https://doi.org/10.32388/5VXK5K

Possibility of Boltzmann was able to derive the entropy as Noether's charge

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October 3, 2023

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Abstract

Boltzmann'virgin paper is rivisited. we disscuss the possibility that he derive entropy as Nother's charge and introduce Planck's constant.

1 Introduction

In recent years, Sasa and Yokokura [4] has derived entropy as a Noether's charge in a semi-static assumption. However, 150 years ago, Boltzmann is a possibility that could derive entropy as Noether charge at the time of the virgin paper 1866[1].Then, we would like to explore the possibility that introduction of Planck's constant was possible at this point. Before the discovery of Noether's theorem[2] and Planck's constant[3] which is 30-40 years ago that these considerations performed is surprising.

2 Noether's theorem

We start to introduce Nother's theorem.

Let's consider of action ,

$$I[x] = \int_{t_2}^{t_1} L(x, \dot{x}, t)dt. \tag{2.1}$$

As a variation of this action,

$$\delta I[x] = \delta \int_{t_2}^{t_1} L(x, \dot{x}, t)dt. \tag{2.2}$$

Now, we take also variation of dt, then,

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$$\delta I[x] = \int_{t_2}^{t_1} \delta L(x,\dot{x},t)dt - \int_{t_2}^{t_1} dL(x,\dot{x},t)\delta t + [L(x,\dot{x},t)\delta t]_{t_2}^{t_1}. \tag{2.3}$$

This first term become

$$\int_{t_2}^{t_1} \left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} + \frac{\partial L}{\partial t}\delta t\right) dt = \int_{t_2}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)\delta x dt + \left[\frac{\partial L}{\partial \dot{x}}\delta x\right]_{t_2}^{t_1} + \int_{t_2}^{t_1} \frac{\partial L}{\partial t}\delta t dt, \tag{2.4}$$

where

$$\delta x(t) \equiv x'(t') - x(t) = x'(t+\delta t) - x(t) = x'(t) + \dot{x}'(t)\delta t - x(t) = \delta^L x(t) + \dot{x}\delta t, \tag{2.5}$$

$$\delta^L x(t) \equiv x'(t) - x(t). \tag{2.6}$$

The second term become

$$\begin{aligned} dL(x,\dot{x},t)\delta t = &\frac{\partial L(x,\dot{x},t)}{\partial x}dx\delta t + \frac{\partial L(x,\dot{x},t)}{\partial \dot{x}}d\dot{x}\delta t + \frac{\partial L(x,\dot{x},t)}{\partial t}dt\delta t \\ =&\frac{\partial L(x,\dot{x},t)}{\partial x}\dot{x}dt\delta t + \frac{\partial L(x,\dot{x},t)}{\partial \dot{x}}\ddot{x}dt\delta t + \frac{\partial L(x,\dot{x},t)}{\partial t}dt\delta t \\ =&\left(\frac{\partial L(x,\dot{x},t)}{\partial x} - \frac{d}{dt}\frac{\partial L(x,\dot{x},t)}{\partial \dot{x}}\right)\dot{x}dt\delta t + \frac{d}{dt}\left(\frac{\partial L(x,\dot{x},t)}{\partial \dot{x}}\dot{x}\delta t\right)dt + \frac{\partial L(x,\dot{x},t)}{\partial t}dt\delta t \end{aligned} \tag{2.7}$$

Summary, (2.3) is

$$\begin{aligned} \delta I[x] &= \int_{t_2}^{t_1} \left(\frac{\partial L(x,\dot{x},t)}{\partial x} - \frac{d}{dt}\frac{\partial L(x,\dot{x},t)}{\partial \dot{x}}\right)\delta^L x dt + \left[\frac{\partial L}{\partial \dot{x}}\delta^L x + L\delta t\right]_{t_2}^{t_1} \\ &= \int_{t_2}^{t_1} \partial_{EL} L \delta^L x dt + [p(x,\dot{x},t)\delta x - E(x,\dot{x},t)\delta t]_{t_2}^{t_1}, \end{aligned} \tag{2.8}$$

where

$$\partial_{EL} L \equiv \left(\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right) L, \tag{2.9}$$

$$p \equiv \frac{\partial L}{\partial \dot{x}}, \tag{2.10}$$

$$E \equiv p\dot{x} - L. \tag{2.11}$$

Then, we introduce Θ by

$$\Theta \equiv p\delta x + E\delta t. \tag{2.12}$$

We get

$$\delta I[x] = \int_{t_2}^{t_1} \partial_{EL} L \delta^L x dt + [\Theta]_{t_2}^{t_1}. \tag{2.13}$$

If $\delta I[x]$ take form boundary term only,

$$\delta I[x] = [F]_{t_2}^{t_1}, \tag{2.14}$$

we get final relation,

$$-\int_{t_2}^{t_1} \partial_{EL} L \delta^L x dt = [C]_{t_2}^{t_1}, \tag{2.15}$$

and Nother's charge,

$$C \equiv \Theta - F. \tag{2.16}$$

3 Discussion on dynamical derivation of Boltzmann's entropy second law[1]

Let describe his argument with the current notation, A free particle action is

$$I[x] = \int_{t_1}^{t_2} \frac{1}{2} m\dot{x}^2 dt. \tag{3.1}$$

Variating (3.1), we get

$$\delta I[x] = \int_{t_2}^{t_1} m\dot{x}\delta\dot{x} + \int_{t_2}^{t_1} \frac{1}{2} m\dot{x}^2 \delta dt. \tag{3.2}$$

In original paper, he introduced $s = \sqrt{x^2 + y^2 + z^2}$ and calulated $d\delta s$. (3.2) is organaized like

$$\delta I[x] = -\int_{t_2}^{t_1} m\ddot{x}\delta x dt + [m\dot{x}\delta x]_{t_2}^{t_1} - \int_{t_2}^{t_1} d\left(\frac{1}{2} m\dot{x}^2\right)\delta t + \left[\frac{1}{2} m\dot{x}^2 \delta t\right]_{t_2}^{t_1}, \tag{3.3}$$

$$\delta I[x] = \int_{t_2}^{t_1} m\ddot{x}\delta x dt - m\ddot{x}\dot{x} dt \delta t + \left[m\dot{x}\delta x + \frac{1}{2} m\dot{x}^2 \delta t\right]_{t_2}^{t_1}, \tag{3.4}$$

$$\delta I[x] = \int_{t_2}^{t_1} m\ddot{x}\delta^L x + \left[m\dot{x}\delta x + \frac{1}{2} m\dot{x}^2 \delta t\right]_{t_2}^{t_1}. \tag{3.5}$$

Now, we think the case of $\delta x = 0$ and the equation of motion $m\ddot{x} = 0$ is established.

Boltzmann imposed the following assumption from quasi-static adiabatic process,

$$\left[\frac{1}{2}m\dot{x}^2\delta t\right]_{t_2}^{t_1} = \int_{t_2}^{t_1}\epsilon dt \tag{3.6}$$

By this change, $\delta I[x]$ bacome

$$\delta I[x] = [\epsilon t]_{t_2}^{t_1} = \epsilon(t_1 - t_2), \tag{3.7}$$

then, we get

$$\epsilon = \frac{\delta \int \frac{1}{2}m\dot{x}^2}{t_1 - t_2}. \tag{3.8}$$

Now, Boltzmann use the definition of temperature that was introduced at the beginning of his thesis,

$$T = \frac{\int \frac{1}{2}m\dot{x}^2}{t_1 - t_2} = \frac{I}{t_1 - t_2}. \tag{3.9}$$

This definition corresponds to Energy equality distribution law.
Using (3.9), we get

$$\frac{\epsilon}{T} = \frac{\delta \int \frac{1}{2}m\dot{x}^2}{\int \frac{1}{2}m\dot{x}^2} = \delta \log I. \tag{3.10}$$

Then Boltzmann derived entropy S by

$$S = \frac{\epsilon}{T} = \beta^{-1}\epsilon = \delta \log \int_{t_2}^{t_1} \frac{1}{2}m\dot{x}^2. \tag{3.11}$$

3.1 Disscussion

The change (3.6) corenponds replacement of energy to microcanonical distribution by Sasa and Yokokura [4].Therefore, macro variable coresponding Noether's charge is

$$C = \epsilon\delta t = TS\delta t, \tag{3.12}$$

using

$$t_1 - t_2 = \delta t. \tag{3.13}$$

Then we get

$$S = \frac{\beta C}{\delta t}. \tag{3.14}$$

If we put

$$\delta t = \hbar\beta, \tag{3.15}$$

then we get entropy by Noether 's charge,

$$S = C. \tag{3.16}$$

It seems that there was a possibility that Boltzmann had anticipated the era.

References

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