The FLRW Geometries and the Expansion of the Universe

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Abstract The usual presentation of the FLRW geometries is given in coordinates that makes it difficult to assess the nature of the expansion of space. There exists, however, alternative representations defined by a (canonical) diffeomorphism that supports more direct descriptions; in particular, these representations can be shown to be subspaces of five-dimensional ”space-times” where the ”space” is either the four-dimensional Euclidean space or the four-dimensional hyperbolic space, and with space-time metrics that are induced from the metrics on the englobing ”space-times”. The paper present the explicit constructions and, in addition similar geometries which appear as natural alternatives to the FLRW geometries, but which is compatible with the theory of relativity in the sense that it implies that the rate of expansion must be smaller than the velocity of light. The two kinds of geometries give the same description of the evolution of the universe for the period of which we have empirical information.

Keywords Universe · Expansion · FLRW Geometries · General Relativity

1 Introduction

The empirical knowledge we possess about the global structure of the universe, taken to impose the view that we live in an expanding space, is constituted by the existence of redshift of the light from distant galaxies and the corresponding slow dilution of the cosmic microwave background radiation [1]. This knowledge is taken to be compatible with the description given by the standard model of cosmology. This model is based on three hypotheses:

Hypothesis 1 the geometrical structure of space-time is described by the Friedman-Lemaître-Robertson-Walker metrics $(\kappa = 1, 0, -1)$

$$\left(\hat{g}_{\kappa\mu\nu}\left(t, y^i\right)\right) = \begin{pmatrix} c^2 & 0 \\ 0 & \hat{g}_{\kappa ij}\left(t, y^i\right) \end{pmatrix} \quad (1)$$

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$$\hat{g}_{\kappa ij}\left(t,y^i\right) = -R^2\left(t\right)\left(\delta_{ij} + \frac{\kappa}{1-\kappa r^2\left(y^i\right)}y_i y_j\right) \tag{2}$$

where $y_i = \delta_{ij}y^j$, $R(t) > 0$ has the dimension of spatial length, y^i is dimensionless and $r^2(y^i) = \delta_{ij}y^i y^j$ [2,3]. They are expressed in the Cartesian coordinates (t,y^i) on the coordinate domains $\left\{(t,y^i) \in \mathbb{R}_+ \times \mathbb{R}^3 | \delta_{ij}y^i y^j < 1\right\}$ for $\kappa = 1$ and $\mathbb{R}_+ \times \mathbb{R}^3$ for $\kappa = 0,-1$. These are the coordinate domains of the quasi-Riemannian space-times $\hat{S}_1 = \mathbb{R}_+ \times M_1$ where M_1 is the sphere of radius 1, $\hat{S}_0 = \mathbb{R}_+ \times \mathbb{R}^3$ is the flat space and $\hat{S}_{-1} = \mathbb{R}_+ \times M_{-1}$ where M_{-1} is the one sheet hyperbolic space with semi-minor axis 1. The spaces M_1 and M_{-1} are homogeneous and isotropic because they support the transitive actions of the groups $SU(2)$ and $SO(3,1)$, respectively. The third possible homogeneous and isotropic subspace of E_4 is the E_3 for which the group is $SO(3) \times T^3$.

Hypothesis 2 the universe is a "perfect fluid", i.e. described by the contravariant energy-momentum tensor

$$\hat{\mathscr{T}}_\kappa^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)\hat{v}_\kappa^\mu \hat{v}_\kappa^\nu - p\hat{g}_\kappa^{\mu\nu} \tag{3}$$

According to the theory of general relativity the equations determining the evolution of the universe are

$$\hat{\mathscr{R}}_{\kappa\mu\nu} - \frac{1}{2}\hat{\mathscr{R}}_{\kappa\lambda}^{\lambda}\hat{g}_{\kappa\mu\nu} + \frac{c^4}{8\pi G}\hat{\mathscr{T}}_{\kappa\mu\nu} + \Lambda\hat{g}_{\kappa\mu\nu} = 0 \tag{4}$$

or

$$\hat{\mathscr{R}}_{\kappa\mu\nu} = -\frac{8\pi G}{c^4}\left(\hat{\mathscr{T}}_{\kappa\mu\nu} - \frac{1}{2}\hat{\mathscr{T}}_{\kappa\lambda}^{\lambda}\hat{g}_{\kappa\mu\nu}\right) - \Lambda\hat{g}_{\kappa\mu\nu} \tag{5}$$

and

$$\hat{\mathscr{T}}_{\kappa;\nu}^{\mu\nu} = 0 \tag{6}$$

Hypothesis 3 the velocity of the fluid elements is $\left(\hat{v}_\kappa^\mu\right) = ((1.0,0,0)$

The three kinds of FLRW geometries described under *Hypothesis* 1 are associated with static spaces, thus, there is no direct expansion of space, but due to the form of the metrics the distances between points increase as time passes. In popular texts as for example [4], the evolution of the universe is pictured by the act of inflating a rubber balloon. In this picture the space is expanding and the distance between the points on its surface are increasing as a consequence of the expansion. The "rubber balloon" model is associated with a different conception of expansion than the FLRW spherical model which indicates that different coordinatizations and the corresponding descriptions of the expansion should be considered. The presentation thus starts by defining a change of coordinates and to derive the corresponding representation of the FLRW geometries and to show that there are englobing semi-Riemannian "space-times" from which they can be derived. The next section presents an alternative but similar space time geometry. In section 4 the motions due to the expansion are described. The main results of the paper is summarized in the final section.

2 The FLRW Geometries

Consider the coordinate map

$$\Psi : \mathbb{R}_+ \times V_\kappa \to \{U_{\kappa t} | t \in \mathbb{R}_+\}; \left(t,y^i\right) \mapsto \left(t,x^i\right) = \left(t,R\left(t\right)y^i\right) \tag{7}$$

where

$$U_{1t} = \left\{x^i \in \mathbb{R}^3 | \delta_{ij} x^i x^j < R^2(t)\right\} \tag{8}$$

$$U_{0t} = U_{-1t} = \mathbb{R}^3 \tag{9}$$

then

$$g_{\kappa\mu\nu}\left(t, x^i\right) = \left(\Psi_{,\mu}^{-1\rho}\right)\left(t, x^i\right)\left(\Psi_{,\nu}^{-1\lambda}\right)\left(t, x^i\right) g_{\rho\lambda} \circ \Psi^{-1}\left(t, x^i\right) \tag{10}$$

or

$$\left(g_{\kappa\mu\nu}\left(t, x^i\right)\right) = \begin{pmatrix} c^2 - \frac{\kappa \dot{R}^2(t) r^2\left(x^i\right)}{R^2(t) - \kappa r^2\left(x^i\right)} & \frac{R(t)\dot{R}(t)}{R^2(t) - \kappa r^2\left(x^i\right)} x_j \\ \frac{R(t)\dot{R}(t)}{R^2(t)\kappa K r^2\left(x^i\right)} x_i & g_{\kappa ij}\left(t, x^i\right) \end{pmatrix} \tag{11}$$

$$g_{\kappa ij}\left(t, x^i\right) = -\delta^{ij} - \frac{\kappa}{R^2(t) - \kappa r^2\left(x^i\right)} x_i x_j \tag{12}$$

for $\dot{R}(t) = \frac{dR(t)}{dt}$, defined on the coordinate domains $\{U_{\kappa t} | t \in \mathbb{R}_+\}$. The space-times are $S_\kappa = \mathbb{R} \times M_{\kappa t}$ where $M_0 = E_3$ and M_{1t}. Moreover, the spaces $M_0 = E_3$ and M_{1t} are subspaces of the four-dimensional Euclidean space E_4, and M_{-1t} is a subspace of the four-dimensional hyperbolic space $H_{3,1}$, i.e.

$$M_{\kappa t} = \left\{x^i | u^2 + \kappa \delta_{ij} x^i x^j - R^2(t) = 0 | \left(x^i, u\right) \in \mathbb{R}^4\right\} \tag{13}$$

or $M_t(1)$ is a three-dimensional sphere whose radius is $R(t)$, i.e. if the radius is increasing with time t the sphere is expanding its area and the distance between separate points are also increasing. For the space $M_t(-1)$ $R(t)$ is the semi-minor axis and the space is flattening for increasing $R(t)$.

Thus, let the FLRW-space-times in this representation be given by

$$S_\kappa = \{M_{\kappa t} | t \in \mathbb{R}_+\} \tag{14}$$

then the metrics on S_κ induced from the metrics

$$ds^2 = \left(c^2 + \kappa \dot{R}^2\right) dt^2 - \delta_{ij} dx^i dx^j - \kappa du^2 \tag{15}$$

on $\mathbb{R}_+ \times E_4$ for $\kappa = 1$ and $H_{3,1}$ for $\kappa = -1$ are those given by eq. 10. I will refer to these representations of as the Eulerian representations of the FLRW geometries.

Of the three possible spaces only M_{1t} is of finite extension (volume) for each moment of time and thus, the only space that restricts the dilution of the cosmic microwave background radiation. In the spaces M_{0t} and M_{-1t} any background radiation will radiate off very rapidly. Notice also that $S_0 = \mathbb{R}_+ \times E_3 = \mathbb{R}^3$ and

$$\left(g_{0\mu\nu}\left(t, x^i\right)\right) = \begin{pmatrix} c^2 & 0 \\ & -\delta_{ij} \end{pmatrix} \tag{16}$$

i.e. no expansion. For these reasons the rest of my exposition will limit to the spherical case.

3 An Alternative Geometry

Let the space-time be S_1 and the metric on $\mathbb{R}_+ \times E_4$

$$ds^2 = c^2 dt^2 - \delta_{ij} dx^i dx^j - u^2 \tag{17}$$

then the induced metric on S_1 is

$$\left(g_{\mu\nu}\left(t,x^i\right)\right) = \begin{pmatrix} c^2 - \frac{R^2(t)\dot{R}^2(t)}{R^2(t)-r^2\left(x^i\right)} & \frac{R(t)\dot{R}(t)x_i}{R^2(t)-r^2\left(x^i\right)} \\ \frac{R(t)\dot{R}(t)x_j}{R^2(t)-r^2\left(x^i\right)} & g_{ij}\left(t,x^i\right) \end{pmatrix} \tag{18}$$

$$g_{ij}\left(t,x^i\right) = -\delta_{ij} - \frac{1}{R^2(t) - r^2\left(x^i\right)} x_i x_j \tag{19}$$

on the coordinate domain $\{U_t | t \in \mathbb{R}_+\}$. As for the FLRW geometry this is an Eulerian representation.

In the FLRW coordinates defined by Ψ^{-1} (eq. 7), the metric is

$$\left(\tilde{g}_{\mu\nu}\left(t,y^i\right)\right) = \begin{pmatrix} c^2 - \dot{R}^2(t) & 0 \\ 0 & \hat{g}_{ij}\left(t,y^i\right) \end{pmatrix} \tag{20}$$

Thus, in order for the space-time to be semi-Riemannian $\dot{R}^2(t) < c^2$, i.e. the rate of expansion must be smaller than the velocity of light.

For the interval of time P for which we have any empirical information about the evolution of the universe $\dot{R}^2(t) << c^2$, the two models coincide and the metrics become

$$\left(\breve{g}_{\mu\nu}\left(t,x^i\right)\right) = \begin{pmatrix} c^2 & 0 \\ 0 & g_{ij}\left(t,x^i\right) \end{pmatrix} \tag{21}$$

$$g_{ij}\left(t,x^i\right) = -\delta_{ij} - \frac{1}{R^2(t) - r^2\left(x^i\right)} x_i x_j \tag{22}$$

in the Euler representation. In fact, the the line element

$$ds^2 = \left(1 - \frac{\dot{R}^2(t)}{c^2} \frac{R^2(t)}{R^2(t) - r^2\left(x^i\right)}\right) dx^{02} \tag{23}$$

$$+ \quad 2\frac{\dot{R}(t)}{c} \frac{R(t)\, x_i}{R^2(t) - r^2\left(x^i\right)} dx^i dx^0 + g_{ij}\left(t,x^i\right) dx^i dx^j \tag{24}$$

which gives eq. 17.

4 Motions

For the period for which we have empirical information about the universe it is sufficient to consider the model $\left(S, \breve{g}_{\mu\nu}\right)$. The geodesics on this space are solutions of the Lagrange equations for the Lagrange functions

$$L\left(t, x^i, \frac{dt}{ds}, \frac{dx^i}{ds}\right) = \sqrt{\left(\frac{dt}{ds}\right)^2 c^2 + \breve{g}_{ij}\left(t,x^i\right) \frac{dx^i}{ds} \frac{dx^j}{ds}} \tag{25}$$

with the constraint

$$L\left(t,x^i,\frac{dt}{ds},\frac{dx^i}{ds}\right)=\varepsilon \tag{26}$$

where $\varepsilon=0$ for photons and $\varepsilon=c$ for massive objects. s is the proper time. The Lagrange equations are

$$\frac{d^2t}{ds^2}-\frac{1}{2c^2}\frac{d}{ds}\left(\breve{g}_{ij}\left(t,x^i\right)\frac{dx^i}{ds}\frac{dx^j}{ds}\right)=0 \tag{27}$$

$$\frac{d}{ds}\left(\breve{g}_{ij}\left(t,x^i\right)\frac{dx^j}{ds}\right)-\partial_{x^i}\breve{g}_{jk}\left(t,x^i\right)\frac{dx^j}{ds}\frac{dx^k}{ds}=0 \tag{28}$$

It is easy to show that the Lagrangian $L\left(t,x^i,\frac{dt}{ds},\frac{dx^i}{ds}\right)$ is constant along the geodesic, moreover, the equation 22 is the equation for the geodesics on $\{U_t|t\in\mathbb{R}_+\}$.

For $\varepsilon=0$ the projection of a geodesic in space-time is therefore a geodesic on U_t for any t. Therefore, a small section of the geodesic of a light ray is projected to a small section of a great circle, and if a light ray with wave length λ_1 is emitted at time $t_1\in$P span a section of a great circle of radius $R(t_1)$, then at the later time $t_2\in$P the wave length λ_2 of the same light wave span a section on the great circle of radius $R(t_2)$, i.e.

$$\frac{\lambda_1}{\lambda_2}=\frac{R(t_1)}{R(t_2)} \tag{29}$$

which describes the redshift of light for intervals of time in P. Thus, turning the argument we can conclude that the condition $\dot{R}^2(t)<<c^2$ holds good for this period. It is an empirical justification of eq. 16.

For each moment of time the increase of the radius of the sphere determines a normal vector field N_t to the sphere in the englobing space E_4. The projection of N_t at time t onto the tangent space TU_t is

$$n^i\left(t,x^i\right)=\frac{\dot{R}(t)}{R(t)}x^i \tag{30}$$

The contravariant representations of N_t on $T\left(\mathbb{R}_+\times U_t\right)$ is

$$\left(n^\mu\left(t,x^i\right)\right)=\left(1,\frac{\dot{R}(t)}{R(t)}x^i\right) \tag{31}$$

and its covariant representation in the homogeneous coordinates is

$$\left(\hat{n}^\mu\left(t,y^i\right)\right)=\left(\Psi^{-1\mu}_{,\nu}\left(t,y^i\right)\right)^{-1}\left(n^\nu\circ\Psi^{-1}\left(t,y^i\right)\right)=(1,0,0,0) \tag{32}$$

This is in accordance with the fact that $V=\left\{y^i\in\mathbb{R}^3|\delta_{ij}y^iy^j<1\right\}$ is a static space.

On the other hand, the vector field P on the M_{1t} due to the expansion satisfies the geodesic equations 21 and 22 for $\varepsilon=c$. Now, eq. 20 gives

$$\frac{dt}{ds}=\frac{1}{\sqrt{1+\frac{1}{c^2}\breve{g}_{ij}\left(t,x^i\right)\dot{x}^i\dot{x}^j}}=\gamma\left(t,x^i,\dot{x}^i\right) \tag{33}$$

Thus, in terms of the time t the geodesic equations are

$$\frac{1}{\gamma(t,x^i,\dot{x}^i)}\frac{d}{dt}\gamma\left(t,x^i,\dot{x}^i\right)-\frac{1}{2c^2}\partial_t\breve{g}_{ij}\left(t,x^i\right)\dot{x}^i\dot{x}^j=0 \tag{34}$$

$$\frac{1}{\gamma(t,x^i,\dot{x}^i)}\frac{d}{dt}\left(\breve{g}_{ij}\left(t,x^i\right)\gamma\left(t,x^i,\dot{x}^i\right)\dot{x}^j\right)-\partial_{x^i}\breve{g}_{jk}\left(t,x^i\right)\dot{x}^j\dot{x}^k=0 \tag{35}$$

The local representative of the vector field P is $u^i=\frac{\dot{R}}{R}\sqrt{R^2-r^2}\frac{x^i}{r}$, i.e.

$$\dot{x}^i(t)=\frac{\dot{R}(t)}{R(t)}\sqrt{R^2(t)-r^2\left(x^i(t)\right)}\frac{x^i(t)}{r\left(x^i(t)\right)} \tag{36}$$

is a solution of eqs 28. Notice that in this case

$$\gamma(t)=\frac{1}{\sqrt{1-\frac{\dot{R}^2(t)}{c^2}}} \tag{37}$$

Fluid elements must move with the velocity $\dot{x}^i$. Moreover, let

$$\left(v^\mu\left(t,x^i,\dot{x}^i\right)\right)=\left(\gamma(t),\gamma(t)u^i\right) \tag{38}$$

be the contravariant four-velocity field in the Euclidean representation, then its representation $\hat{v}^\mu\left(t,y^i\right)$ in homogeneous coordinates is the four-vector that should enter in the covariant energy momentum tensor for a perfect fluid (eq. 2), i.e. *Hypothesis* 3 is not valid.

5 Final Remarks

The Eulerian representation pictures explicitly the expansion of the universe if we assume that the time and space coordinates $\left(t,x^i\right)$ are measured according to the operational definitions laid down by the Systeme Internationaland used by astronomers. The description inherent in the Eulerian representation, moreover, excludes the flat and hyperbolic cases on empirical grounds since they cannot support the slow dilution of the background radiation, and it contradicts *Hypothesis* 3.

I have presented two alternative metrics for the space-time S_1, being the induced metrics derived from $ds^2=\left(c^2+\dot{R}^2(t)\right)dt^2-\delta_{ij}dx^idx^j-u^2$ and $ds^2=c^2dt^2-\delta_{ij}dx^idx^j-u^2$ on $\mathbb{R}_+\times E_4$, the first of which gives the FLRW geometry. When $\dot{R}^2(t)<<c^2$ the two models coincide. For the period for which we have empirical evidence, i.e. the last 13 billion years, we can therefore at present not distinguish between the two geometries on empirical grounds. It still seems to me that the second choice is the most satisfying since it complies with the theory of special relativity by excluding the possibility for expansion rates exceeding the velocity of light. This choice will, however, exclude the possibility for an inflationary period after a possible Big Bang as well as making it necessary to reconsider the estimate for the age of the universe.

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