

Peer Review

Review of: "Encoding Sequences in Intuitionistic Real Algebra"

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General Remarks

- The paper is brief and concise, which is generally a positive aspect. However, it is challenging to understand for non-experts and contains some logical gaps. Furthermore, several typos and inaccuracies in the presentation of proofs contribute to the impression that this article is unfinished and requires substantial revision.
- I strongly recommend adding more detailed explanations throughout the paper to enhance its accessibility for a broader audience.
- Additionally, if feasible, providing illustrative interpretations or examples of the defined concepts would greatly aid readers in understanding the definitions and theorems more clearly.

Detailed Remarks:

Abstract:

The abstract is currently just one sentence, making it likely understandable only to experts. Expanding it into several sentences to provide more context and explain the problem in greater detail would make it more accessible to a wider audience. Additionally, highlighting the significance of the paper's results within the broader research landscape would help readers better understand its contributions and relevance.

Paragraph 1:

- Line 3: Could you please clarify what you mean by "species"?
- Line 4: Where is the Appendix?
- Line 5: Which theorem in [1] are you referring to? Theorem 10?

- Line 7: What do you mean by "the" axiom system in [2]? Could you clarify which specific system you are referring to?
- Line 13: The word "peculiarities" should be corrected to "peculiarities."

Paragraph 2:

- Line 2: What do you mean by "choice sequences"?
- Please check the sources. I believe [6] should be [2].
- Please specify the exact location in the sources. I could not find (i) and (ii) in [2] and [6]. Kindly provide precise references, such as [2, Theorem x] or [6, Page y], to help locate the information.
- In (ii) "such that from the axioms $\exists k \in \mathbb{N}^+ (x=k) \equiv N(x)$ follows" should be "such that $\exists k \in \mathbb{N}^+ (x=k) \equiv N(x)$ follows from the axioms"
- Line before Proposition 1: "The folowing" should be corrected to "The following."

Proposition 1:

- Could you please clarify what you mean by a "uniform definition"?
- Do you mean the theory $\mathcal{T}$ from [3]?
- What do you mean by "The restriction ... works"?
- As a non-100%-expert, it is not clear to me why this should be an immediate corollary of the statements above.

Last paragraph of the introduction:

- Given a choice sequence (let's say c) with range in the set $\{0, 1\}$, do you mean that not the sequence itself, but rather $(\sum_{i=1}^n c(i)2^i)_n$ is a real number generator? Otherwise, the real number would always be 0.
- In the definitions of $\exists^R$, $\forall^R$ and $\exists^{\mathbb{N}^*}$, $\forall^{\mathbb{N}^*}$ please add more space after the semicolons.
- At the end, do you just mean that $f_n f = f_n \cdot f$ is the ordinary multiplication on real numbers? Why is this something special?

Chapter 2, Paragraph 1:

- There is no Lemma (7) and Lemma (?), only Proposition (7).
- Line 12 has a typo: "reaql" should be "real".

Lemma 2:

- Line 2: It should be $\$z'\$$ and not just $\$z\$$.
- Do you mean $\exists^{\mathbb{N}^+}$ instead of $\exists^+$? Perhaps you should revise it in the definition.
- Add a separator (for example, the word "assume") between y and $\exists$.

Proof of Lemma 2:

- Could you at least mention that (i) is trivial?
- Do you mean "iii." instead of "2."?
- It is not entirely clear what you intend to prove. First, it would be necessary to prove that $x'(p) = 0$, and then show that $y'(j) = x'(j)$ for all $j < p$. As far as I can see, the first part is already assumed to be true without explicitly mentioning it. However, please state it explicitly.

After the proof of Lemma 2:

- Maybe you could describe more clearly why $[x', p]$ is only equivalent to the formula that follows, and why the notation includes x' even though the formula itself only contains x .
- In the formula, there should be parentheses following the existential quantifier for y :

$$\exists y(\exists^+ q(y \cdot 2^{p-1} = q) \wedge 0 \leq x - y < 2^{-p}).$$

Proof of Lemma 3:

At the end of the proof, you apply Lemma 2.(iii) to obtain $y'(j) = x'(j)$ for all $j \leq 2m$. However, to apply this lemma, you would need $x'(2m + 1) = 0$, whereas the assumption states that $x'(2m + 1) = 1$. Therefore, it is unclear why Lemma 2.(iii) can be applied in this case.

Definition 4

- It is unclear what you are defining and what you are stating to be true. Are you defining ξ' or asserting that it is a code? Perhaps you could simply highlight the defined notions more clearly.
- Why is there a box at the end of the definition?
- In the definition of $CODE(x)$ by $\forall m[x, 2m + 1]$, I believe it should be x' instead of x , i.e.

$$\forall m[x', 2m + 1].$$

Corollary 5

Please describe in more detail how it follows. For example, how do you define u for a given α , and vice versa?

Definition 6

- I like this definition. I assume that it represents the central message of the paper. If so, could you emphasize it more clearly?
- Additionally, you could consider discussing its possible implications and applications further, perhaps in an outlook section.
- Why do you use ξ_i in the definition instead of just ξ ?

Proposition 7

- Is ξ an arbitrary 0-1 sequence, or is it the 0-1 sequence from Definition 4? I think the first one, but please make this clear.
- It is not entirely clear how $\tau(\xi_i(t_1 = t_2))$, which is equivalent to $C(\xi', t_1, t_2)$, follows from Corollary 5, as this formula does not appear there at all.
- The source [6] has 13 pages. Could you please specify more precisely where the quantifier elimination is presented there? For the sake of clarity, perhaps you could elaborate on it directly instead of only referring to it.
- Could you please also consider the cases of negation and logical connectives?

Reference

- The source [4] was already published in 1999:
<https://www.sciencedirect.com/science/article/pii/S0168007298000591>
- The page numbers of [5] and [6] should be swapped:
<https://onlinelibrary.wiley.com/doi/10.1002/malq.19980440305>
<https://onlinelibrary.wiley.com/doi/10.1002/malq.202000048>
- Perhaps you could provide additional sources to demonstrate the general interest in the topic of this paper? Currently, only eight sources are cited, four of which are self-citations.

Declarations

Potential competing interests: No potential competing interests to declare.