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# The FLRW Geometries and the Expansion of Space

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ABSTRACT

The usual presentation of the FLRW geometries is given in coordinates that makes it difficult to assess the nature of the expansion of space for these geometries. There exists, however, alternative representations defined by a (canonical) diffeomorphism that introduces coordinates giving more accessible descriptions. The paper presents these coordinates and in addition similar geometries which appear as natural alternatives to the FLRW geometries. The two kinds of geometries give the same description of the evolution of the universe for the period of which we have empirical information. A number of consequences of this observation is presented for the spherical model.

*Keywords:* Cosmology — FLRW Geometries — Expansion of Universe

## 1. INTRODUCTION

The empirical knowledge we possess about the global structure of the universe, taken to impose the view that we live in an expanding space, is constituted by the existence of redshift of the light from distant galaxies and the corresponding slow dilution of the cosmic microwave background radiation (1). This knowledge is taken to be compatible with the description given by the standard models of cosmology. The modes are based on three hypotheses:

*Hypothesis 1*—the geometrical structure of space-time is described by the Friedman-Lemaître-Robertson-Walker metrics ($\kappa = 1, 0, -1$)

$$\left(\hat{g}_{\kappa\mu\nu}\left(t, y^i\right)\right) = \begin{pmatrix} c^2 & 0 \\ 0 & \hat{g}_{\kappa ij}\left(t, y^i\right) \end{pmatrix} \tag{1}$$

$$\hat{g}_{\kappa ij}\left(t, y^i\right) = -R^2\left(t\right)\left(\delta_{ij} + \frac{\kappa}{1 - \kappa r^2\left(y^i\right)} y_i y_j\right) \tag{2}$$

where $y_i = \delta_{ij} y^j$, $R\left(t\right) > 0$ has the dimension of spatial length, $y^i$ is dimensionless and $r^2(y^i) = \delta_{ij} y^i y^j$ (2; 3). They are expressed in the Cartesian coordinates $(t, y^i)$ on the coordinate domains $\left\{(t, y^i) \in \mathbb{R}_+ \times \mathbb{R}^3 | \delta_{ij} y^i y^j < 1\right\}$ for $\kappa = 1$ and $\mathbb{R}_+ \times \mathbb{R}^3$ for $\kappa = 0, -1$. These are coordinate domains of the quasi-Riemannian space-times $\hat{S}_1 = \mathbb{R}_+ \times M_1$ where $M_1$ is the sphere of radius 1, $\hat{S}_0 = \mathbb{R}_+ \times M_0 = \mathbb{R}_+ \times \mathbb{R}^3$ and $\hat{S}_{-1} = \mathbb{R}_+ \times M_{-1}$ where $M_{-1}$ is the one sheet hyperbolic space with semi-minor axis 1. The spaces $M_1$, $M_0$ and $M_{-1}$ are homogeneous and isotropic because they support transitive actions of the groups $SU\left(2\right)$, $SO\left(3\right) \times T^3$ and $SO\left(3, 1\right)$, respectively.

*Hypothesis 2*—the universe is a "perfect fluid", i.e. described by the contravariant energy-momentum tensor

$$\hat{\mathcal{T}}^{\mu\nu}_{\kappa} = \left(\rho + \frac{p}{c^2}\right) \hat{v}^{\mu}_{\kappa} \hat{v}^{\nu}_{\kappa} - p \hat{g}^{\mu\nu}_{\kappa} \tag{3}$$

According to the theory of general relativity the equations determining the evolution of the universe are

$$\hat{\mathcal{R}}_{\kappa\mu\nu} - \frac{1}{2} \hat{\mathcal{R}}^{\lambda}_{\kappa\lambda} \hat{g}_{\kappa\mu\nu} + \frac{c^4}{8\pi G} \hat{\mathcal{T}}_{\kappa\mu\nu} + \Lambda \hat{g}_{\kappa\mu\nu} = 0 \tag{4}$$

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or

$$\hat{\mathcal{R}}_{\kappa\mu\nu} = -\frac{8\pi G}{c^4}\left(\hat{\mathcal{T}}_{\kappa\mu\nu} - \frac{1}{2}\hat{\mathcal{T}}^{\lambda}_{\kappa\lambda}\hat{g}_{\kappa\mu\nu}\right) - \Lambda\hat{g}_{\kappa\mu\nu} \tag{5}$$

and

$$\hat{\mathcal{T}}^{\mu\nu}_{\kappa;\nu} = 0 \tag{6}$$

*Hypothesis 3*—the velocity of the fluid elements is $(\hat{v}^{\mu}_{\kappa}) = ((1, 0, 0, 0)$

The three kinds of FLRW geometries described under *Hypothesis* 1 are associated with static spaces, thus, there is no direct expansion of space, but due to the form of the metrics the distances between points increase, as time passes, with respect to the coordinates $y^i$. In popular texts as for example (4), the evolution of a spherical universe is exemplified by the act of inflating a rubber balloon. Though this example could correspond to the conceptual picture provided by empirical observations it does not correspond to the picture given by the FLRW coordinates for the spherical case.

In the following I define a change of coordinates and derive the corresponding representation of the FLRW geometries. In this representation the spherical model of space expands like the ballon example, however, the flat space is static as is its metric, while the hyperbolic space is flattening in time. The next section presents an alternative but similar space-time geometry for the spherical case. In section 4 consequences of the expansion are described for the spherical case. The main results of the paper is summarized in the final section.

## 2. THE FLRW GEOMETRIES

Consider the coordinate map

$$\Psi : \mathbb{R}_+ \times V_\kappa \to \{U_{\kappa t} | t \in \mathbb{R}_+\}\,;\, \left(t, y^i\right) \mapsto \left(t, x^i\right) = \left(t, R\left(t\right) y^i\right) \tag{7}$$

where

$$U_{1t} = \left\{x^i \in \mathbb{R}^3 | \delta_{ij} x^i x^j < R^2\left(t\right)\right\} \tag{8}$$

$$U_{0t} = U_{-1t} = \mathbb{R}^3 \tag{9}$$

then

$$g_{\kappa\mu\nu}\left(t, x^i\right) = \left(\Psi^{-1\rho}_{,\mu}\right)\left(t, x^i\right)\left(\Psi^{-1\lambda}_{,\nu}\right)\left(t, x^i\right) g_{\rho\lambda} \circ \Psi^{-1}\left(t, x^i\right) \tag{10}$$

or

$$\left(g_{\kappa\mu\nu}\left(t, x^i\right)\right) = \begin{pmatrix} c^2 - \frac{\dot{R}^2(t) r^2\left(x^i\right)}{R^2(t) - \kappa r^2(x^i)} & \frac{R(t)\dot{R}(t)}{R^2(t) - \kappa r^2(x^i)} x_j \\ \frac{R(t)\dot{R}(t)}{R^2(t)\kappa K r^2(x^i)} x_i & g_{\kappa ij}\left(t, x^i\right) \end{pmatrix} \tag{11}$$

$$g_{\kappa ij}\left(t, x^i\right) = -\delta^{ij} - \frac{\kappa}{R^2\left(t\right) - \kappa r^2\left(x^i\right)} x_i x_j \tag{12}$$

for $\dot{R}\left(t\right) = \frac{dR(t)}{dt}$, defined on the coordinate domains $\{U_{\kappa t} | t \in \mathbb{R}_+\}$. The space-times are $S_\kappa = \{M_{\kappa t} | t \in \mathbb{R}_+\}$.

I will refer to this representations of as the Eulerian representations of the FLRW geometries because in this representation the coordinates $x^i$ are those measured according to the operational definitions set down by the Systeme International and used by astronomers to give the distances to galaxies. In this representation $M_{1t}$ is a sphere of radius $R\left(t\right)$, $M_{0t}$ is the static Euclidean space and $M_{-1t}$ is the one sheet hyperbolic space with semi-minor axis $R\left(t\right)$.

In the following I will consider only the spherical case.

## 3. AN ALTERNATIVE GEOMETRY FOR THE SPHERICAL MODEL

The space $M_{1t}$ is a subspace of the four-dimensional Euclidean space $E_4$, i.e.

$$M_{1t} = \left\{x^i \in \mathbb{R}^4 | \delta_{ij} x^i x^j + u^2 - R^2\left(t\right) = 0\right\} \tag{13}$$

or $M_{1t}$ is a three-dimensional sphere whose radius is $R\left(t\right)$. The metric 11 on $S_1$ is induced from the metric on $\mathbb{R}_+ \times E_4$ defined by

$$ds^2 = \left(c^2 + \dot{R}^2\right) dt^2 - \delta_{ij} dx^i dx^j - du^2 \tag{14}$$

on $\mathbb{R}_+ \times E_4$.

Let the space-time be $S_1$ and the metric on $\mathbb{R}_+ \times E_4$ be defined by

$$ds^2 = c^2 dt^2 - \delta_{ij} dx^i dx^j - du^2 \tag{15}$$

then the induced metric on $S_1$ is

$$\left(g_{\mu\nu}\left(t, x^i\right)\right) = \begin{pmatrix} c^2 - \frac{R^2(t)\dot{R}^2(t)}{R^2(t) - r^2(x^i)} & \frac{R(t)\dot{R}(t)x_i}{R^2(t) - r^2(x^i)} \\ \frac{R(t)\dot{R}(t)x_j}{R^2(t) - r^2(x^i)} & g_{ij}\left(t, x^i\right) \end{pmatrix} \tag{16}$$

$$g_{ij}\left(t, x^i\right) = -\delta_{ij} - \frac{1}{R^2(t) - r^2(x^i)} x_i x_j \tag{17}$$

on the coordinate domain $\{U_t | t \in \mathbb{R}_+\}$. As for the FLRW geometry this is an Eulerian representation.

In the FLRW coordinates defined by $\Psi^{-1}$ (7), the metric is

$$\left(\tilde{g}_{\mu\nu}\left(t, y^i\right)\right) = \begin{pmatrix} c^2 - \dot{R}^2(t) & 0 \\ 0 & \hat{g}_{ij}\left(t, y^i\right) \end{pmatrix} \tag{18}$$

Thus, in order for the space-time to be semi-Riemannian $\dot{R}^2(t) < c^2$, i.e. the rate of expansion must be smaller than the velocity of light.

For the interval of time I for which we have any empirical information about the evolution of the universe $\dot{R}^2(t) << c^2$, the two models coincide, and the metrics become

$$\left(\breve{g}_{\mu\nu}\left(t, x^i\right)\right) = \begin{pmatrix} c^2 & 0 \\ 0 & g_{ij}\left(t, x^i\right) \end{pmatrix} \tag{19}$$

$$g_{ij}\left(t, x^i\right) = -\delta_{ij} - \frac{1}{R^2(t) - r^2(x^i)} x_i x_j \tag{20}$$

in the Euler representation. In fact, the line element

$$ds^2 = \left(1 - \frac{\dot{R}^2(t)}{c^2} \frac{R^2(t)}{R^2(t) - r^2(x^i)}\right) dx^{02} \tag{21}$$

$$+2\frac{\dot{R}(t)}{c} \frac{R(t)\, x_i}{R^2(t) - r^2(x^i)} dx^i dx^0 + g_{ij}\left(t, x^i\right) dx^i dx^j \tag{22}$$

which approximates 19.

## 4. REDSHIFTS AND MOTIONS

For the period I it is sufficient to consider the model $(S_1, \breve{g}_{\mu\nu})$. The geodesics on this space are solutions of the Lagrange equations for the Lagrange functions

$$L\left(t, x^i, \frac{dt}{ds}, \frac{dx^i}{ds}\right) = \sqrt{\left(\frac{dt}{ds}\right)^2 c^2 + \breve{g}_{ij}\left(t, x^i\right) \frac{dx^i}{ds}\frac{dx^j}{ds}} \tag{23}$$

with the constraint

$$L\left(t, x^i, \frac{dt}{ds}, \frac{dx^i}{ds}\right) = \epsilon \tag{24}$$

where $\epsilon = 0$ for photons and $\epsilon = c$ for massive objects. s is the proper time. The Lagrange equations are

$$\frac{d^2t}{ds^2} - \frac{1}{2c^2}\frac{d}{ds}\left(\breve{g}_{ij}\left(t, x^i\right) \frac{dx^i}{ds}\frac{dx^j}{ds}\right) = 0 \tag{25}$$

$$\frac{d}{ds}\left(\breve{g}_{ij}\left(t, x^i\right) \frac{dx^j}{ds}\right) - \partial_{x^i} \breve{g}_{jk}\left(t, x^i\right) \frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \tag{26}$$

It is easy to show that the Lagrangian $L\left(t, x^i, \frac{dt}{ds}, \frac{dx^i}{ds}\right)$ is constant along the geodesic, moreover, 25 and 26 are the equations for the geodesics on $\{U_t | t \in \mathbb{R}_+\}$.

For $\epsilon = 0$ the projection of a geodesic in space-time is therefore a geodesic on $U_t$ for any $t$, i.e. a section of a light ray is projected to a section of a great circle. Thus, if a light ray with wav length $\lambda_1$ is emitted at time $t_1 \in$ P covers a section of a great circle of radius $R(t_1)$, then at the later time $t_2 \in$P the wavelength $\lambda_2$ of the same light wave covers a section on the great circle of radius $R(t_2)$, i.e.

$$\frac{\lambda_1}{\lambda_2} = \frac{R(t_1)}{R(t_2)} \tag{27}$$

which describes the redshift of light for intervals of time in P. Thus, turning the argument we can conclude that the condition $\dot{R}^2(t) << c^2$ holds good for this period. It is an empirical justification of. 19.

For each moment of time the increase of the radius of the sphere determines a normal vector field $N_t$ to the sphere in the englobing space-time $R+ \times E_4$. The projection of $N_t$ at time $t$ onto the tangent space $TU_t$ is

$$n^i\left(t, x^i\right) = \frac{\dot{R}(t)}{R(t)} x^i \tag{28}$$

This result is easily obtained by considering the great circles on the sphere orthogonal to the coordinate domain and noticing that the speed of radial extension is $\dot{R}(t)$.

The contravariant representations of the projection of $N_t$ on $T(\mathbb{R}_+ \times U_t)$ is

$$\left(n^\mu\left(t, x^i\right)\right) = \left(1, \frac{\dot{R}(t)}{R(t)} x^i\right) \tag{29}$$

and its covariant representation in the homogeneous coordinates is

$$\left(\hat{n}^\mu\left(t, y^i\right)\right) = \left(\Psi^{-1\mu}_{,\nu}\left(t, y^i\right)\right)^{-1}\left(n^\nu \circ \Psi^{-1}\left(t, y^i\right)\right) = (1,0,0,0) \tag{30}$$

This is in accordance with the fact that $V = \left\{y^i \in \mathbb{R}^3 | \delta_{ij} y^i y^j < 1\right\}$ is a static space.

On the other hand, the vector field $U$ on the $M_{1t}$ due to the expansion satisfies the geodesic equations for $\epsilon = c$. Now, 24 gives

$$\frac{dt}{ds} = \frac{1}{\sqrt{1 + \frac{1}{c^2}\breve{g}_{ij}\left(t, x^i\right)\dot{x}^i\dot{x}^j}} = \gamma\left(t, x^i, \dot{x}^i\right) \tag{31}$$

Thus, in terms of the time $t$ the geodesic equations are

$$\frac{1}{\gamma\left(t, x^i, \dot{x}^i\right)}\frac{d}{dt}\gamma\left(t, x^i, \dot{x}^i\right) - \frac{1}{2c^2}\partial_t \breve{g}_{ij}\left(t, x^i\right)\dot{x}^i\dot{x}^j = 0 \tag{32}$$

$$\frac{1}{\gamma\left(t, x^i, \dot{x}^i\right)}\frac{d}{dt}\left(\breve{g}_{ij}\left(t, x^i\right)\gamma\left(t, x^i, \dot{x}^i\right)\dot{x}^j\right) - \partial_{x^i}\breve{g}_{jk}\left(t, x^i\right)\dot{x}^j\dot{x}^k = 0 \tag{33}$$

The local representative of the vector field $U$ is $u^i\left(t, x^i\right) = \frac{\dot{R}}{R}\sqrt{R^2 - r^2}\frac{x^i}{r}$. It is the projection of the tangent vectors describing the increase of the length of the great circles on the sphere to the coordinate domain. Since the length of the circumference of the great circles is $2\pi R(t)$ the speed of the increase is $\dot{R}(t)$. Thus,

$$\dot{x}^i(t) = \frac{\dot{R}(t)}{R(t)}\sqrt{R^2(t) - r^2\left(x^i(t)\right)}\frac{x^i(t)}{r\left(x^i(t)\right)} \tag{34}$$

satisfies 32 and 33. Notice that in this case

$$\gamma(t) = \frac{1}{\sqrt{1 - \frac{\dot{R}^2(t)}{c^2}}} \tag{35}$$

Fluid elements will move with the velocity $u^i\left(t, x^i\right)$on the sphere. Let

$$\left(u^\mu\left(t, x^i\right)\right) = \left(1, u^i\left(t, x^i\right)\right) \tag{36}$$

be the contravariant four-velocity field in the Euclidean representation, then its representation $\hat{v}^\mu\left(t, y^i\right)$ in FLRW coordinates is the four-vector that should enter in the covariant energy momentum tensor for a perfect fluid (5); thus, since $\hat{v}^i \neq 0$ *Hypothesis* 3 is not valid.

## 5. FINAL REMARKS

I have shown that if we assume that the time and space coordinates $\left(t, x^i\right)$ are measured according to the operational definitions laid down by the Systeme International and used by astronomers, the Eulerian representation pictures explicitly the expansion of a spherical universe. Moreover, I have shown that the expansion of the universe in the model is incompatible with *Hypothesis* 3. This is due to the fact that the normal vector field to the spherical space, describing its evolution, is an element in the tangent bundle of the englobing space $R_+ \times E_4$ not the tangent bundle of the of space-time and that the spatial projection of the transformed $(1, 0, 0, 0)$ to the Euclidean representation is the normal vector field on the sphere. In the framework of the theory of general relativity it is the tangent vector field on space induced by normal vector field that describes the evolution of the spherical space, its extension.

I have also presented two alternative metrics for the space-time $S_1$, being the induced metrics derived from $ds^2 = \left(c^2 + \dot{R}^2\left(t\right)\right) dt^2 - \delta_{ij} dx^i dx^j - du^2$ and $ds^2 = c^2 dt^2 - \delta_{ij} dx^i dx^j - du^2$ on $\mathbb{R}_+ \times E_4$, the first of which gives the FLRW geometry. When $\dot{R}^2\left(t\right) << c^2$ the two models coincide. For the period for which we have empirical evidence, i.e. the last 13 billion years, we can therefore at present not distinguish between the two geometries on empirical grounds. The second choice will, however, exclude the possibility for an inflationary period after a possible Big Bang as well as making it necessary to reconsider the estimate for the age of the universe.

## REFERENCES

[1] Bennett, C. L, Cosmology from start to finish, Nature. 440 (7088)

[2] Weinberg, S., Cosmology, Oxford University Press, Oxford (2008)

[3] Hobson, M.P., Efstathiou, G., and Lasenby, A.N., General Relativity, Cambridge University Press, Cambridge (2006)

[4] Hawking, S.W., A Brief History of Time, Bantham (1988)