x(t, x_0)) = \\ &= \frac{e}{4\pi} \int_{\mathbb{R}^3} d^3 z \int_{\Omega_0} \frac{d^3 x_0}{|z|} [\partial \rho(\tilde{t} - |z|, x(\tilde{t}, x_0) - z)/\partial \tilde{t} + \\ &\quad + \langle \nabla | \rho(\tilde{t} - |z|, x(\tilde{t}, x_0) - z) \dot{x}(\tilde{t}, x_0) \rangle \text{Jac}(x(t, x_0))] = \\ &= \frac{e}{4\pi} \int_{\mathbb{R}^3} d^3 z \int_{\Omega_t} \frac{d^3 x(\tilde{t}, x_0)}{|z|} [\partial \rho(\tilde{t} - |z|, x(\tilde{t}, x_0) - z)/\partial \tilde{t} + \\ &\quad + \langle \nabla | J(\tilde{t} - |z|, x(\tilde{t}, x_0) - z) \rangle] = 0, \end{aligned}$$

where we put, by definition,  $\text{Jac}(x(\tilde{t}, x_0)) := |\det \partial x(\tilde{t}, x_0)/\partial x_0|$  - the usual Jacobian of the transformation  $\Omega_0 \ni x_0 \rightarrow x(\tilde{t}, x_0) \in \Omega_{\tilde{t}}$ , for all  $x_0 \in \Omega_0$ ,  $\tilde{t} \in \mathbb{R}_{\tilde{t}}$ . On the other hand, the expression (5.14) can be respectively recalculated as

$$(5.15) \quad \begin{aligned} dI[\varphi; \Omega_{\tilde{t}}]/d\tilde{t} &= \frac{d}{d\tilde{t}} \int_{\Omega_{\tilde{t}}} \varphi(\tilde{t}, x(\tilde{t}, x_0)) d^3 x(\tilde{t}, x_0) = \\ &= \frac{d}{d\tilde{t}} \int_{\Omega_0} \varphi(\tilde{t}, x(\tilde{t}, x_0)) \text{Jac}(x(\tilde{t}, x_0)) d^3 x_0 = \\ &= \int_{\Omega_0} [\partial \varphi(\tilde{t}, x(\tilde{t}, x_0))/\partial \tilde{t} + \langle \nabla | u(\tilde{t}, x) \varphi(\tilde{t}, x(\tilde{t}, x_0)) \rangle \text{Jac}(x(\tilde{t}, x_0))] d^3 x_0 = \\ &= \int_{\Omega_{\tilde{t}}} [\partial \varphi(\tilde{t}, x(\tilde{t}, x_0))/\partial \tilde{t} + \langle \nabla | u(\tilde{t}, x) \varphi(\tilde{t}, x(\tilde{t}, x_0)) \rangle] d^3 x(\tilde{t}, x_0) = 0, \end{aligned}$$

what is exactly equivalent to the condition (5.10) at all points  $(x, \tilde{t}) \in M^4$ .

c) Having defined the electric field tension  $E(\tilde{t}, x) := -\nabla \varphi(\tilde{t}, x) - \partial A(\tilde{t}, x)/\partial \tilde{t}$  and the magnetic field tension  $B(\tilde{t}, x) := \nabla \times A(\tilde{t}, x)$  at all points  $(\tilde{t}, x) \in M^4$ , upon applying to them curl-operation  $\nabla \times$  one easily obtains

$$(5.16) \quad \begin{aligned} \nabla \times E(\tilde{t}, x) &= \nabla \times (-\nabla \varphi(\tilde{t}, x) - \partial A(\tilde{t}, x)/\partial \tilde{t}) = -\partial B(\tilde{t}, x)/\partial \tilde{t}, \\ \nabla \times B(\tilde{t}, x) &= \nabla \times (\nabla \times A(\tilde{t}, x)) = \nabla \langle \nabla | A(\tilde{t}, x) \rangle - \Delta A. \end{aligned}$$

Taking now into account the Lorenz condition (5.10), one obtains

$$(5.17) \quad \begin{aligned} \nabla \times E(\tilde{t}, x) &= -\partial B(\tilde{t}, x)/\partial \tilde{t}, \quad \nabla \times B(\tilde{t}, x) := \nabla \times (\nabla \times A(\tilde{t}, x)) = \\ &= \partial E(\tilde{t}, x)/\partial \tilde{t} + [\partial^2 A(\tilde{t}, x)/\partial \tilde{t}^2 - \Delta A(\tilde{t}, x)]. \end{aligned}$$

Similarly, taking into account the wave propagation equation (5.2) jointly with the charge continuity equation (5.5), one obtains

$$(5.18) \quad \langle \nabla | \partial^2 A(\tilde{t}, x)/\partial \tilde{t}^2 - \Delta A(\tilde{t}, x) \rangle = \langle \nabla | J(\tilde{t}, x) \rangle,$$

that upon substituting it into (5.17) gives rise to the system

$$(5.19) \quad \begin{aligned} \nabla \times E(\tilde{t}, x) &= -\partial B(\tilde{t}, x)/\partial \tilde{t}, \\ \nabla \times B(\tilde{t}, x) &= \partial E(\tilde{t}, x)/\partial \tilde{t} + J(\tilde{t}, x), \end{aligned}$$

coinciding exactly with the classical Faraday and Bio-Savart laws in the Minkowski space-time  $M^4$ . Similarly, having applied the gradient operation  $\nabla$  to the vector magnetic and electric potentials, one simply derives from the wave propagation equation (5.2) and the Lorenz condition (5.10) the corresponding boundary conditions

$$(5.20) \quad \langle \nabla | B(\tilde{t}, x) \rangle = 0, \quad \langle \nabla | E(\tilde{t}, x) \rangle = \rho(\tilde{t}, x)$$

for the above evolution equations (5.19) at all points  $(\tilde{t}, x) \in M^4$ . Combining the differential relationships (5.19) with those of (5.20), we obtain the classical Maxwell equations, proving the proposition.  $\square$

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