# An Alternative to the Merton Jump-Diffusion Model: A Simple, Explicit Formula

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**Abstract**: In this paper, we provide an alternative to the Merton jump-diffusion model. In doing so, we provide a simple, explicit formula that doesn't require a computational method. Furthermore, we introduce a new, simple method for solving partial integro- differential equations.

Keywords: Option pricing, Merton model, jump-diffusion, closed-form solution, partial integro- differential equation, the Black-Scholes formula.

# 1 Introduction

In response to some of the limitations of the Black-Scholes model, Merton (1976) introduced a seminal jump-diffusion model for the price of the European call option. However, his well-known and highly cited formula is not a closed form; it is an infinite sum (approximation) that requires a computational method. Furthermore, some of the parameters and probability distribution assumptions can be eliminated.

Later models on jump diffusions such as Kou (2002), Zhang and Wang (2013), Zhu et al (2013) and Gong and Zhuang (2016) have some limitations. These models require computational or numerical methods. Alghalith (2020) adopted a different process and a different approach.

In addition, previous models do not clearly capture the intuitive and desirable features captured by the model we introduce in this paper.

In this paper, we overcome these limitations. In doing so, without a loss of generality, we provide a far simpler, explicit formula that doesn't require any numerical/computational methods. Furthermore, our formula is perfectly intuitive. Also, our formula is a small modification of the Black-Scholes formula. Thus, it is also easily and directly comparable to the Black-Scholes formula.

Moreover, we introduce a new method for solving partial differential-difference equations. In doing so, we devise a simple method to transform a partial integro- differential equation to a partial differential equation.

## 2 The methods

The following is a brief description of the Merton model. The stock price is given by

$$S\left(t\right)=\Pi_{j=1}^{n}Y_{j}Se^{\left(\alpha-\frac{\sigma^{2}}{2}-\lambda k\right)t+\sigma Z\left(t\right)}, \tag{1}$$

where $S=S\left(0\right)$, $Z$ is a Gaussian variable, $n$ is a Poisson Process, $\lambda t$ is its intensity,$Y_j$ are identically and independently distributed, $\alpha$ is expected return rate, and $\sigma$ is the volatility.

The dynamics of the stock price are given by

$$dS=S\left[\left(\alpha-\lambda k\right)dt+\sigma dZ+\left(Y-1\right)dn\right], \tag{2}$$

where $Y-1$ is the relative jump size (independent of $dn_u$), and $k$ is its mean.

The dynamics of the option price are given by

$$dC(t,S) = C(t,S)\left((\alpha_c - \lambda k_c)\,dt + \sigma_c dZ + dq_c\right), \tag{3}$$

where

$$\alpha_c = \left(C_t + (\alpha - \lambda k)\,SC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} + \lambda E_Y\left[C(t,YS) - C(t,S)\right]\right)/C(t,S), \tag{4}$$

$$\sigma_c = \sigma SC_S / C(t,S),$$

and the subscripts of $C$ are partial derivatives, and $q_c$ is an independent Poisson process.

Using Merton's assumptions such as a diversifiable jump risk and risk neutrality, we obtain Merton's well-known partial differential-difference equation

$$C_t + (r - \lambda k)\,SC_S - rC + \frac{1}{2}\sigma^2 S^2 C_{SS} + \lambda E_Y\left[C(t,YS) - C(t,S)\right] = 0, \tag{5}$$

where $r$ is the interest rate (a constant), and $YS$ is the price of the underlying asset after the jump.

The Merton pricing formula (under the assumption of the log-normality of the jump size) is

$$C(t,S)=\sum_{i=0}^{\infty}e^{-\bar{\lambda}T}\frac{\left(\bar{\lambda}T\right)^{i}}{i!}C_{BS}\left(\sigma_i,r_i,S,T\right), \tag{6}$$

where $C_{BS}$ is the Black-Scholes price, $\bar{\lambda}=(1+k)\lambda$, $r_i=r-\lambda k+\frac{i\ln(1+k)}{T}$, and $\sigma_i^2=\sigma^2+\frac{i\delta^2}{T}$, where $\delta^2=Var(\ln Y)$ and $T$ is the time to expiry.

## 2.1 The revision

The dynamics of the price of the underlying asset are given by

$$dS=S\left[rdt+\sigma dZ+(Y-1)\,dn\right], \tag{7}$$

where $Z\sim N(-\lambda kt/\sigma,t)$. There is no loss of generality in assuming that the mean is $\lambda kt/\sigma$. Therefore, the Merton partial integro- differential equation (with a slight modification) is

$$C_t+r\left(SC_S-C\right)+\frac{1}{2}\sigma^2S^2C_{SS}+\lambda E_Y\left[C(t,YS)-C(t,S)\right]=0. \tag{8}$$

Now, let $\frac{E_Y[C(t,YS)-C(t,S)]}{C(t,S)}=\varphi_t$, so that $E_Y\left[C(t,YS)-C(t,S)\right]=\varphi_tC(t,S)$.

Thus, (8) can be given by

$$C_t + r\left(SC_S - C\right) + \frac{1}{2}\sigma^2 S^2 C_{SS} + \lambda\varphi_t C\left(t, S\right) = 0. \tag{9}$$

Therefore,

$$C_t + rSC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} + \left(\lambda\varphi_t - r\right) C = 0, C\left(T, S\left(T\right)\right) = g\left(S\right), \tag{10}$$

where $g$ is the payoff of the option. This is a generalized Black-Scholes partial differential equation. Conditioning on each value of $\varphi$ (given $\varphi = \varphi_i$ ), its solution is[1]

$$\bar{C}\left(0, S, \varphi_i\right) = e^{\lambda\varphi_i T}\left[SN\left(d_1\right) - e^{-rT}KN\left(d_2\right)\right] = e^{\lambda\varphi_i T}C_{BS}, \tag{13}$$

where $d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$, $K$ is the strike price, and $C_{BS}$ is the Black-Scholes price.

[1] The generalized Black-Scholes partial differential equation is

$$C_t + \theta SP_S + \frac{1}{2}\sigma^2 S^2 C_{SS} - \gamma C = 0, C\left(T, S\left(T\right)\right) = g\left(S\right). \tag{11}$$

Its solution is

$$e^{(\theta-\gamma)T}SN\left(d_1\right) - e^{-\gamma T}KN\left(d_2\right), \tag{12}$$

where $d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(S/K\right) + \left(\theta + \sigma^2/2\right)T\right]$ and $d_2 = d_1 - \sigma\sqrt{T}$.

Now, the option price can be expressed as a weighted average of these prices conditional on $\varphi$ as follows

$$C(0,S) = \int_{\varphi_i} e^{\lambda\varphi_i T} C_{BS} dF(\varphi_i), \tag{14}$$

where $F$ is the cumulative density. By the continuity, the expected value is a specific value of $\bar{C}(\varphi_i)$ denoted by $\hat{C}(\varphi_i) = C(\hat{\varphi}_i) = e^{\lambda\hat{\varphi}_i T} C_{BS}$, where $\hat{\varphi}_i$ is a value (outcome) of $\varphi$.

Thus, the price of the option is

$$C(0,S) = e^{\lambda\hat{\varphi}_i T} C_{BS} = e^{\lambda\hat{\varphi}_i T}\left[SN(d_1) - e^{-rT} KN(d_2)\right], \tag{15}$$

where $d_1 = \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

Similar to the parameters of the classical model, the parameter $\hat{\varphi}_i$ can be estimated using similar methods or other methods. In addition, the implied value of $\hat{\varphi}_i$ can be calculated and then used in the estimation of $\hat{\varphi}_i$. To illustrate this, if the market price of the option $C_m = 20$, $S = 100$, $K = 90$, $r = .05$, $T = 1$, $\sigma = .25$, $\lambda = 1$, then $C_{BS} = 18.14$. We can calculate the implied value of $\hat{\varphi}_i$ using $C_m = e^{\lambda\hat{\varphi}_i T} C_{BS}$, and thus $20 = 18.14 e^{\hat{\varphi}_i}$. Therefore,

$\hat{\varphi}_i = .097$. Furthermore, using (9), $\hat{\varphi}_i$ can be estimated as $E\left(\frac{\int_t^T \varphi_u du}{T-t}\right)$ (the expected value of the average of $\varphi_u$).

**A verification:**

A simple way to verify the result is to let $C^*$ be the true Merton option price, and $\bar{C}(r, s, \sigma, \phi, T) = e^{\phi T} C_{BS}$ be the generalized Black-Scholes price of the European option. By the continuity of $\bar{C}$, there is a specific value of the parameter $\phi$ such as $\hat{\phi}$, such that $C^* = \bar{C}(r, s, \sigma, \hat{\phi}, T) = e^{\hat{\phi} T} C_{BS} = e^{\lambda \hat{\varphi}_i T} C_{BS}$.

**Practical example:**

If $S = 100$, $K = 90$, $r = .05$, $T = 1$, $\sigma = .25$, $\lambda = 1$ and $\hat{\varphi}_i = .01$, then the price of the European call $C(0, S) = 18.14 e^{.01} = \$18.32$.

# 3 Conclusion

In sum, this result is perfectly intuitive since if there is no jump, $\lambda = 0$ and thus the price will be equal to the Black-Scholes price. Moreover, the option price increases in $\lambda$. Aside from the simplicity, it is intuitively very appealing. This method can be applied to other models in finance or mathematics in the future. Furthermore, future research can introduce methods to estimate the parameter $\hat{\varphi}_i$.

There is no conflict of interest.

## References

[1] Alghalith, M. (2020). Pricing options under simultaneous stochastic volatility and jumps: A simple closed-form formula without numerical/computational methods, Physica A: Statistical Mechanics and its Applications, 540, 2020, 123100.

[2] Gong, X.L., Zhuang, X.T., 2016. Option pricing for stochastic volatility model with infinite activity Lévy jumps. Physica A: Statistical Mechanics & Its Applications, 455, 1–10.

[3] Kou, S.G., 2002. A jump-diffusion model for option pricing. Management Science, 48, 1086–1101.

[4] Merton, R.C., 1976. Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics, 3, 125–144.

[5] Zhang, S.M., Wang, L.H., 2013. A fast numerical approach to option pricing with stochastic interest rate, stochastic volatility and dou-

ble jumps. Communications in Nonlinear Science and Numerical Simulation, 18, 1832–1839.

[6] Zhou, W., He, J.M., Yu, D.J., 2013. Double-jump diffusion model based on the generalized double exponential distribution of the random jump and its application. System Engineering Theory and Practice, 33, 2746–2756.