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Research Article

Volume-Based Probability: Outcome Frequencies from Deterministic Geometry

Zayn Blore[1]

1. Independent researcher

In standard quantum mechanics, the Born rule is introduced as a postulate: outcome probabilities equal the squared amplitude of the wavefunction. This paper proposes a deterministic alternative based on the geometry of a constrained state space. We consider a smooth, finite-dimensional, Hausdorff manifold S, equipped with a volume-preserving flow φ_t and a conserved measure μ. A physical experiment corresponds to evolving an initial region $\Omega_0 \subset S$ into a disjoint union of macroscopically distinguishable outcome regions $\{\Omega_i\}$, each defined by both dynamical separation and observational distinguishability. We show that for almost every microstate in Ω_0, repeated experiments yield long-run frequencies matching the ratios $\mu(\Omega_i)/\mu(\Omega_0)$. This result requires no probability postulate, wavefunction, or stochastic process, only deterministic dynamics and geometric structure. This result lays the foundation for Paper B, which shows why this becomes $|\Psi|^2$ in quantum mechanics.

Corresponding author: Zayn Blore, zblore@gmail.com

1. Introduction

Quantum mechanics assigns probabilities to physical outcomes through the Born rule. In its standard form, this rule is introduced as a postulate: the probability of a measurement outcome is given by the squared amplitude of the corresponding component of the wavefunction. Despite its empirical success, the rule remains mathematically opaque and conceptually unsatisfying. Why should probabilities relate to squared amplitudes? Why should randomness enter at all, if the underlying dynamics (e.g., via Schrödinger evolution) are deterministic?

This paper aims to reformulate that problem in purely geometric terms. We investigate whether a deterministic, volume-based structure can yield outcome statistics consistent with the Born rule, not by inserting probabilities, but by deriving them from the evolution of state space regions under well-defined, volume-preserving dynamics.

The framework begins with a smooth, finite-dimensional, Hausdorff manifold S, interpreted as the set of all microstates. Each microstate represents a full specification of the system's underlying degrees of freedom. We assume that S is equipped with:

- A volume-preserving flow $\varphi_t : S \to S$, capturing deterministic evolution.
- A conserved volume measure μ, analogous to Liouville measure in classical mechanics.
- A method for defining macroscopically distinguishable outcome regions $\{\Omega_i\}$, constructed from physical observables.

An experiment is represented as preparing a region $\Omega_0 \subset S$, then evolving it under φ_t such that it branches into disjoint outcome regions. If these outcome regions are well-defined (in the sense of dynamical separation and observational distinguishability), we show that the observed relative frequencies converge to the volume ratios $\mu(\Omega_i)/\mu(\Omega_0)$, without assuming any probabilistic mechanism.

Importantly, this approach does not assume Hilbert space, amplitudes, or the wavefunction. These structures may emerge later, but they are not required here. Instead, the focus is on whether a minimal geometric foundation can explain the statistical behaviour of quantum experiments.

The result does not replace quantum mechanics but it reframes the Born rule as a deterministic emergent pattern from volume-based dynamics. This opens the door to reconciling determinism and quantum statistics without invoking collapse, branching universes, or hidden variables.

Historical Context

The idea that statistical regularities can arise from deterministic dynamics has deep roots in physics. In classical contexts, Boltzmann pioneered the use of coarse-grained volumes to explain thermodynamic behaviour, and Birkhoff's ergodic theorem[1] formalized the convergence of time and space averages. Jaynes later reframed these ideas in terms of information theory[2], viewing entropy as a principle of logical inference. In quantum theory, Born's postulate[3] introduced the amplitude-squared rule for probabilities, assigning statistical weights $|\langle i \mid \Psi\rangle|^2$ to measurement outcomes. This was an axiom rather than a derived consequence and it remains a focal point for foundational inquiry. More recent work on

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typicality[4][5][6] shows that frequency patterns can emerge from geometry alone. The present work follows this line by demonstrating that deterministic, volume-preserving flow yields outcome frequencies proportional to region volumes, and that this suffices to match amplitude-squared predictions if the geometry aligns. Unlike probabilistic interpretations, this approach locates empirical regularities in the structure of the state space itself.

2. Deterministic-Volume Framework

2.1. State Space and Outcome Decomposition

We begin by formalising the state space introduced in Section 1. Let S denote the set of all physical microstates, a smooth, finite-dimensional, Hausdorff manifold. Each point $s \in S$ represents a complete specification of the system's microscopic degrees of freedom.

We define $\varphi_t : \Sigma \to \Sigma$ to be a smooth one-parameter family of diffeomorphisms, representing the deterministic evolution of microstates on a finite-dimensional constraint surface Σ. We assume that φ_t preserves a smooth volume form μ on Σ, such that for all measurable regions $\Omega \subset \Sigma$, we have

$$\mu(\varphi_t(\Omega)) = \mu(\Omega).$$

This ensures Liouville-like conservation of phase-space volume under evolution and provides the foundation for assigning stable frequency weights to branching outcomes.

Measurement settings may influence the dynamical flow itself, by altering the system-apparatus interaction, or may instead define a coarse-grained observable f applied after evolution. The present framework accommodates both: we allow for different φ_t depending on the physical setup, as well as multiple outcome maps f applied to a fixed flow.

In physical terms, distinct measurement contexts may alter the overall evolution φ_t by modifying the interaction Hamiltonian or boundary conditions that govern the joint system-apparatus dynamics. For instance, rotating a Stern-Gerlach device or changing a detector's basis setting corresponds to a different coupling, which in turn alters how microstates evolve toward macroscopically distinct regions. The framework accommodates both cases: where φ_t changes due to different dynamics, and where φ_t is fixed but the outcome map f defines alternative coarse-grainings of the same flow. This dual flexibility reflects the structure of real experiments, where both dynamical and observational elements contribute to outcome generation.

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In classical systems, a natural choice is the Liouville measure, which is uniquely preserved by Hamiltonian flow. In quantum contexts, however, there is no canonical phase space, and the appropriate measure is less obvious. We hypothesize that μ must obey structural constraints, such as symmetry invariance or preservation under relevant dynamical flows, to yield physically meaningful outcome frequencies. These constraints are explored further in follow-up work. For the purposes of this paper, we assume only that μ is φ-invariant, Borel-regular, and coarse-grainable into observable partitions.

In this framework, a physical experiment corresponds to preparing an ensemble of initial conditions within a measurable subset $\Omega_0 \subset S$ and evolving them via φ_t. Over time, the deterministic flow may separate these initial conditions into disjoint outcome regions $\{\Omega_i\} \subset S$, where each Ω_i represents a macroscopically distinguishable result (e.g., a detector click, spin-up, or particle position).

While the Hausdorff property ensures that for any two distinct points in S there exist disjoint open neighbourhoods, this topological separability alone is not physically meaningful.

Outcome regions must arise from the deterministic evolution of Ω_0 under the flow φ_t, and must correspond to macroscopically distinguishable outcomes, such as pointer positions, detector clicks, or classical field configurations. Arbitrary disjoint sets, even if well-defined topologically, do not qualify as outcome regions without such dynamical and observational grounding.

In practice, outcome regions are defined relative to a coarse-grained observable, a function on S whose level sets correspond to macroscopically distinct, classically recordable values. These values define the meaningful experimental outcomes. Outcome partitions are further constrained by empirical repeatability: only regions that yield stable, reproducible observational results qualify as valid branches.

Instead, we define $\{\Omega_i\}$ as a valid outcome decomposition only if:

1. The flow φ_t maps Ω_0 into the disjoint Ω_i deterministically.
2. Each Ω_i corresponds to a macroscopically distinguishable result in the physical sense (i.e., an observable pointer reading or detector outcome).
3. The partition is experimentally reproducible, such that repeated preparations of Ω_0 yield outcome assignments aligned with the $\{\Omega_i\}$.

These physical criteria rule out trivial counterexamples, such as disjoint neighbourhoods constructed via topology alone. In our framework, geometry and observability jointly determine the valid outcome structure.

The manifold S can be viewed as encoding hidden variables in the sense of encoding microstates not specified by macroscopic observables. These variables are not assumed to be position or momentum per se (as in Bohmian mechanics), but rather generalized degrees of freedom in a smooth state space to account for all experimental contexts.[1]

Definition (Core Functions and Structures):

Let S denote the full classical state space of the system. We define:

- $\mu : \mathcal{B}(S) \to [0, \infty)$ as a Borel measure on S, representing geometric volume. We assume μ is invariant under the flow φ_t.
- $\varphi_t : S \to S$ is a deterministic, volume-preserving flow that evolves the state continuously in time.
- $\Omega_0 \subset S$ is the measurable region representing the initial macrostate prior to branching. Under φ_t it is mapped into disjoint measurable outcome regions $\{\Omega_1, \ldots, \Omega_n\} \subset S$.
- $O : S \to \{1, \ldots, n\}$ is the observable outcome function, defined by: $O(x) = i \iff x \in \Omega_i$. It maps each microstate to the corresponding macroscopic outcome region.
- $f : S \to R^k$ is a coarse-grained observable used to define outcome regions via preimages:

  $\Omega_i := f^{-1}(\Delta_i)$,

  where the Δ_i are disjoint regions of observable values.

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Symbol	Definition / Role
S	Full classical state space of the system
μ	Volume measure on S; assumed to be Borel and φ-invariant
φ_t	Deterministic flow: $\varphi : S \to S$ volume-preserving
Ω_0	Initial measurable region (pre-branching macrostate)
$\{\Omega_i\}$	Disjoint outcome regions from deterministic evolution
$O(x)$	Outcome function: maps microstate $x \in S$ to outcome index i
$f(x)$	Optional observable map: $f : S \to R^k$, used for coarse-graining
Δ_i	Region in observable space corresponding to outcome i
w_i	Outcome weight: $\mu(\Omega_i)/\mu(\Omega_0)$
$\widehat{w}_i^N$	Empirical frequency of outcome i after N trials

Table 1. Core Symbols and Definitions

2.2. Volume-preserving flow φ_t

Time evolution is governed by a one-parameter family of smooth, invertible maps $\varphi_t \colon S \to S$, $t \in \mathbb{R}$, where S is a smooth, finite-dimensional, second-countable, Hausdorff manifold equipped with a Borel σ-algebra and a non-negative, σ-additive measure μ. Each map φ_t is assumed to be a C^1 diffeomorphism, that is, continuously differentiable with a continuously differentiable inverse. This flow satisfies:

- **Determinism:** Every microstate $x \in S$ evolves along a unique trajectory $\varphi_t(x)$.
- **Invertibility:** No two distinct microstates merge; the flow is bijective at all times.
- **Volume Preservation:** The measure μ is invariant under the flow, so for every measurable region $\Omega \subset S$ and all $t \in \mathbb{R}$, $\mu(\varphi_t[\Omega]) = \mu(\Omega)$.

This identity defines φ_t as a measure-preserving diffeomorphism on (S, μ), a standard structure in ergodic theory and classical dynamics[7][8]. It arises naturally in systems with conserved phase-space structure, whether classical (Hamiltonian) or geometric and ensures that the "size" of state space regions is invariant under evolution.

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At the infinitesimal level, this corresponds to Liouville's condition that the Jacobian determinant of the flow is unity throughout the state space[8]:

$$\det(D\varphi_t(x)) = 1 \quad \forall x \in S$$

meaning the Jacobian determinant of the flow equals one at every point. This condition underlies all subsequent results, as it guarantees that outcome weights based on region volumes remain fixed once branching has occurred.

In the physical context of any deterministic microscopic theory, the volume-preserving flow may emerge from an action principle, a divergence-free vector field, or a structure-preserving constraint on admissible dynamics. This flow acts not only on the microscopic degrees of freedom of the system but also on any subsystems, such as measurement apparatus or observers, that record macroscopic outcomes. Once a branching event occurs, the observer's internal configuration becomes entangled with a specific outcome region Ω_i, and hence evolves deterministically within that region. This ensures that empirical frequencies, as recorded within any single branch, reflect the geometric structure of state space and remain stable under time evolution.

2.3. Branching as a geometric partition

A branching event is defined as a deterministic evolution in which the initial region $\Omega_0 \subset S$ is mapped into a disjoint union of outcome regions:

$$\varphi_t[\Omega_0] = \bigsqcup_i \Omega_i,$$

where each $\Omega_i \subset S$ is a measurable subset corresponding to a macroscopically distinguishable experimental result. In physical systems, the observable f is constrained by empirical resolution, instrumentation, and repeatability. Only decompositions aligned with such observables are operationally meaningful. Although multiple valid coarse-grainings may exist, each determines a unique set of outcome weights. This flexibility mirrors the diversity of experimental contexts, without undermining the internal consistency of outcome assignment within any single setup.

To ensure that these outcome regions are both mathematically meaningful and operationally relevant, we impose the following constraints:

1. **Measurability:** Each outcome region Ω_i must be a Borel-measurable subset of the state space S, ensuring that the volume $\mu(\Omega_i)$ is well-defined.

2. **Dynamical Coherence:** The outcome regions must arise from the deterministic evolution of disjoint measurable subsets of Ω_0. That is, for each i, there exists a subset $A_i \subset \Omega_0$ such that $\varphi_t[A_i] \subseteq \Omega_i$.
3. **Operational Coarse-Graining:** Each region Ω_i should be definable in terms of a macroscopic observable $f: S \to R^k$. Specifically, Ω_i is constructed as a coarse-grained preimage:

   $\Omega_i := \{x \in S | f(x) \in \Delta_i\}$

   where the sets $\Delta_i \subset R^k$ are disjoint and correspond to distinct observable outcomes.

These constraints ensure that outcome weights are derived from the deterministic geometry of the system and are not dependent on arbitrary or subjective partitions. They also make clear how branching relates to physically observable distinctions.

This partition structure under a measure-preserving map is standard in deterministic dynamical systems and underpins frequency analysis in ergodic theory.

To qualify as a valid outcome decomposition, the following three conditions must be satisfied:

1. **Deterministic Assignment:** The flow φ_t maps every point in Ω_0 into exactly one Ω_i. That is, for each $x \in \Omega_0$, there exists a unique i such that $\varphi_t(x) \in \Omega_i$, and the outcome regions are pairwise disjoint.
2. **Macroscopic Distinguishability:** Each region Ω_i corresponds to a stable, observable microstate, such as a distinct detector click, pointer position, or classical field configuration. This implies the existence of a coarse-grained observable

   $f: S \to O$,

   where O is a finite outcome set, and $\Omega_i = f^{-1}(i)$ defines the region of state space associated with outcome i.

   Since f is coarse-grained and O is finite, each Ω_i is measurable by construction[9].

   The existence of such a map guarantees that the decomposition is measurable and tied to operationally meaningful data. Observables of this type are standard in classical and quantum measurement theory[5][10].
3. **Empirical Repeatability:** The partition must yield reproducible outcome statistics under repeated trials, starting from Ω_0. This condition rules out trivial or arbitrary decompositions constructed via topological separation alone and ensures that branching reflects stable physical processes.

Together, these criteria ensure that branching events are not merely mathematical artifacts but correspond to physically meaningful processes capable of generating distinct, classically recordable outcomes. Because the regions Ω_i are pairwise disjoint and collectively exhaustive, and the flow φ_t preserves volume, the subsequent assignment of outcome weights will be unambiguous and conserved over time.

The observer, regarded as a subsystem embedded in S, becomes dynamically correlated with a particular Ω_i during the branching event. From that point forward, the observer's records, expectations, and perceived statistics are entirely determined by the trajectory of $\varphi_t(x)$ within that region.

Uniqueness and Partition Ambiguity

While the construction of outcome regions $\{\Omega_i\}$ is constrained by measurability, dynamical coherence, and coarse-graining via observables, it is not necessarily unique. Multiple valid decompositions of Ω_0 into macroscopic outcomes may exist, especially when different levels of resolution are chosen or different observables are deemed relevant. However, this non-uniqueness does not undermine the use of outcome weights: once a branching structure is fixed, through a specific observable f or coarse-graining scheme, the associated weights $w_i = \mu(\Omega_i)/\mu(\Omega_0)$ are objectively defined. The volume-typicality theorem applies to any such valid decomposition, provided it satisfies the geometric and operational criteria outlined above. This flexibility reflects the fact that different experimental configurations may give rise to different, but equally legitimate, branching structures.

Thus, branching is understood as a geometrically clean, empirically grounded decomposition of state space that underlies the emergence of outcome frequencies from purely deterministic evolution.

3. Outcome Weights and the Typicality Theorem

Section 2 shows an invariant volume μ on state space and a clean partition $\Omega_0 \to \{\Omega_i\}$. Section 3 details how geometry alone turns into empirical probability.

3.1. Outcome Weights

Consider a branching event at time t, where an initial region $\Omega_0 \subset S$ evolves under the volume-preserving flow φ_t into disjoint outcome regions $\Omega_1, \Omega_2, \ldots, \Omega_n$, such that

$$\varphi_t[\Omega_0] = \bigsqcup_{i=1}^{n} \Omega_i.$$

Each $\Omega_i \subset S$ corresponds to a macroscopically distinguishable and observable outcome, as defined in Section [sec:2.3]. Because φ_t is assumed to be a measure-preserving diffeomorphism[7][8], the total

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measure of the evolved region must be conserved. Since the outcome regions Ω_i are pairwise disjoint and jointly exhaustive, we have

$$\mu(\Omega_0) = \sum_{i=1}^{n} \mu(\Omega_i)$$

This conservation law ensures that volume is merely redistributed, not created or destroyed, under deterministic evolution.

We now define the outcome weight w_i of region Ω_i relative to the initial region Ω_0 as:

$$w_i = \frac{\mu(\Omega_i)}{\mu(\Omega_0)}$$

This ratio satisfies $w_i \in [0,1]$ and $\Sigma_i w_i = 1$ by construction. It represents the normalized geometric size of each outcome region, analogous in structure to a probability measure but derived without stochastic assumptions.

This formula presupposes that the evolution under φ_t maps the initial region Ω_0 into a finite collection of disjoint outcome regions $\{\Omega_i\}$, each of which is macroscopically distinguishable. These regions correspond to observable outcomes, such as detector clicks or pointer positions. The weight assigned to each outcome is then given by the preserved volume fraction that enters Ω_i. Since μ is invariant under the flow, this assignment remains stable over time. The outcome probabilities are interpreted as empirical frequencies assuming that the experiment samples Ω_0 uniformly over repeated trials. This approach aligns with standard reasoning in statistical mechanics, where typicality arguments justify identifying relative volumes with long-run frequencies (see e.g.,[11][12]. The key assumption is that the coarse-grained branching structure corresponds to physical observables, and that the sampling of Ω_0 is unbiased at the scale of μ.

We assume that the flow φ_t induces a disjoint partition of the initial region Ω_0 into a finite collection of subsets $\{\Omega_i\}$, corresponding to distinct macroscopic outcomes. These subsets must not merely be topologically or mathematically disjoint (as ensured, for instance, by the Hausdorff property), but must arise from the dynamical evolution of the system in a way that is physically meaningful. Specifically, each Ω_i must map to a distinguishable measurement outcome, such as a specific pointer position, detector result, or classical field configuration, under coarse-graining by a real-world observer. The boundary between outcome regions reflects a branching structure determined by the system-environment interaction, not an arbitrary slicing of Σ.

These weights quantify the typicality of each outcome: the proportion of microstates in Ω_0 that evolve into Ω_i under φ_t.

This is a standard move in deterministic statistical mechanics, where ensemble frequencies are grounded in phase-space volume[10][13].

If one samples an initial microstate $x_0 \in \Omega_0$ according to the uniform measure μ, then the image $\varphi_t(x_0)$ will land in region Ω_i with typicality w_i. In this view, outcome weights express how common each outcome is, not how likely it is in any stochastic sense.

For embedded observers correlated with outcome regions (as defined in Section [sec:2.3]), these weights match the expected long-run frequencies recorded across an ensemble of identically prepared trials.

This link between geometry and observer experience is central to later arguments concerning empirical convergence (Section [sec:5]) and quantum compatibility[14]. Thus, the emergence of statistical regularities in measurement does not require any probabilistic postulates: it arises directly from the volume-preserving dynamics of deterministic flows acting on observable partitions of state space.

Proposition (Typicality of Volume-Based Frequencies)

Let φ_t be a deterministic, volume-preserving flow on a smooth state space Σ with invariant measure μ. Let $\Omega_0 \subset \Sigma$ be a measurable initial region, and let $\{\Omega_j\}$ be a disjoint partition of Ω_0 corresponding to macroscopically distinguishable outcomes.

Then, for almost all sequences of trials drawn from Ω_0 with uniform measure, the relative frequencies $f_i^{(N)}$ of outcomes in Ω_i after N repetitions satisfy:

$$\lim_{N\to\infty} f_i^{(N)} = \frac{\mu(\Omega_i)}{\mu(\Omega_0)}$$

Justification sketch

This result reflects deterministic typicality: sampling $x_k \in \Omega_0$ independently and uniformly implies, by measure-theoretic concentration, that long-run frequencies converge to volume ratios. No stochasticity, ergodicity, or ensemble averaging is required. The key assumptions are volume conservation under φ_t, a fixed branching structure $\{\Omega_i\}$, and unbiased sampling across Ω_0. This aligns with classical arguments in Boltzmannian statistical mechanics[12][13].

3.2. Basic properties

The outcome weights $w_i := \mu(\Omega_i)/\mu(\Omega_0)$, defined in Section [sec:3.1], possess several key properties that follow directly from the structure of the flow and measure:

1. **Normalization:** Because the outcome regions Ω_i are disjoint and exhaustive, and the measure μ is additive, the weights satisfy
$\sum_{i=1}^{n} w_i = \frac{1}{\mu(\Omega_0)} \sum_{i=1}^{n} \mu(\Omega_i) = 1$
This ensures that outcome weights form a complete partition of unit volume relative to the initial region.
2. **Time Invariance:** Because the flow φ_t is volume-preserving, the outcome weights w_i are independent of the particular branching time t.
$\mu(\varphi_t[\Omega]) = \mu(\Omega) \quad \Longrightarrow \quad w_i(t) = w_i(0)$
This means the relative sizes of outcome regions are conserved by evolution, even as their locations in S may change.
3. **Observer Independence:** The values of w_i depend only on the geometry of state space and the volume measure, not on any assumptions about probability, subjective belief, or measurement collapse. Because they are defined purely from the structure of S and φ_t, they apply equally to any embedded observer whose internal state becomes correlated with Ω_i.
4. **No Frequency Assumption:** The weights are not assumed to match empirical frequencies by postulate. Rather, they are used as geometric inputs to derive such frequencies in the next section via a typicality argument. This distinction is crucial: we do not assume that w_i are probabilities; instead, we demonstrate why they behave like probabilities under deterministic evolution.
5. **Deterministic Origin:** The outcome weights arise from the deterministic evolution of microstates under φ_t because each initial condition in Ω_0 is mapped into exactly one Ω_i, and the sizes of those sets determine the frequency structure of outcomes. No randomness or wavefunction structure is required.

These properties show that outcome weights provide a natural and robust foundation for understanding relative frequencies without invoking probabilistic axioms. They inherit their meaning entirely from the geometry and dynamics of the system.

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3.3. Volume-Typicality Theorem

We now show why the outcome weights

$$w_i = \frac{\mu(\Omega_i)}{\mu(\Omega_0)}$$

correspond to the long-run frequencies of outcomes observed by embedded agents, under deterministic evolution. Let $\Omega_0 \subset S$ be an initial region, and suppose the flow φ_t deterministically evolves Ω_0 into disjoint, measurable outcome regions:

$$\phi(t)[\Omega_0] = \bigsqcup_{i=1}^{n} \Omega_i$$

where each Ω_i corresponds to a macroscopically distinct and coarse-grained observable state, as defined in Section [sec:2.3].

Let $x^{(1)}, x^{(2)}, \ldots, x^{(N)} \in \Omega_0$ be a sequence of independent initial microstates, sampled according to the measure μ. Each evolves via φ_t into some outcome region. Define the empirical frequency of outcome i across this sequence as:

$$\widehat{w}_i^{(N)} := \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}_{\Omega_i}(\varphi_t(x^{(j)})),$$

where $\mathbf{1}_{\Omega_i}$ is the indicator function for region Ω_i. Then, for every $\epsilon > 0$, the law of large numbers for measure spaces implies:

$$\lim_{N \to \infty} \mu \left\{ (x^{(1)}, \ldots, x^{(N)}) \in \Omega_0^N : |\widehat{w}_i^{(N)} - w_i| > \epsilon \right\} = 0.$$

This result shows that, for almost all sequences of microstates drawn from μ, the observed frequencies converge to the volume weights.

Because the flow is deterministic and volume-preserving, and because outcome regions are disjoint and coarse-grained, each outcome accumulates frequency proportional to its invariant volume.

No stochasticity or quantum assumptions are needed. The emergence of relative frequencies is a purely geometric phenomenon, arising from the deterministic redistribution of volume in state space.

The convergence result assumes that the flow φ_t distributes microstates across outcome regions in a representative way. While this does not require full ergodicity, systems with pathological structure or conserved subspaces may violate the conditions needed for typicality. In practice, weak mixing or effective chaoticity suffice to ensure convergence in most coarse-grained systems.

This theorem forms the basis of volume-based probability: observed frequencies match geometric weights because of how deterministic flows allocate trajectories across macroscopic outcomes. In this framework, probability is not a primitive concept but a consequence of measure structure and repeated evolution.

Theorem (Volume-Based Typicality of Outcomes)

Let S be a state space equipped with a Borel measure μ, and let $\varphi_t : S \to S$ be a deterministic, volume-preserving flow. Let $\Omega_0 \subset S$ be a measurable initial region that branches into disjoint measurable outcome regions $\{\Omega_1, \ldots, \Omega_n\}$ such that:

$$\varphi_t[\Omega_0] = \bigsqcup_{i=1}^{n} \Omega_i$$

Define the outcome weights by:

$$w_i := \frac{\mu(\Omega_i)}{\mu(\Omega_0)}$$

Let $x^{(1)}, \ldots, x^{(N)} \in \Omega_0$ be independently sampled microstates according to the measure μ, and define the empirical frequency of outcome i as:

$$\widehat{w}_i^{(N)} := \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}_{\Omega_i}(\varphi_t(x^{(j)}))$$

Then, for every $\epsilon > 0$,

$$\lim_{N\to\infty} \mu^N \left\{ (x^{(1)}, \ldots, x^{(N)}) \in \Omega_0^N : |\hat{w}_i^{(N)} - w_i| > \epsilon \right\} = 0$$

That is, for almost all sequences of initial microstates drawn from μ, the observed relative frequencies of outcomes converge to their geometric weights.

3.4. Toy Model: Two-Outcome Branching System

To illustrate how outcome frequencies emerge from region volume, consider a toy model with a two-dimensional state space:

$$S := \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1, 0 \le y \le 1\}$$

equipped with the standard Lebesgue measure μ. Let the initial macrostate be the entire unit square: $\Omega_0 := S$.

Define a deterministic branching map $\varphi : S \to S$ as follows:

- If $x < \frac{2}{3}$, map the point to region Ω_1.
- If $x \geq \frac{2}{3}$, map it to region Ω_2.

Explicitly, let:

$$\Omega_1 := \left\{(x, y) \in S \mid 0 \leq x < \frac{2}{3}\right\}, \quad \Omega_2 := \left\{(x, y) \in S \mid \frac{2}{3} \leq x \leq 1\right\}$$

Since the system is volume-preserving and the flow partitions Ω_0 into these two regions, we compute the weights:

$$w_1 = \frac{\mu(\Omega_1)}{\mu(\Omega_0)} = \frac{2}{3}, \quad w_2 = \frac{\mu(\Omega_2)}{\mu(\Omega_0)} = \frac{1}{3}$$

Now let an observer record the outcomes of $N = 300$ independent trials, each starting from a uniformly sampled point in Ω_0. By the typicality theorem, almost every such sequence of microstates will yield:

$$\widehat{w}_1^{(N)} \approx \frac{2}{3}, \quad \widehat{w}_2^{(N)} \approx \frac{1}{3}$$

as $N \to \infty$, with deviations decreasing exponentially in N.

While measurement contexts can vary, the underlying state space S and measure μ need not change. Rather, outcome regions are defined differently per observable. The degrees of freedom of S are sufficient to encode all such variations, avoiding any implication of infinite new parameters per measurement context.

This toy model demonstrates how deterministic flow and coarse partitioning of the state space produce outcome frequencies that match the relative volumes, without invoking probability, randomness, or quantum postulates.

Section 3 has now furnished everything an observer needs to turn conserved geometry into rock-solid predictions. Sections [sec:4] and [sec:5] will illustrate the theorem in explicit toy models and discuss why branches stay independent after the split.

4. Embedded Observers and Frequency Records

To complete the link between volume weights and empirical statistics, we now consider observers embedded within the deterministic system. The goal is to explain why observers record frequencies that match the geometric weights $w_i = \mu(\Omega_i)/\mu(\Omega_0)$ without invoking any stochastic assumptions.

Let the system be prepared in an initial macrostate represented by a measurable region $\Omega_0 \subset S$. The observer is treated as a physical subsystem whose internal configuration evolves deterministically along with the rest of the state.

Suppose that the full system undergoes a sequence of repeated, isolated evolutions from identically prepared regions Ω_0, and that in each trial the global flow φ_t maps the system to one of several coarse-grained outcome regions $\Omega_i \subset S$. These regions correspond to macroscopically distinct results, such as a pointer position, detector click, or memory encoding.

We define a recorded outcome as a stable, internal physical configuration of the observer subsystem that is correlated with entry into a specific region Ω_i. Because the observer evolves as part of the total system, and because φ_t is deterministic and volume-preserving, the observer's records across repeated trials form a sequence:

$$R = (i_1, i_2, \ldots, i_N)$$

where each i_k satisfies $\varphi_t(x^{(k)}) \in \Omega_{i_k}$ for some microstate $x^{(k)} \in \Omega_0$.

Although each microstate represents a complete specification of the system's degrees of freedom, such distinctions are typically unresolvable in practice. Observers access only coarse-grained outcomes via macroscopic observables. Microstates that map to the same observable result under f are operationally indistinguishable, because no physical process available to the observer can reliably distinguish them. This indistinguishability arises from decoherence, finite detector resolution, and the empirical indistinctness of nearby configurations in S.

Let $\widehat{w}_i^{(N)}$ be the frequency with which outcome i appears in the record R. By the typicality theorem (Section [sec:3.3]), we know that for almost all sequences $x^{(1)}, \ldots, x^{(N)} \in \Omega_0$, these frequencies converge:

$$\lim_{N \to \infty} \widehat{w}_i^{(N)} = w_i$$

Because the observer subsystem evolves according to the same deterministic flow, and because each outcome region has volume $\mu(\Omega_i)$, the internal memory of the observer will reflect the same weight structure, even without invoking probability.

This framework eliminates the need for stochastic postulates. The observer does not assign likelihoods or experience uncertainty; they simply evolve into regions of state space that differ in volume. The record of their experience, when aggregated over repeated trials, aligns with the invariant geometric weights.

Thus, the experience of frequency regularities by embedded observers arises because deterministic flow maps microstates into outcome regions in proportion to volume, and because the observer's internal records are physical encodings of those transitions.

5. Empirical Convergence

We now connect the geometric outcome weights to empirical experience. Specifically, we explain why observers embedded in a deterministic system will observe frequencies that converge to the weights $w_i = \mu(\Omega_i)/\mu(\Omega_0)$, even though no probabilistic assumption has been made.

Let $x^{(1)}, x^{(2)}, \ldots, x^{(N)} \in \Omega_0$ be a finite sequence of initial states drawn independently from the uniform measure μ. Each evolves under the volume-preserving flow φ_t into one of the outcome regions Ω_i. An observer embedded within each trajectory records the macroscopic outcome it enters.

Let $\widehat{w}_i^{(N)}$ denote the observed frequency of outcome i across the sequence. Then by the typicality theorem, we have:

$$\lim_{N \to \infty} \widehat{w}_i^{(N)} = w_i$$

for almost all sequences of initial conditions drawn from μ. Because each outcome region contains a proportion w_i of the initial volume, and because the flow preserves that volume, the set of microstates that lead to outcome i has the same measure as the region itself. This implies that relative frequencies, when measured over many runs, reflect these volume ratios.

Importantly, convergence does not require an external probability law. It results from the distribution of microstates within Ω_0, and the structure of the flow that deterministically transports those states into disjoint regions.

The observer does not need to interpret or assign likelihoods. Their internal record of frequencies, once enough trials have been recorded, will match the outcome weights defined geometrically.

Thus, outcome weights derived from volume ratios are not only mathematically stable but also empirically verifiable. This completes the explanatory arc: from geometry and dynamics to observed regularities, without invoking any stochastic or epistemic assumptions.

Falsifiability

Unlike standard quantum tests, which verify the Born rule's outcome statistics, the present framework predicts that outcome frequencies emerge solely from volume structure, even in classically engineered systems. A falsifying result would arise if volume-conserving systems, such as analog simulators, reversible cellular automata, or chaotic classical systems, consistently failed to reproduce the predicted frequency ratios. Similarly, if experimental interventions on outcome partitioning (e.g., dynamic remapping of coarse-grained observables) showed systematic deviation from volume-based weights, this would challenge the core assumptions of the model.

Analogue tests like those proposed for entanglement-based derivations of the Born rule could offer a path forward. For example, engineered setups with controlled branching geometries could directly probe whether outcome region volume ratios align with long-run frequencies in fully deterministic flows.

6. Compatibility with Amplitude-Squared Weights

The framework developed in this paper makes no reference to quantum mechanics or probabilistic axioms. However, it is useful to ask whether the outcome weights derived from geometric volume can match those observed in physical theories, particularly quantum theory.

In standard quantum mechanics, the probability of obtaining outcome i is given by the squared amplitude:

$$P(i) = |\langle i|\psi\rangle|^2$$

where ψ is the system's state vector and $\{|i\rangle\}$ are orthonormal outcome states. No such structure appears in this paper. Still, we can ask the following: if the geometry of an underlying deterministic system is such that the volume ratios

$$\frac{\mu(\Omega_i)}{\mu(\Omega_0)} = |\langle i|\psi\rangle|^2$$

hold for all measurement contexts, then the present framework guarantees that frequencies will match the Born rule, not by probability postulates, but because frequencies track volume, and volume ratios are preserved under deterministic flow.

This is not a derivation, but a compatibility statement: volume-based typicality will reproduce the Born rule if the outcome regions happen to align with amplitude-squared weights. Why such a correspondence

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should hold is not explained here; that question is addressed in[14].

Thus, the framework presented in this work does not derive the Born rule. It shows only that the statistical predictions of quantum mechanics are consistent with a deterministic, geometric foundation, provided a suitable mapping exists between amplitudes and outcome volumes.

7. Summary

This paper has presented a purely geometric foundation for outcome frequencies in deterministic systems, based on the structure and evolution of volumes in state space.

We began by defining the state space S, its invariant measure μ, and a class of volume preserving flows φ_t that govern deterministic evolution. A branching event was defined as a partition of an initial region $\Omega_0 \subset S$ into disjoint outcome regions $\{\Omega_i\}$, each corresponding to a macroscopically distinguishable and coarse-grained observable state.

We then introduced the notion of outcome weights $w_i := \mu(\Omega_i)/\mu(\Omega_0)$, which quantify the relative volume of each outcome region. Because the flow preserves volume and maps each microstate in Ω_0 into exactly one Ω_i, these weights remain constant over time and are independent of any stochastic interpretation.

The central result, the typicality theorem, showed that these weights determine the long-run frequencies of outcomes across repeated evolutions of microstates. This occurs not by assumption, but because larger regions in state space necessarily contain more microstates and are therefore encountered more often in deterministic sampling. The law of large numbers applies to the measure μ, not to any concept of chance.

This leads to a formulation of volume-based probability, in which observed frequencies arise from deterministic geometry. No randomness, collapse, or interpretation of quantum mechanics is invoked. The framework explains how stable statistical patterns emerge from structure alone: from volume, flow, and macroscopic branching.

This framework is currently limited to systems with finite-dimensional state spaces. Extending the volume-typicality result to infinite-dimensional systems, such as quantum fields or gravitational degrees of freedom, remains an open challenge. Approaches based on effective coarse-graining or truncation may offer one route forward, but no such generalisation is attempted here.

This account remains minimal and geometric by design. It does not attempt to explain why specific outcomes occur, nor does it rely on physical interpretations of measurement. It provides only what is required to understand the origin of relative frequencies from deterministic evolution in measurable state spaces.

This paper is self-contained. While related investigations (e.g., symmetry-based derivations of region volume or quantum structure) are developed separately, no results from those works are assumed or required here.

Footnotes

[1] In quantum systems, branching is typically gradual and mediated by decoherence. The idealisation of perfectly disjoint regions used here assumes that such decoherence is sufficiently complete to render outcomes macroscopically distinguishable. This abstraction captures the effective behavior of realistic systems without explicitly modelling environmental interactions.

References

1. ^Birkhoff GD (1931). "Proof of the Ergodic Theorem." Proc. Natl. Acad. Sci. U.S.A. **17**(12):656–660. doi:10.1073/pnas.17.2.656.
2. ^Jaynes ET (1957). "Information Theory and Statistical Mechanics." Phys. Rev. **106**(4):620–630. doi:10.1103/physrev.106.620.
3. ^Born M (1926). "Quantenmechanik der Stoßvorgänge" [Quantum Mechanics of Collision Processes]. Z. Physik [Journal of Physics]. **38**(11-12):803–827. doi:10.1007/BF01397184.
4. ^Goldstein S (2001). "Boltzmann's Approach to Statistical Mechanics." In: Chance in Physics. Springer Berlin Heidelberg. pp. 39–54. doi:10.1007/3-540-44966-3 3. ISBN 9783540449669.
5. a, bWallace D (2012). The Emergent Multiverse: Quantum Theory According to the Everett Interpretation. OUP Oxford.
6. ^Werndl C, Frigg R (2015). "Reconceptualising Equilibrium in Boltzmannian Statistical Mechanics and Characterising Its Existence." Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics. **49**:19–31. doi:10.1016/j.shpsb.2014.12.002.
7. a, bWalters P (2000). An Introduction to Ergodic Theory. New York, NY: Springer.
8. a, b, cArnol'd VI (2001). Mathematical Methods of Classical Mechanics.
9. ^Halmos PR (2013). Measure Theory. 1950th ed. New York, NY: Springer.
10. a, bKhinchin AY (1949). Mathematical Foundations of Statistical Mechanics. Mineola, NY: Dover Publications.

11. [^] *Sharp K, Matschinsky F (2015). "Translation of Ludwig Boltzmann's Paper "On the Relationship Between the Second Fundamental Theorem of the Mechanical Theory of Heat and Probability Calculations Regarding the Conditions for Thermal Equilibrium" Sitzungberichte der Kaiserlichen Akademie der Wissenschaften. Mathematisch-Naturwissen Classe. Abt. II, LXXVI 1877, pp 373-435 (Wien. Ber. 1877, 76:373-435). Reprinted in Wiss. Abhandlungen, Vol. II, Reprint 42, p. 164-223, Barth, Leipzig, 1909." Entropy.* **17**(4):1971–2009. doi:10.3390/e17041971.
12. [a, b] *Goldstein S (2011). "Typicality and Notions of Probability in Physics." In: Probability in Physics. Springer Berlin Heidelberg. pp. 59–71. doi:10.1007/978-3-642-21329-84. ISBN 9783642213298.*
13. [a, b] *Lebowitz JL (1993). "Boltzmann's Entropy and Time's Arrow." Physics Today.* **46**(9):32–38. doi:10.1063/1.881363.
14. [a, b] *Blore Z (2025). "FIXING THE MEASURE: DERIVING |Ψ|^2 FROM SYMMETRY IN DETERMINISTIC GEOMETRY." Zenodo. doi:10.5281/ZENODO.16746829.*

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