Review of: "The smallest gap between primes"

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Potential competing interests: No potential competing interests to declare.

This paper proves the twin prime conjecture,

that there are infinitely many prime numbers such that $p_{k+1}=p_k+2$,

where

the primes are numbered

$$p_1 = 2, \quad p_2 = 3, \quad p_3 = 5, \quad p_4 = 7, \ \ldots.$$

A central role is played by the function

$$H_a(x) = \log\left(\frac{x}{x-1}\right) - \frac{1}{x+a} + \log\left(\frac{x^2 - \sqrt[\log(x)+1]{x}}{x^2}\right).$$

The root in this formula can be understood as follows,

$$\sqrt[\log(x)+1]{x} = e^{\frac{\log x}{1+\log x}} = e\cdot\exp\left(-\frac{1}{1+\log x}\right).$$

I do not understand the role of this function in the proof.

In particular,

could it be replaced by a constant?

It is a bounded function for $x\geq1$,

slowly increasing from 1 to e.

It follows that

$$H_a(x) \approx \frac{1}{x^2}\left(\frac{1}{2} + a - e\right), \qquad \text{as } x \to \infty.$$

In particular,

$H_2(x)<0$ for large x and $H_4(x)>0$.

The main insight of the paper is Lemma 5,

$$\sum_{p_k\geq p_n}\left(\log(\frac{p_k}{p_k-1}-\log(1+\frac{1}{p_{k+1}})\right)=\log(1+\frac{1}{p_n})+\log(\prod_{p_k\geq p_n}\frac{p_k^2}{p_k^2-1}).$$

This could be verified directly,

but in the paper it is proved by an appeal to the Euler product for $\zeta(2)$,

$$\frac{\pi^2}{6}=\prod_{k=1}^{\infty}\frac{p_k^2}{p_k^2-1}.$$

If the twin prime conjecture were false then $p_{k+1}\geq p_k+4$ for $k\geq n$,

where p_n is a prime number after the last twin prime.

This makes the connection with H_4,

and a contradiction is derived with the positivity of H_4.

The proof of this starts with a computation that could be formulated as a separate lemma:

for p_n large enough,

$$\frac{p_n+1}{p_n}\cdot\prod_{p_k\geq p_n}\frac{p_k^2}{p_k^2-1}\leq\prod_{p_k\geq p_n}\frac{p_k^2}{p_k^2-\frac{\log(p_k)+1}{\sqrt{p_k}}}.$$

The proof of this lemma ends with the statement that it is implied by the inequality

$$2\cdot\left(1-\prod_{p_k>p_n}\frac{p_k^2-1}{p_k^2}\right)>\log\zeta(2).$$

The left-hand side is close to 0 for large n,

whereas the right-hand side is a positive number.

The inequalities from this point in the paper are in the opposite direction,

and there is no contradiction.