Possibility of Boltzmann was able to derive the entropy as Noether’s charge

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Abstract

Boltzmann’virgin paper is revisited. we discuss the possibility that he derive entropy as Noether’s charge and introduce Planck’s constant.

1 Introduction

In recent years, Sasa and Yokokura [4] has derived entropy as a Noether’s charge in a semi-static assumption. However, 150 years ago, Boltzmann is a possibility that could derive entropy as Noether charge at the time of the virgin paper 1866[1]. Then, we would like to explore the possibility that introduction of Planck’s constant was possible at this point. Before the discovery of Noether’s theorem[2] and Planck’s constant[3] which is 30-40 years ago that these considerations performed is surprising.

2 Noether’s theorem

We start to introduce Noether’s theorem.

Let’s consider of action ,

\[ I[x] = \int_{t_2}^{t_1} L(x, \dot{x}, t) dt. \] (2.1)

As a variation of this action,

\[ \delta I[x] = \delta \int_{t_2}^{t_1} L(x, \dot{x}, t) dt. \] (2.2)

Now, we take also variation of \( dt \), then,

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\[
\delta I[x] = \int_{t_2}^{t_1} \delta L(x, \dot{x}, t) dt - \int_{t_2}^{t_1} dL(x, \dot{x}, t) \delta t + [L(x, \dot{x}, t) \delta t]_{t_2}^{t_1}.
\]

This first term becomes
\[
\int_{t_2}^{t_1} \left( \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} + \frac{\partial L}{\partial t} \delta t \right) dt = \int_{t_2}^{t_1} \left( \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \delta x dt + \left[ \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right]_{t_2}^{t_1} + \int_{t_2}^{t_1} \frac{\partial L}{\partial t} \delta t dt,
\]

where
\[
\delta x(t) \equiv x'(t') - x(t) = x'(t + \delta t) - x(t) = x'(t) + \dot{x}'(t) \delta t - x(t) = \delta^L x(t) + \dot{x} \delta t,
\]

\[
\delta^L x(t) \equiv x'(t) - x(t).
\]

The second term becomes
\[
dL(x, \dot{x}, t) \delta t = \frac{\partial L}{\partial x} dx \delta t + \frac{\partial L}{\partial \dot{x}} d\dot{x} \delta t + \frac{\partial L}{\partial t} dt \delta t
\]
\[
= \left( \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \dot{x} dt \delta t + \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \dot{x} \delta t \right) dt + \frac{\partial L}{\partial t} \delta t dt \delta t.
\]

Summary, (2.3) is
\[
\delta I[x] = \int_{t_2}^{t_1} \left( \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \delta^L x dt + \left[ \frac{\partial L}{\partial \dot{x}} \delta \dot{x} + L \delta t \right]_{t_2}^{t_1}
\]
\[
= \int_{t_2}^{t_1} \partial_{EL} L \delta^L x dt + [p(x, \dot{x}, t) \delta x - E(x, \dot{x}, t) \delta t]_{t_2}^{t_1},
\]

where
\[
\partial_{EL} L \equiv \left( \frac{\partial}{\partial x} - \frac{d}{dt} \frac{\partial}{\partial \dot{x}} \right) L,
\]
\[
p \equiv \frac{\partial L}{\partial \dot{x}},
\]
\[
E \equiv p \dot{x} - L.
\]

Then, we introduce \( \Theta \) by
\[
\Theta \equiv p \dot{x} + E \delta t.
\]

We get
\[ \delta I[x] = \int_{t_2}^{t_1} \partial_{EL} \delta^L x dt + [\Theta]^t_{t_2}. \]  

(2.13)

If \( \delta I[x] \) take form boundary term only,

\[ \delta I[x] = [F]^t_{t_2}, \]  

(2.14)

we get final relation,

\[ -\int_{t_2}^{t_1} \partial_{EL} \delta^L x dt = [C]^t_{t_2}, \]  

(2.15)

and Noether’s charge,

\[ C \equiv \Theta - F. \]  

(2.16)

3 Discussion on dynamical derivation of Boltzmann’s entropy second law[1]

Let describe his argument with the current notation, A free particle action is

\[ I[x] = \int_{t_1}^{t_2} \frac{1}{2} m \dot{x}^2 dt. \]  

(3.1)

Variating (3.1), we get

\[ \delta I[x] = \int_{t_2}^{t_1} m \dot{x} \dot{\delta x} + \int_{t_2}^{t_1} \frac{1}{2} m \dot{x}^2 \delta t. \]  

(3.2)

In original paper, he introduced \( s = \sqrt{x^2 + y^2 + z^2} \) and calculated \( d\delta s \).

(3.2) is organized like

\[ \delta I[x] = -\int_{t_2}^{t_1} m \ddot{x} \delta x dt + [m \dot{x} \delta x]^t_{t_2} - \int_{t_2}^{t_1} d \left( \frac{1}{2} m \dot{x}^2 \right) \delta t + \left[ \frac{1}{2} m \dot{x}^2 \delta t \right]^t_{t_2}, \]  

(3.3)

\[ \delta I[x] = \int_{t_2}^{t_1} m \ddot{x} \delta x dt - m \ddot{x} \delta x \delta t + \left[ m \dot{x} \delta x + \frac{1}{2} m \dot{x}^2 \delta t \right]^t_{t_2}, \]  

(3.4)

\[ \delta I[x] = \int_{t_2}^{t_1} m \ddot{x}^{\delta x} x + \left[ m \dot{x} \delta x + \frac{1}{2} m \dot{x}^2 \delta t \right]^t_{t_2}. \]  

(3.5)

Now, we think the case of \( \delta x = 0 \) and the equation of motion \( m \ddot{x} = 0 \) is established.

Boltzmann imposed the following assumption from quasi-static adiabatic process,
\left[ \frac{1}{2} m \dot{x}^2 \right]_{t_2}^{t_1} = \int_{t_2}^{t_1} \epsilon dt \quad (3.6)

By this change, \( \delta I[x] \) become

\[ \delta I[x] = [\epsilon t]_{t_2}^{t_1} = \epsilon (t_1 - t_2), \quad (3.7) \]

then, we get

\[ \epsilon = \frac{\delta \int \frac{1}{2} m \dot{x}^2}{t_1 - t_2}. \quad (3.8) \]

Now, Boltzmann use the definition of temperature that was introduced at the beginning of his thesis,

\[ T = \frac{\int \frac{1}{2} m \dot{x}^2}{t_1 - t_2} = \frac{I}{t_1 - t_2}. \quad (3.9) \]

This definition corresponds to Energy equality distribution law.

Using (3.9), we get

\[ \frac{\epsilon}{T} = \frac{\delta \int \frac{1}{2} m \dot{x}^2}{\int \frac{1}{2} m \dot{x}^2} = \delta \log I. \quad (3.10) \]

Then Boltzmann derived entropy \( S \) by

\[ S = \frac{\epsilon}{T} = \beta^{-1} \epsilon = \delta \log \int_{t_2}^{t_1} \frac{1}{2} m \dot{x}^2. \quad (3.11) \]

### 3.1 Discussion

The change (3.6) corresponds replacement of energy to microcanonical distribution by Sasa and Yokokura [4]. Therefore, macro variable corresponding Noether’s charge is

\[ C = \epsilon \delta t = TS \delta t, \quad (3.12) \]

using

\[ t_1 - t_2 = \delta t. \quad (3.13) \]

Then we get

\[ S = \frac{\beta C}{\delta t}. \quad (3.14) \]

If we put

\[ \delta t = \hbar \beta, \quad (3.15) \]

then we get entropy by Noether’s charge,
\[ S = C. \]  

(3.16)

It seems that there was a possibility that Boltzmann had anticipated the era.

References


