Short Communication

Non-Hermitian Operators, Biorthogonal Basis Sets, Normal Operators, Antiunitary Symmetry, and Exceptional Points: Everything in a Small Packet

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A simple 2×2 parameter-dependent matrix is suitable for the illustration of several features of non-Hermitian operators, like biorthogonal basis sets, normal operators, antiunitary symmetry and exceptional points.

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1. Introduction

In the last decades there has been an increasing interest in non-Hermitian operators because they are relevant for the discussion of a great number of physical problems (photonics, mechanics, electrical circuits, acoustics, active matter, unidirectional invisibility, enhanced sensitivity, topological energy transfer, coherent perfect absorption, single-mode lasing, robust biological transport, etc.)[1][2][3] (and references there). The treatment of non-Hermitian eigenvalue equations requires special mathematical tools like, for example, biorthogonal basis sets[3][4]. A particular feature of non-Hermitian equations is the occurrence of exceptional points[3][5][6][7][8][9] that have been the subject of a number of pedagogical papers[10][11][12][13][14].

Normal operators[15] are a particular class of non-Hermitian operators that do not require the use of biorthogonal basis sets. Some non-Hermitian operators exhibit antiunitary symmetry[16] and may have real eigenvalues when this antiunitary symmetry is not broken.

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The purpose of this paper is the discussion of all the features of non-Hermitian operators just mentioned by means of an extremely simple example. In section 2 we outline some mathematical aspects of biorthogonal basis sets. In section 3 we briefly address the particular classes of normal operators and non-Hermitian operators with antiunitary symmetry. In section 4 we show that a simple 2×2 non-Hermitian matrix exhibits all the features of non-Hermitian operators mentioned above. Finally, in section 5 we summarize the main results of the paper and draw conclusions.

2. Biorthogonal basis sets

In this section we outline some of the results put forward by Brody[4] some time ago about the eigenvalues and eigenvectors of a non-Hermitian operator H and its adjoint $H^{\dagger}$:

$$H\left|u_i\right\rangle = E_i\left|u_i\right\rangle, H^{\dagger}\left|v_i\right\rangle = W_i\left|v_i\right\rangle, i = 1, 2, \ldots . \tag{1}$$

For simplicity, we assume that both sets of eigenvectors $\{\left|u_i\right\rangle\}$ and $\{\left|v_i\right\rangle\}$ are complete.

It follows from

$$\left\langle v_i\middle|H\middle|u_j\right\rangle = E_j\left\langle v_i\middle|u_j\right\rangle = \left\langle u_j\middle|H^{\dagger}\middle|v_i\right\rangle^* = \left\langle u_j\middle|W_i\middle|v_i\right\rangle^* = W_i^*\left\langle v_i\middle|u_j\right\rangle, \tag{2}$$

where the asterisk denotes complex conjugation, that

$$\left(E_j - W_i^*\right)\left\langle v_i\middle|u_j\right\rangle = 0. \tag{3}$$

Since $\left|v_i\right\rangle$ cannot be orthogonal to all the eigenvectors $\left|u_j\right\rangle$ (unless $\left|v_i\right\rangle = 0$) then $\left\langle v_i\middle|u_j\right\rangle \neq 0$ for some value of j. We arrange the vectors' labels so that $\left\langle v_i\middle|u_j\right\rangle = \left\langle v_i\middle|u_i\right\rangle\delta_{ij}$. In that case $E_i = W_i^*$.

If we expand an arbitrary vector $\left|\psi\right\rangle$ as

$$\left|\psi\right\rangle = \sum_j c_j\left|u_j\right\rangle, \tag{4}$$

then we have $\left\langle v_i\middle|\psi\right\rangle = c_i\left\langle v_i\middle|u_i\right\rangle$ and

$$\left|\psi\right\rangle = \sum_j \frac{\left|u_j\right\rangle\left\langle v_j\middle|\psi\right\rangle}{\left\langle v_j\middle|u_j\right\rangle}, \tag{5}$$

from which we can formally conclude that

$$\sum_j \frac{\left|u_j\right\rangle\left\langle v_j\right|}{\left\langle v_j\middle|u_j\right\rangle} = \hat{1}, \tag{6}$$

where $\hat{1}$ is the identity operator.

By means of exactly the same argument we can also prove that

$$\sum_{j}\frac{|v_j\rangle\langle u_j|}{\langle u_j|v_j\rangle}=\hat{1}. \tag{7}$$

In general, the basis sets $\{|u_i\rangle\}$ and $\{|v_i\rangle\}$ are not orthogonal and for this reason it is convenient to resort to a biorthogonal set of states as discussed by Brody[4].

3. Particular cases

There are two classes of non-Hermitian operators that are of particular interest which will be discussed in what follows.

3.1. Normal operators

If $\left[H,H^{\dagger}\right]=0$ then H is said to be a normal operator. In this case, $H^{\dagger}|u_j\rangle=E_j^{*}|u_j\rangle$ as follows from[15]

$$\begin{aligned}\left\langle\left(H^{\dagger}-E_j^{*}\right)u_j\middle|H^{\dagger}-E_j^{*}u_j\right\rangle &= \left\langle u_j\middle|(H-E_j)\left(H^{\dagger}-E_j^{*}\right)\middle|u_j\right\rangle \\ &= \left\langle u_j\middle|\left(H^{\dagger}-E_j^{*}\right)(H-E_j)\middle|u_j\right\rangle=0.\end{aligned} \tag{8}$$

The result mentioned above is a direct consequence of the fact that $\langle f|f\rangle=0$ if an only if $|f\rangle=0$.

It follows from

$$\langle u_i|H|u_j\rangle=E_j\langle u_i|u_j\rangle=\left\langle u_j\middle|H^{\dagger}\middle|u_i\right\rangle^{*}=\left\langle u_j|E_i^{*}|u_i\right\rangle^{*}=E_i\langle u_i|u_j\rangle, \tag{9}$$

that $\langle u_i|u_j\rangle=0$ if $E_i\neq E_j$[15].

3.2. Antiunitary symmetry

An antiunitary operator A satisfies[16]

$$\begin{aligned}A\left(a|f\rangle+b|g\rangle\right)&=a^{*}A|f\rangle+b^{*}A|g\rangle, \\ \langle Af|A|g\rangle&=\langle g|f\rangle,\end{aligned} \tag{10}$$

and can be written as $A=UK$, where U is a unitary operator and K is the complex conjugation operator. If $AHA^{-1}=H$ (or $AH=HA$) we say that H exhibits an antiunitary symmetry. It follows from these expressions that,

$$AH|u_j\rangle=AE_j|u_j\rangle=E_j^{*}A|u_j\rangle=HA|u_j\rangle. \tag{11}$$

If $A|u_j\rangle=a_j|u_j\rangle$ then $HA|u_j\rangle=a_jE_j|u_j\rangle$ and equation (11) leads to $E_j=E_j^{*}$. In this case we say that the antiunitary symmetry is unbroken; otherwise, we say that it is broken.

In order to obtain U it is commonly convenient to resort to the obvious expression $UH^{*}U^{\dagger}=H$ or

$$UH^* - HU = 0 \tag{12}$$

3.3. Hellmann-Feynman theorem

Suppose that H depends on a parameter λ. If we differentiate $H\left|u\right\rangle = E\left|u\right\rangle$ with respect to λ we have

$$\frac{dH}{d\lambda}\left|u\right\rangle + H\frac{d}{d\lambda}\left|u\right\rangle = \frac{dE}{d\lambda}\left|u\right\rangle + E\frac{d}{d\lambda}\left|u\right\rangle$$

If $H^\dagger\left|v\right\rangle = E^*\left|v\right\rangle$ then

$$\begin{aligned}
\left\langle v\left|\frac{dH}{d\lambda}\right|u\right\rangle + \left\langle v\left|H\frac{d}{d\lambda}\right|u\right\rangle &= \left\langle v\left|\frac{dH}{d\lambda}\right|u\right\rangle + \left\langle H^\dagger v\left|\frac{d}{d\lambda}\right|u\right\rangle, \\
&= \left\langle v\left|\frac{dH}{d\lambda}\right|u\right\rangle + E\left\langle v\left|\frac{d}{d\lambda}\right|u\right\rangle, \\
&= \frac{dE}{d\lambda}\left\langle v|u\right\rangle + E\left\langle v\left|\frac{d}{d\lambda}\right|u\right\rangle,
\end{aligned} \tag{13}$$

from which it follows the Hellmann-Feynman theorem (HFT)[17][18] for a non-Hermitian operator[19]

$$\frac{dE}{d\lambda} = \frac{\left\langle v\left|\frac{dH}{d\lambda}\right|u\right\rangle}{\left\langle v|u\right\rangle}. \tag{14}$$

4. Example

It is surprising that we can illustrate all the general concepts outlined in sections 2 and 3 by means of the simple 2×2 matrix

$$\mathbf{H} = \begin{pmatrix} 1 & i \\ i & \beta \end{pmatrix}, \tag{15}$$

where β is a real parameter.

4.1. Normal matrix

To begin with, note that $\mathbf{H}$ is normal for $\beta = 1$:

$$\left[\mathbf{H}, \mathbf{H}^\dagger\right] = \begin{pmatrix} 0 & 2i(\beta - 1) \\ 2i(1 - \beta) & 0 \end{pmatrix}. \tag{16}$$

For $\beta = 1$ we have

$$\begin{aligned}
\mathbf{Hu}_i &= E_i\mathbf{u}_i, i = 1, 2, \\
E_1 &= E_2^* = 1 + i, \mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\
\mathbf{H}^\dagger\mathbf{u}_i &= E_i^*\mathbf{u}_i,
\end{aligned} \tag{17}$$

that agree with the general results of subsection 3.1.

Since

$$\mathbf{H}\cdot\mathbf{H}^{\dagger}=\mathbf{H}^{\dagger}\cdot\mathbf{H}=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}, \tag{18}$$

we conclude that $2^{-1/2}\mathbf{H}$ is a unitary operator.

4.2. Antiunitary symmetry

The eigenvalues of the matrix (15)

$$E_{\pm}=\frac{1+\beta\pm\sqrt{(1+\beta)\,(\beta-3)}}{2}, \tag{19}$$

are complex for $-1<\beta<3$ and real otherwise as illustrated in Figure 1.

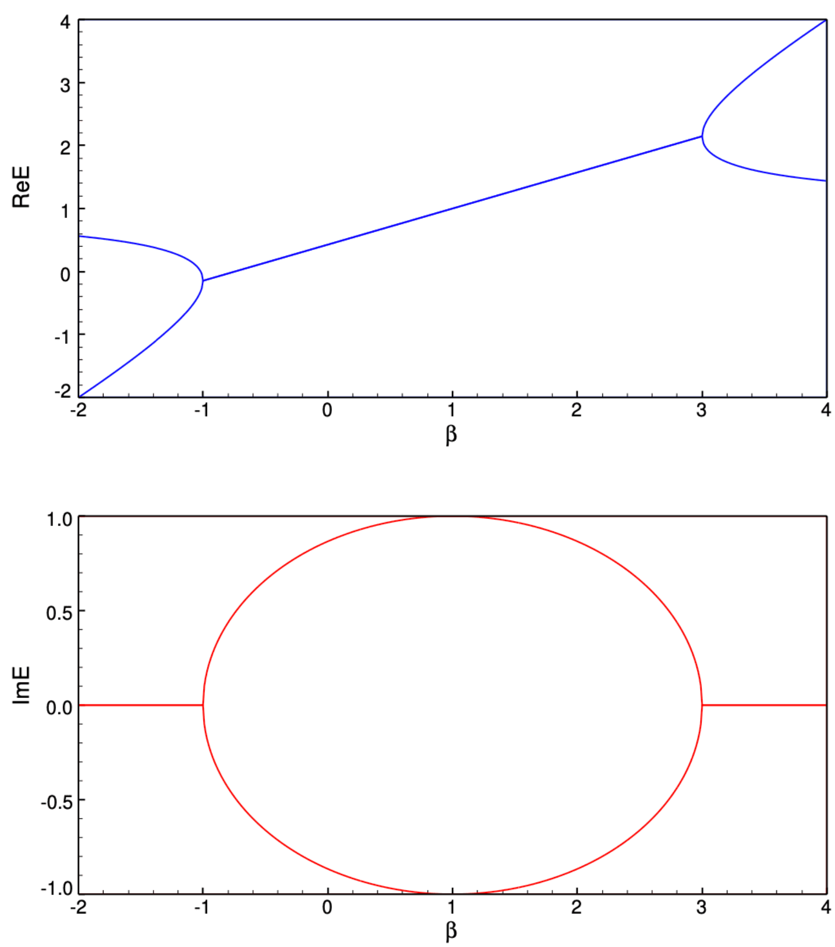

Figure 1. Real and imaginary parts of the eigenvalues of the matrix (15)

The occurrence of real eigenvalues suggests the existence of an antiunitary symmetry. It follows from equation (12) that

$$\mathbf{U}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{20}$$

Clearly, this antiunitary symmetry is broken for all $-1<\beta<3$, where $\beta=-1$ and $\beta=3$ are exceptional points[5][6][7][8][9][10][11][12][13][14]. One can easily verify that there are no solutions to the equations $\mathbf{A}\mathbf{u}_i=\mathbf{U}\mathbf{u}_i^*=a\mathbf{u}_i$ for the vectors $\mathbf{u}_i$ in equation (22). In that case ($\beta=1$) the antiunitary symmetry is broken and the eigenvalues are complex.

As an example of unbroken antiunitary symmetry we choose $\beta=4$ for which we have

$$\begin{aligned}E_1&=\frac{5}{2}-\frac{\sqrt{5}}{2},\mathbf{u}_1=\begin{pmatrix}1\\ \left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)i\end{pmatrix},\\ E_2&=\frac{5}{2}+\frac{\sqrt{5}}{2},\mathbf{u}_2=\begin{pmatrix}1\\ -\left(\frac{\sqrt{5}}{2}+\frac{3}{2}\right)i\end{pmatrix},\end{aligned} \tag{21}$$

and

$$\begin{aligned}W_1&=\frac{5}{2}-\frac{\sqrt{5}}{2},\mathbf{v}_1=\begin{pmatrix}1\\ \left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)i\end{pmatrix},\\ W_2&=\frac{5}{2}+\frac{\sqrt{5}}{2},\mathbf{v}_2=\begin{pmatrix}1\\ \left(\frac{\sqrt{5}}{2}+\frac{3}{2}\right)i\end{pmatrix}.\end{aligned} \tag{22}$$

By means of these expressions we verify that

$$\begin{aligned}\frac{\mathbf{u}_1\cdot\mathbf{v}_1^\dagger}{\mathbf{v}_1^\dagger\cdot\mathbf{u}_1}+\frac{\mathbf{u}_2\cdot\mathbf{v}_2^\dagger}{\mathbf{v}_2^\dagger\cdot\mathbf{u}_2}&=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\\ \frac{\mathbf{v}_1\cdot\mathbf{u}_1^\dagger}{\mathbf{u}_1^\dagger\cdot\mathbf{v}_1}+\frac{\mathbf{v}_2\cdot\mathbf{u}_2^\dagger}{\mathbf{u}_2^\dagger\cdot\mathbf{v}_2}&=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\end{aligned} \tag{23}$$

It is worth noting that the complex numbers $\mathbf{v}_i^\dagger\cdot\mathbf{u}_i$ and $\mathbf{u}_i^\dagger\cdot\mathbf{v}_i$ are equivalent to $\langle v_i|u_i\rangle$ and $\langle u_i|v_i\rangle$, respectively. On the other hand, the 2×2 matrices $\mathbf{u}_i\cdot\mathbf{v}_i^\dagger$ and $\mathbf{v}_i\cdot\mathbf{u}_i^\dagger$ are equivalent to $|u_i\rangle\langle v_i|$ and $|v_i\rangle\langle u_i|$, respectively. In addition to it, the scalar products $\mathbf{u}_1^\dagger\cdot\mathbf{u}_2=\mathbf{v}_1^\dagger\cdot\mathbf{v}_2=2\neq0$ illustrate the fact that the basis sets $\{|u_i\rangle\}$ and $\{|v_i\rangle\}$ are not orthogonal. All these results are in agreement with the general ones shown in section 2. In this case, the eigenvalues are real because the antiunitary symmetry is unbroken as shown by

$$\mathbf{U}\mathbf{u}_i^*=-\mathbf{u}_i,\mathbf{U}\mathbf{v}_i^*=-\mathbf{v}_i,i=1,2. \tag{24}$$

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4.3. Exceptional points

Finally, we analyze one of the exceptional points. For $\beta=-1$ we have

$$\begin{aligned}\mathbf{H}\mathbf{u}_1&=\begin{pmatrix}0\\0\end{pmatrix},\mathbf{u}_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix},\\ \mathbf{H}^\dagger\mathbf{v}_1&=\begin{pmatrix}0\\0\end{pmatrix},\mathbf{v}_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}.\end{aligned}\tag{25}$$

The matrices $\mathbf{H}(-1)$ and $\mathbf{H}^\dagger(-1)$ are defective (non-diagonalizable) as shown by the fact that each one has only one eigenvector. These unique eigenvectors are also eigenvectors of $\mathbf{A}$:

$$\mathbf{U}\mathbf{v}_1^*=-\mathbf{v}_1,\mathbf{U}\mathbf{v}_1^*=-\mathbf{v}_1.\tag{26}$$

On the other hand, the solutions for $\beta=3$ are

$$\begin{aligned}\mathbf{H}\mathbf{u}_1&=2\mathbf{u}_1,\mathbf{u}_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix},\\ \mathbf{H}^\dagger\mathbf{v}_1&=2\mathbf{v}_1,\mathbf{v}_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix},\end{aligned}\tag{27}$$

and the matrix is also defective.

4.4. Hellmann-Feynman theorem

For $\beta=2$ we have

$$\begin{aligned}E_1&=\frac{3}{2}-\frac{\sqrt{3}}{2}i,\mathbf{u}_1=\begin{pmatrix}1\\-\frac{\sqrt{3}}{2}-\frac{i}{2}\end{pmatrix},\\ E_2&=E_1^*,\mathbf{u}_2=\begin{pmatrix}1\\\frac{\sqrt{3}}{2}-\frac{i}{2}\end{pmatrix},\\ W_1&=E_1^*,\mathbf{v}_1=\begin{pmatrix}1\\-\frac{\sqrt{3}}{2}+\frac{i}{2}\end{pmatrix},\\ W_2&=E_1,\mathbf{v}_2=\begin{pmatrix}1\\\frac{\sqrt{3}}{2}+\frac{i}{2}\end{pmatrix},\end{aligned}\tag{28}$$

which come from the arbitrary choice $E_1(\beta)=E_-(\beta)$ and $E_2(\beta)=E_+(\beta)$. One can easily verify that

$$\lim_{\beta\to 2}\frac{dE_i}{d\beta}=\frac{\mathbf{v}_i^\dagger\frac{d\mathbf{H}}{d\beta}\mathbf{u}_i}{\mathbf{v}_i^\dagger\mathbf{u}_i},\tag{29}$$

in agreement with equation (14).

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5. Conclusions

The purpose of this paper is a pedagogical discussion of mathematical concepts like biorthogonal basis sets, normal operators, antiunitary symmetry and exceptional points. The main contribution is a parameter-dependent 2×2 matrix that exhibits all the features just mentioned as the parameter is varied. It is remarkable that such a simple model illustrates so many mathematical concepts associated to non-Hermitian operators.

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