Review Article

Identifying a Pair of Skew Lines Using Scalar Triple Product: A Short Review

Yongsik Jang[1], **Emmanuelle Romano**[2], **Natanael Karjanto**[1]

1. Department of Mathematics, University College, Sungkyunkwan University, Seoul, South Korea; 2. Department of Mathematics and Informatics, Chiba University, Japan

This article explores a pair of skew lines in three-dimensional (3D) space using the scalar triple product (STP), which is a combination of vector operations dot and cross products acting on three vectors in 3D space. Using STP, we formulate practical theorems not only to determine whether two distinct lines in 3D space are skew but also to calculate the distance between a pair of skew lines. Based on the main results, we provide a comparison as evidence to show that our findings in this paper align with the existing literature on related topics. Several examples of skew lines in daily life and potential applications related to rectilinear dynamics are also presented.

Correspondence: papers@team.qeios.com — Qeios will forward to the authors

1. Introduction

Although two lines in a two-dimensional (2D) plane can either coincide, parallel, or intersect, two lines in a three-dimensional (3D) space can also be skew, in addition to the three possible cases found in the plane geometry. In 3D geometry, *skew lines* are a pair of lines that do not intersect and are not parallel to each other. They are also non-coplanar because the lines do not lie in the same plane [1][2][3].

Skew lines can be found in many real-life situations. For example, a line that is formed by a floor and a wall and another line that is formed by a ceiling and a different wall that is perpendicular to the first wall, say, the one with a window in an office room, form a pair of skew lines. Geometrically, this situation can be represented as a pair of lines through adjacent or opposite edges of a regular tetrahedron.

Another set of examples can be found outside houses or office buildings. Consider our modern cities, where they feature different types of roads, such as highways and overpasses. They are considered to be on different levels on the earth's surface. Notice that two lines on different levels of a structure, like the lines painted on a road on an overpass and the lines on the road below, form a set of skew lines, where they are neither parallel nor intersecting because they exist on different planes. Figure 1 shows additional examples of skew lines found in everyday life.

Given any arbitrary two lines in $\mathbb{R}^3$ in their parametric forms, determining whether they are skew is straightforward. First, by comparing whether their direction vectors are parallel and, second, by finding parameter values in an overdetermined system. If they do not match, then the lines do not intersect. This approach can be found in standard textbooks in multivariable calculus [4][5][6].

Although the topic of lines and planes in 3D space is usually presented in conjunction with vectors and their operations, most textbooks, with notable exceptions like Boas' book [7], rarely utilize vector operations to identify whether two lines are skew or to calculate the shortest distance between them. In this article, we approach the problems using a vector operation known as the *scalar triple product* (STP), sometimes also referred to as *triple scalar product* (TSP), that is, the dot product of a vector with the cross product of two other vectors [8][9].

STP is a standard vector operation and has numerous well-established applications in mathematics, physics, and engineering, such as testing the linear independence and coplanarity of three vectors, to calculate the volume of a parallelepiped, to determine the flux integral of a vector field across the parametrically defined surface, and to calculate the magnitude of the torque produced by a force at an angle acting on a lever arm. However, there is relatively little work in formalizing theorems that utilize STP as a structured tool to verify the skewness of two lines in 3D space.

This article fills the literature gap by formulating a practical theorem to determine whether a pair of lines in 3D space is skew. Some of the references in the literature were mentioned to verify our results. Furthermore, the proposed theorem considers a more general setup for the conditions of such lines in 3D space.

Figure 1. Some examples of skew lines in real life. From the top left clockwise direction (color online): The lines pass through two adjacent lateral hexagonal sides from different longitudinal sides of a bolt. The yellow median lines of the bridge road on the top and the highway below it. The green lines on the white band run across the top of the tennis net and the white center line divides the tennis court beneath it. Two red lines in an office space, one in the middle of the window side and the other on the top of the wall near the ceiling. Figure courtesy to SplashLearn, Pexels Kindel Media, Pexels Sami Abdullah, and Pexels Pixabay, respectively.

Although some dictionaries mention that the first known use of the term ”skew lines” was in the mid-20th century [10][11], a body of published literature suggests that the phrase appeared much earlier. For example, English mathematician Thomas T. Tate defined skew lines in his geometry monograph in 1860 [12]. An engineer from Philadelphia, Joseph M. Wilson, mentioned the term once in 1877 when describing the construction of the Pennsylvania railroad that intersected Belmont Avenue in Philadelphia, United States [13].

The use of the phrase during the first half of the 20th century is also abundant in the literature on solid geometry, both in textbooks and research articles. In their first monograph on projective geometry ever written in English, Veblen and Young (1910) discussed the properties of skew lines in multiple places in

doi.org/10.32388/7A0J4T

their books [14]. Hobbs (1921), Hawkes et al. (1922), Haertter (1930), and Frame (1948) provided a definition of skew lines in their monographs [15][16][17][18].

In particular, Hawkes et al. (1922) demonstrated the construction of a line that is perpendicular to a pair of skew lines [16] and Hobbs (1921) expanded further by including some theorems related to skew lines, such as "if two skew lines are perpendicular to each other, then a plane can be drawn through either line perpendicular to the other", "through either of two skew lines, one and only one plane parallel to the other can be drawn", "between two skew lines, one and only one common perpendicular can be drawn" [15].

However, an explanation of finding the distance between a pair of skew lines is found in an engineering drawing handbook, albeit in the absence of the distance formula [19]. Similarly, without providing any formulas, a mathematics dictionary from the 1940s provided a descriptive definition for skew lines and the distance between them [20]. Furthermore, although the notion can be challenging for many students, Davidson and Pressland (1926) and Green (1941) thoroughly cover the definition and calculation of the angle between two skew lines [21][22].

Hedrick and Ingold (1914) use skew lines as a tool to construct other geometric concepts in 3D space, such as pencils and doublets [2]. At a more advanced level, Rao (1920) uses skew lines as a fundamental element in constructing and understanding imaginary elements in geometry, particularly their connection to imaginary lines and quadric surfaces associated with them [23]. As one possible application of vector concepts in analytic geometry, Byrne (1935) derived a formula for the perpendicular distance between two skew lines where it involves direction angles [24]. Although the "skew lines" terminology was not explicitly mentioned, Bell (1938) provided the condition where two lines are coplanar and a formula for calculating the shortest distance where two lines are not coplanar [25].

The use of the term "skew lines" during the second half of the 20th century and the first quarter of the 21st century is abundant. Exhausting all of them in this rather short review article is neither constructive nor necessary, as many of them are written at an advanced level for specialists in geometry. However, some notable work is accessible to undergraduate students and secondary school mathematics teachers.

For example, building upon the treatise from Bell (1938), Clarke (1951) algebraically verified the uniqueness of the shortest distance between a pair of skew lines, which is also their common perpendicular [26]. Using the minimization of a two-variable function, Givens (1970) also derived a necessary condition for the shortest distance between a pair of skew lines [27]. In solid geometry, Smith and Henderson (1985) argued that the existence of a tetrahedron depends on the presence of skew lines, demonstrating that its volume remains constant whenever segments of fixed lengths are moved along a pair of skew lines [28].

Beyond a pair of skew lines, Viro and Viro (2006) attempted to answer the question of how skew lines can be arranged in 3D space, writing in the form of an introduction that is accessible to advanced high school students. Analogously to arranging a bunch of perfectly straight and infinitely long uncooked spaghetti noodles, the paper explores whether different arrangements of these lines are truly distinct, or if they can all be transformed into one another through continuous movements (isotopies) without touching or becoming parallel [29].

We end this rather long introduction by presenting the organization of this article. The preliminary section, i.e., Section 2, introduces conventions for notations and some necessary definitions. Section 3 follows by presenting the main result of this article, i.e., a theorem that serves as a systematic criterion to determine whether a given pair of lines in 3D space is skew and a formula to calculate their distance. Section 4 provides some potential applications beyond geometry and pure mathematics, such as in maritime navigation and skew bridge construction. Finally, Section 5 concludes our discussion.

2. Preliminaries

In this section, we provide several definitions and theorems related to a pair of skew lines and STP in 3D space.[1]

Definition 2.1. (Plane sharing). Let $A\left(\mathbb{R}^3\right)$ be the set of all planes in $\mathbb{R}^3$. For any two lines ℓ_1 and ℓ_2 in $\mathbb{R}^3$, we say that ℓ_1 and ℓ_2 **share a plane** if and if only if there exists a plane $\alpha \in A\left(\mathbb{R}^3\right)$ such that both ℓ_1 and ℓ_2 lie on α. The relationship is denoted $\ell_1 \sim \ell_2$.

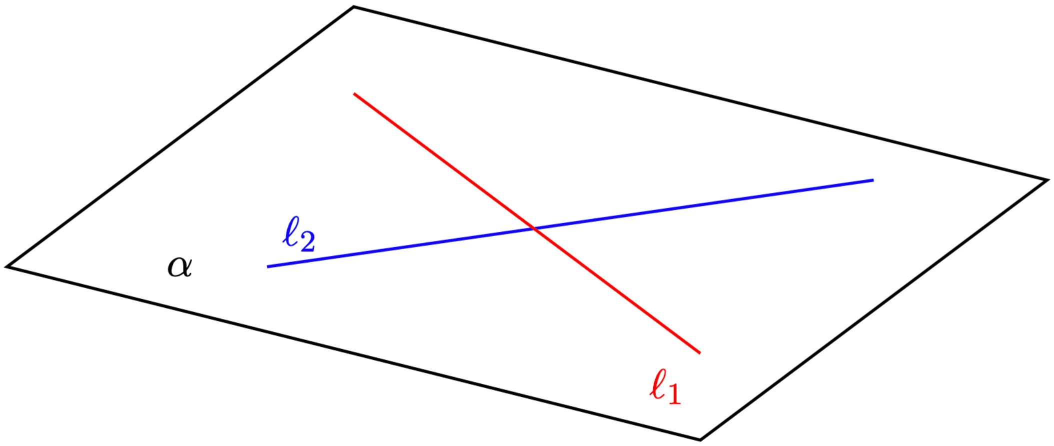

Figure 2. Two lines ℓ_1 and ℓ_2 lying on a plane α.

Definition 2.2. (Skew lines). Two lines ℓ_1 and ℓ_2 in $\mathbb{R}^3$ are **skew lines** if and only if ℓ_1 and ℓ_2 are neither parallel nor intersecting. We denote this relationship by $\ell_1 \nsim \ell_2$.[2]

Note that the binary relation $\sim$ satisfies the properties of an equivalence relation while $\nsim$ does not. Furthermore, for any two lines ℓ_1 and ℓ_2, exactly one of the conditions $\ell_1 \sim \ell_2$ or $\ell_1 \nsim \ell_2$ always holds.

Definition 2.3. (Dot/scalar product). Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$. The **dot/scalar product** of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u} \cdot \vec{v}$, is defined as

$$\vec{u} \cdot \vec{v} := |\vec{u}|\,|\vec{v}| \cos\alpha, \tag{1}$$

where α is the angle between $\vec{u}$ and $\vec{v}$.

Alternatively, the dot product is given by

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{2}$$

We can verify that the definition of the dot product given by equation (1) is equivalent to equation (2).

Definition 2.4. (Cross product). Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$. The **cross product** of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u} \times \vec{v}$, is a vector perpendicular to both $\vec{u}$ and $\vec{v}$, defined as [30]

$$\vec{u} \times \vec{v} := |\vec{u}|\,|\vec{v}| \sin(\alpha)\hat{\mathbf{n}}, \tag{3}$$

where α is the angle between $\vec{u}$ and $\vec{v}$, and $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$, following the right-hand rule.

Alternatively, in component form, the cross product is given by

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1). \tag{4}$$

We can verify that the definition of the cross product given by equation (3) is equivalent to equation (4). Based on this definition, we obtain the following proposition:

Corollary 2.5. *Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$. From Definition 2.4, the magnitude of the cross product $\vec{u} \times \vec{v}$ gives the **area** of the parallelogram $\mathcal{A}$ spanned by $\vec{u}$ and $\vec{v}$:*

$$|\vec{u} \times \vec{v}| = \text{Area}(\mathcal{A}) = |\vec{u}|\,|\vec{v}| \sin\alpha,$$

where α is the angle between $\vec{u}$ and $\vec{v}$. In particular, when $\vec{u}$ and $\vec{v}$ are linearly dependent, the area of the parallelogram is zero, i.e., $\text{Area}(\mathcal{A}) = 0$.

Definition 2.6. (Determinant). Let $\vec{a} = (a_1, a_2), \vec{b} = (b_1, b_2)$ be vectors in $\mathbb{R}^2$, and $\vec{u} = (u_1, u_2, u_3), \vec{v} = (v_1, v_2, v_3), \vec{w} = (w_1, w_2, w_3)$ be vectors in $\mathbb{R}^3$. The **determinant** of 2×2 or 3×3 matrices formed by these vectors is defined as follows:

$$\det \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} = \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} := a_1 b_2 - a_2 b_1,$$

and

$$\det \begin{bmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{bmatrix} = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} := u_1 \det \begin{bmatrix} v_2 & v_3 \\ w_2 & w_3 \end{bmatrix} - u_2 \det \begin{bmatrix} v_1 & v_3 \\ w_1 & w_3 \end{bmatrix} + u_3 \det \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}.$$

These determinants are related to the area (in $\mathbb{R}^2$) and the volume (in $\mathbb{R}^3$) by representing the signed area and volume of a **parallelogram** and **parallelepiped**, respectively, formed by the vectors that form rows (or columns) of the matrix, as discussed in [31]. Consequently, we obtain the following theorem:

Theorem 2.7. *Let $\vec{a}, \vec{b}, \vec{u}, \vec{v}, \vec{w}$ be the vectors defined in Definition 2.6. Then, the area $\mathcal{A}$ and volume $\mathcal{V}$ spanned by these vectors are given by their corresponding determinants:*

$$\text{Area}(\mathcal{A}) = \left| \det \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} \right| \quad and \quad \text{Volume}(\mathcal{V}) = \left| \det \begin{bmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{bmatrix} \right|,$$

respectively, where the absolute value ensures that the measure is non-negative. Specifically, the area $\mathcal{A}$ and volume $\mathcal{V}$ are described by the following sets:

$$\mathcal{A} = \left\{ x\vec{a} + y\vec{b} \mid (x, y) \in [0, 1]^2 \right\},$$

*which represents the **parallelogram** spanned by $\vec{a}$ and $\vec{b}$, and*

doi.org/10.32388/7A0J4T

$$\mathcal{V} = \left\{ x\vec{u} + y\vec{v} + z\vec{w} \mid (x, y, z) \in [0, 1]^3 \right\},$$

which represents the ***parallelepiped*** *spanned by $\vec{u}$, $\vec{v}$, and $\vec{w}$, respectively.*

Definition 2.8. (Scalar triple product, STP). Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^3$. The **scalar triple product** $T\left[\vec{u}, \vec{v}; \vec{w}\right]$ is defined as

$$T\left[\vec{u}, \vec{v}; \vec{w}\right] := \left(\vec{u} \times \vec{v}\right) \cdot \vec{w},$$

that is, the scalar product of the cross product of $\vec{u}$ and $\vec{v}$ with $\vec{w}$.

Ostermann et al. [31] also note that the STP of given vectors is equivalent to the volume of the parallelepiped spanned by those vectors.

Corollary 2.9. *Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$. Then,*

$$T\left[\vec{u}, \vec{v}; \vec{w}\right] = \mathrm{Volume}(\mathcal{V}) = \left| \det \begin{bmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{bmatrix} \right|,$$

here, $\mathcal{V}$ represents the parallelepiped spanned by the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$.

3. Verifying skew lines and calculating their distance

In this section, we verify using STP whether a pair of lines in $\mathbb{R}^3$ is skew and provide a formula to calculate their shortest distance.

Theorem 3.1. (Skew Lines Test). *Suppose there are two lines ℓ_1 and ℓ_2 in $\mathbb{R}^3$. Let $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ be directional vectors of ℓ_1 and ℓ_2, respectively. Let P_1 be a point on ℓ_1 and P_2 be a point on ℓ_2, then*

$$T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right] = 0 \Leftrightarrow \ell_1 \sim \ell_2, \tag{5}$$

and

$$T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right] \neq 0 \Leftrightarrow \ell_1 \nsim \ell_2. \tag{6}$$

Proof. We start by proving equation (5). First, assume $\ell_1 \sim \ell_2$.

Consider two cases: ℓ_1 and ℓ_2 are parallel to each other, and ℓ_1 and ℓ_2 are not parallel.

Case 1.

- ℓ_1 and ℓ_2 are parallel. Clearly, we have
$T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right] = \left(\overrightarrow{u_1} \times \overrightarrow{u_2}\right) \cdot \overrightarrow{P_1P_2} = \mathbf{0} \cdot \overrightarrow{P_1P_2} = 0.$

 doi.org/10.32388/7A0J4T

Case 2.

- ℓ_1 and ℓ_2 are not parallel.

  By Definition 2.1, there exists a plane $\alpha \in A\left(\mathbb{R}^3\right)$ containing both lines ℓ_1 and ℓ_2.

  If $P_1 = P_2$, then $\overrightarrow{P_1P_2} = \mathbf{0}$, which directly yields

  $T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right] = 0.$

  Thus, we shall let $P_1 \neq P_2$. Observe that $\overrightarrow{u_1} \times \overrightarrow{u_2}$ is perpendicular to α, and since $\overrightarrow{P_1P_2}$ lies on α, it follows that $\overrightarrow{u_1} \times \overrightarrow{u_2}$ is also perpendicular to $\overrightarrow{P_1P_2}$. Hence,

  $T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right] = 0.$

To prove the converse, assume $T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right] = 0$. Then,

$$0 = \left(\overrightarrow{u_1} \times \overrightarrow{u_2}\right) \cdot \overrightarrow{P_1P_2} = \overrightarrow{u_1} \cdot \left(\overrightarrow{u_2} \times \overrightarrow{P_1P_2}\right). \tag{7}$$

Let $\zeta \in A\left(\mathbb{R}^3\right)$ be the plane containing $\overrightarrow{u_2}$ and $\overrightarrow{P_1P_2}$. Since $\overrightarrow{u_2} \times \overrightarrow{P_1P_2}$ is perpendicular to ζ and $\overrightarrow{u_1}$ is perpendicular to $\overrightarrow{u_2} \times \overrightarrow{P_1P_2}$ by equation (7), it follows that $\overrightarrow{u_1}$ lies in ζ. Therefore, $\overrightarrow{u_1}, \overrightarrow{u_2}$, and $\overrightarrow{P_1P_2}$ lie on the same plane, implying $\ell_1 \sim \ell_2$.

Next, we are going to prove equation (6). Assume $\ell_1 \nsim \ell_2$. Let Q_1 and Q_2 be points on line ℓ_1 and ℓ_2, respectively such that vector $\overrightarrow{Q_1Q_2}$ is perpendicular to both $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$. Thus, there exist real numbers s, t such that

$$\overrightarrow{P_1P_2} = \overrightarrow{Q_1Q_2} + s\overrightarrow{u_1} + t\overrightarrow{u_2}.$$

Therefore,

$$\begin{aligned} T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right] &= T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{Q_1Q_2} + s\overrightarrow{u_1} + t\overrightarrow{u_2}\right] && (8)\\ &= \left(\overrightarrow{u_1} \times \overrightarrow{u_2}\right) \cdot \left(\overrightarrow{Q_1Q_2} + s\overrightarrow{u_1} + t\overrightarrow{u_2}\right) && (9)\\ &= \left(\overrightarrow{u_1} \times \overrightarrow{u_2}\right) \cdot \overrightarrow{Q_1Q_2}. && (10) \end{aligned}$$

The equation (10) comes from the fact that $\left(\overrightarrow{u_1} \times \overrightarrow{u_2}\right) \cdot \overrightarrow{u_1} = \left(\overrightarrow{u_1} \times \overrightarrow{u_2}\right) \cdot \overrightarrow{u_2} = 0$. Notice that both $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ are both nonzero vectors and are not parallel to each other. Therefore, $\overrightarrow{u_1} \times \overrightarrow{u_2}$ is also a nonzero vector. Moreover, both $\overrightarrow{Q_1Q_2}$ and $\overrightarrow{u_1} \times \overrightarrow{u_2}$ are parallel to each other; this results in:

$$T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right] \neq 0.$$

To prove the converse, assume $T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right] \neq 0$. In order to prove $\ell_1 \nsim \ell_2$, by Definition 2.2 it is sufficient to show that ℓ_1 and ℓ_2 are not parallel to each other and do not intersect each other.

First, assume $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ are parallel. Then there exists $c \in \mathbb{R}$ such that $\overrightarrow{u_2} = c\overrightarrow{u_1}$. Then,

$$\begin{aligned} T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right] &= T\left[\overrightarrow{u_1}, c\overrightarrow{u_1}; \overrightarrow{P_1P_2}\right] \\ &= \left(\overrightarrow{u_1} \times c\overrightarrow{u_1}\right) \cdot \overrightarrow{P_1P_2} \\ &= \mathbf{0} \cdot \overrightarrow{P_1P_2} = 0, \end{aligned}$$

which contradicts our initial assumption. Thus, $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ are not parallel, yielding ℓ_1 and ℓ_2 not parallel to each other.

Next, assume ℓ_1 and ℓ_2 intersect each other. Then, there exist $s, t \in \mathbb{R}$ such that

$$\overrightarrow{OP}1 + s\overrightarrow{u_1} = \overrightarrow{OP_2} + t\overrightarrow{u_2} \tag{11}$$

$$\overrightarrow{P_1P_2} = s\overrightarrow{u_1} - t\overrightarrow{u_2}. \tag{12}$$

Substituting equation (12) into the triple scalar product yields

$$\begin{aligned} T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right] &= T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; s\overrightarrow{u_1} - t\overrightarrow{u_2}\right] \\ &= \left(\overrightarrow{u_1} \times \overrightarrow{u_2}\right) \cdot \left(s\overrightarrow{u_1} - t\overrightarrow{u_2}\right) \\ &= s\overrightarrow{u_1} \cdot \left(\overrightarrow{u_1} \times \overrightarrow{u_2}\right) - t\overrightarrow{u_2} \cdot \left(\overrightarrow{u_1} \times \overrightarrow{u_2}\right) \\ &= 0. \end{aligned}$$

The final result 0 comes from the fact that the cross product of a vector with itself is the zero vector $\mathbf{0}$, and the dot product of any vector with the zero vector is 0. Thus, we have

$$T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right] = 0,$$

which contradicts our initial assumption. Therefore, ℓ_1 and ℓ_2 do not intersect each other.

Hence, we conclude $\ell_1 \nsim \ell_2$. □

Observe that equation (5) provides a stronger statement than the one formulated by Gellert et al. [32]. Moreover, our proof does not rely on the formula for the distance between two skew lines, whereas Gellert et al.'s proof explicitly uses it.

Theorem 3.2. (Distance between two skew lines). *Suppose $\ell_1 \nsim \ell_2$. Let $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ be directional vectors of ℓ_1 and ℓ_2, respectively. Let P_1 and P_2 be any points on ℓ_1 and ℓ_2, respectively. Then, the shortest distance between ℓ_1 and ℓ_2, denoted by $d(\ell_1, \ell_2)$, is given by:*

 doi.org/10.32388/7A0J4T

$$d(\ell_1, \ell_2) = \frac{\left| T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right] \right|}{\left|\overrightarrow{u_1} \times \overrightarrow{u_2}\right|}.$$

Proof. Let Q_1 and Q_2 be points on lines ℓ_1 and ℓ_2, respectively, such that vector $\overrightarrow{Q_1Q_2}$ is perpendicular to both $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$. $d(\ell_1, \ell_2)$ is nothing other than $\left|\overrightarrow{Q_1Q_2}\right|$ (Figure 3). Since $\overrightarrow{u_1} \times \overrightarrow{u_2}$ and $\overrightarrow{Q_1Q_2}$ are parallel, we have:

$$\left(\overrightarrow{u_1} \times \overrightarrow{u_2}\right) \cdot \overrightarrow{Q_1Q_2} = \pm\left|\overrightarrow{u_1} \times \overrightarrow{u_2}\right| \left|\overrightarrow{Q_1Q_2}\right|. \tag{13}$$

Thus, by equation (10) and (13):

$$\left| T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right] \right| = \left|\left(\overrightarrow{u_1} \times \overrightarrow{u_2}\right) \cdot \overrightarrow{Q_1Q_2}\right| = \left|\overrightarrow{u_1} \times \overrightarrow{u_2}\right| \left|\overrightarrow{Q_1Q_2}\right|. \tag{14}$$

Finally, equation (14) yields

$$\left|\overrightarrow{Q_1Q_2}\right| = \frac{\left| T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right] \right|}{\left|\overrightarrow{u_1} \times \overrightarrow{u_2}\right|},$$

as desired. □

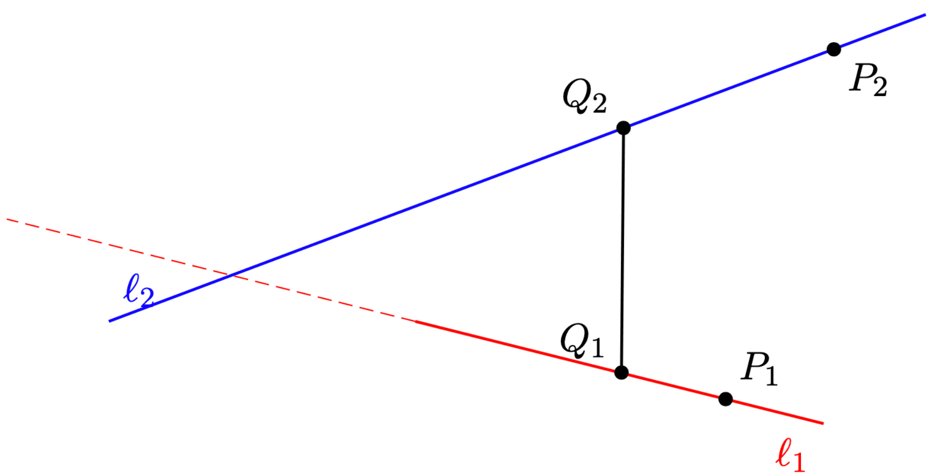

Figure 3. The shortest distance between two skew lines ℓ_1 and ℓ_2 is $|\overrightarrow{Q_1Q_2}|$

Corollary 2.9 allows us to compute the shortest distance between ℓ_1 and ℓ_2 via the determinant of a matrix:

doi.org/10.32388/7A0J4T

Proposition 3.3. *Let ℓ_1, ℓ_2, $\overrightarrow{u_1}$, $\overrightarrow{u_2}$, P_1, and P_2 be as defined in Theorem 3.2. Then, the shortest distance between ℓ_1 and ℓ_2 is given by*

$$d\left(\ell_1, \ell_2\right) = \frac{1}{\left|\overrightarrow{u_1} \times \overrightarrow{u_2}\right|}\left|\det\begin{bmatrix} \overrightarrow{u_1} \\ \overrightarrow{u_2} \\ \overrightarrow{P_1P_2} \end{bmatrix}\right|.$$

Ostermann et al. [31] provide an interesting perspective in calculating the distance between two skew lines using the concept of the volume of a parallelepiped. Consider the lines ℓ_1 and ℓ_2 with directional vectors $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$, respectively. Let P_1 and P_2 be points on the lines ℓ_1 and ℓ_2, respectively. By Theorem (2.7), the volume $\mathcal{V}$ of the parallelepiped spanned by the vectors $\overrightarrow{u_1}, \overrightarrow{u_2}$, and $\overrightarrow{P_1P_2}$ is given by

$$\text{Volume}(\mathcal{V}) = \left|\det\begin{bmatrix} \overrightarrow{u_1} \\ \overrightarrow{u_2} \\ \overrightarrow{P_1P_2} \end{bmatrix}\right|. \tag{15}$$

Moreover, according to Corollary 2.5, the base of this parallelepiped has an area $\mathcal{A}$ of $\left|\overrightarrow{u_1} \times \overrightarrow{u_2}\right|$, as shown in the right diagram of Figure 4. Therefore, comparing equation (15) with the formula for the volume of a parallelepiped, i.e., $\text{Volume}(\mathcal{V}) = \text{Area}(\mathcal{A}) \cdot h$, we obtain the following expression:

$$h = \frac{1}{\left|\overrightarrow{u_1} \times \overrightarrow{u_2}\right|}\left|\det\begin{bmatrix} \overrightarrow{u_1} \\ \overrightarrow{u_2} \\ \overrightarrow{P_1P_2} \end{bmatrix}\right|.$$

This height h is exactly the shortest distance $d\left(\ell_1, \ell_2\right)$ between the skew lines, and thus the preceding proposition follows. From the preceding theorem, we obtain the following corollary.

Corollary 3.4. *Let $\ell_1, \ell_2, \overrightarrow{u_1}, \overrightarrow{u_2}, P_1$, and P_2 be as defined in Theorem 3.2. The shortest distance between ℓ_1 and ℓ_2 can be expressed as*

$$d\left(\ell_1, \ell_2\right) = \left|\hat{\mathbf{n}} \cdot \overrightarrow{P_1P_2}\right|,$$

where $\hat{\mathbf{n}} = \frac{\overrightarrow{u_1} \times \overrightarrow{u_2}}{\left|\overrightarrow{u_1} \times \overrightarrow{u_2}\right|}$ is the unit vector of $\overrightarrow{u_1} \times \overrightarrow{u_2}$.

doi.org/10.32388/7A0J4T

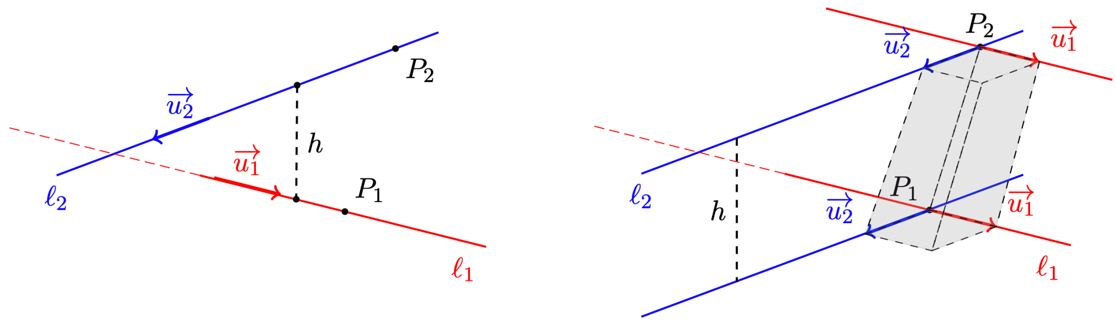

Figure 4. The projections of ℓ_1 and ℓ_2 onto P_1 and P_2, respectively, from the left diagram form a parallelepiped spanned by the directional vectors $\overrightarrow{u_1}$, $\overrightarrow{u_2}$, and $\overrightarrow{P_1P_2}$ as shown in the right diagram.

Observe that this expression agrees with the one formulated in [32][33] even though we do not provide an explicit expression for the line equations in parametric form. Furthermore, Boas [7] specifically mentioned the notion behind selecting arbitrary points P_1 and P_2 in her work and computing the dot product of $\overrightarrow{P_1P_2}$ with its normal vector $\hat{\mathbf{n}}$. Figure 3 shows that the length of $\overrightarrow{Q_1Q_2}$ is the shortest distance between two skew lines in 3D space and the vector $\overrightarrow{Q_1Q_2}$ is perpendicular to both lines ℓ_1 and ℓ_2 [32].

4. Application

In the context of navigating submarines under the sea, if two submarines move in different directions (e.g., not parallel) without colliding with each other, their journey lines can be considered a pair of skew lines in 3D space. The Skew Lines Test (Theorem 3.1) can be deployed to determine whether the paths of two submarines will collide by expressing each journey line as a vector equation. We can check whether the journey lines that represent the submarines' rectilinear dynamics intersect at some point despite operating in the same area.

Furthermore, given the non-colliding journey lines of the two submarines, the shortest distance between these two journey lines can be calculated using the distance formula between two skew lines (Theorem 3.2). Having this knowledge is particularly useful in route planning to avoid journey lines that are too close to each other or any potential hazards.

However, because the Earth is round, the journey lines of a submarine over long distances should ideally be mapped onto a spherical surface, which limits the applicability of this theorem in real-world navigation. This shortcoming arises because the theorem assumes a 3D space, whereas the actual

doi.org/10.32388/7A0J4T

navigational space is affected by the Earth's curvature. This issue becomes significant only for sufficiently long journey paths, as noted in some studies, e.g., [34]. For short journey paths, however, the effect of the Earth's curvature is negligible due to the relatively small size of submarines compared to the vast scale of the Earth. In such cases, we can consider the journey lines as straight lines and apply the Skew Line Test.

A similar geometric framework applies in civil engineering, particularly in the design of skew bridges. Gregory [35] discussed the case of two bridges forming a pair of skew lines with different heights, which is a relatively straightforward scenario when the two bridges cross each other at an orthogonal angle (90°). However, complexity arises when the bridges intersect at a nonperpendicular angle, i.e., either an acute or obtuse angle. The application of Theorem 3.1 can be implemented in the following manner.

Figure 5. An example of skew bridge construction adopted by Gregory [35] near Gospel Oak station, a ward adjacent to Kentish Town in the London Borough of Camden, United Kingdom. The railway tracks heading in the northeast and southeast directions are the Suffragette (heading to Barking) and Mildmay (heading to Stratford) Overgrounds, respectively. The road beneath them is Gordon House Road. Figure courtesy of *Google Maps*.

Let ℓ_1 be a straight line with directional vector $\overrightarrow{u_1}$. For any curve ℓ_2', we define the directional vector based on the tangent at arbitrarily chosen points, forming a sequence of directional vectors $\left\{\overrightarrow{u_{2,\lambda}}\right\}_{\lambda\in\Lambda}$, where Λ represents a set of arbitrary points in ℓ_2', as shown in the diagram depicted in Figure 6. In particular, we use the notation ℓ_2' instead of the standard ℓ_2 to emphasize that it is not a straight line. Note that Figure 6 illustrates the sequence of directional vectors $\left\{\overrightarrow{u_{2,\lambda}}\right\}_{\lambda\in\Lambda}$ for $\Lambda = \{1,2,3,4,5,6,7\}$.

By construction, all directional vectors $\overrightarrow{u_{2,\lambda}}$ of ℓ_2' represent the direction of a straight line $\ell_{2,\lambda}$ that is locally tangent to ℓ_2' at the given point. Since $\ell_{2,\lambda} \nsim \ell_1$ for all $\lambda \in \Lambda$, it follows that for any pair of points P_1 and P_2 on ℓ_1 and ℓ_2', respectively, Theorem 3.1 guarantees

$$T\left[\overrightarrow{u_1}, \overrightarrow{u_{2,\lambda}}; \overrightarrow{P_1P_2}\right] \neq 0, \forall\lambda \in \Lambda.$$

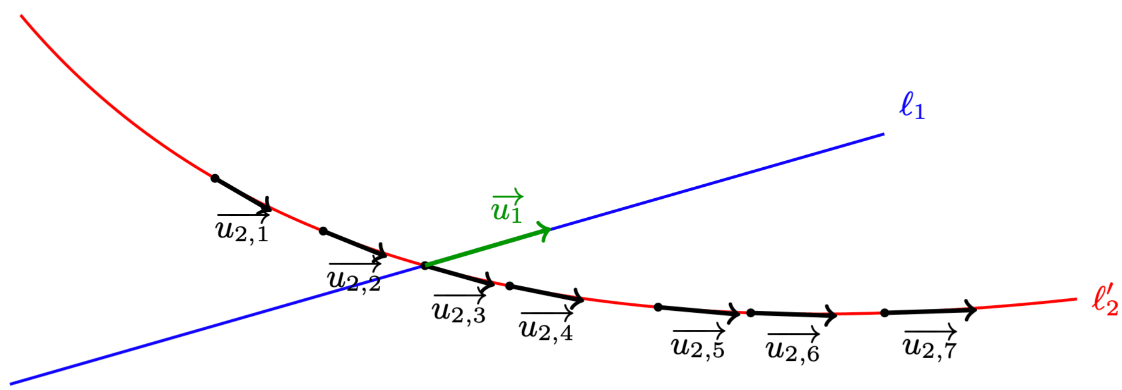

Figure 6. An illustration of an aerial view of two non-orthogonal skew bridges represented by line, curve, and vectors based on Figure 5. The blue line and red curve corresponds to Gordon House Road and Suffragette line, respectively.

Furthermore, by Theorem 3.2, we can deduce that the distance between ℓ_1 and ℓ_2' is given by

$$d\left(\ell_1, \ell_{2,\lambda}\right) = \frac{\left|T\left[\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{P_1P_2}\right]\right|}{\left|\overrightarrow{u_1} \times \overrightarrow{u_{2,\lambda}}\right|}. \tag{16}$$

This formula holds for any choice of $\lambda \in \Lambda$, which means that the distance formula (16) remains invariant regardless of which directional tangent vector is chosen.

Among the arbitrary points in ℓ_2', there exists a particular $\lambda' \in \Lambda$ such that the directional vector $\overrightarrow{u_{2,\lambda'}}$ forms an angle α with $\overrightarrow{u_1}$, satisfying

$$\cos\alpha = \frac{\overrightarrow{u_1} \cdot \overrightarrow{u_{2,\lambda'}}}{\left|\overrightarrow{u_1}\right|\left|\overrightarrow{u_{2,\lambda'}}\right|}.$$

In particular, the second subscript in $\overrightarrow{u_2}$ corresponds to $\lambda' = 3$ in Figure 6.

 doi.org/10.32388/7A0J4T

5. Conclusion

This article reviews some characteristics of skew lines in 3D space, discusses how to verify their skewness, and presents a formula to calculate the shortest distance between them. Using STP, we present an old subject in solid geometry in connection to basic vector operations of the scalar and vector products. This systematic criterion for determining whether a given pair of lines is skew is convenient to implement and simplifies the calculation process.

Although this method is effective for lines in three-dimensional space, it faces some limitations when applied to real-world navigation systems, where the Earth's curvature must be considered. Incorporating Earth's curvature into the model poses a significant difficulty, particularly for long-distance navigation. For short-distance navigation, the curvature effect can be ignored because of the relatively small scale of submarines compared to the Earth's size. In such cases, the Skew Lines Theorem can still be effectively applied as if the journey paths were straight. Alternatively, if we do not want to assume the paths are straight, the skew bridge configuration provides an example of its application under such conditions.

This presents an opportunity for future work to extend the current theorem to accommodate the Earth's spherical nature. One potential direction for development is to adapt the theory from geodesic lines, which represent the shortest paths between two points on a sphere [36].

Footnotes

[1] To maintain clarity, we will use the notation $\mathbb{R}^3$ when referring specifically to three-dimensional Euclidean space within definitions, theorems, propositions, and corollaries. In broader discussions, we will use the term "3D space." We trust that the readers will find this distinction helpful in understanding the intended level of formality and mathematical precision within each context.

[2] For brevity, when a pair of lines ℓ_1 and ℓ_2 are established as skew, we may sometimes use "skew" as an adjective (e.g., "the skew configuration"). In such cases, it should be clear from the context that the previously defined binary relation for skew lines is applied.

References

1. ^ *Altshiller-Court, N. (1979). Modern Pure Solid Geometry. New York, US: Chelsea Publishing.*
2. a, b *Adler, A. A. (1912). The Theory of Engineering Drawing. New York, US: D. van Nostrand Company.*

 doi.org/10.32388/7A0J4T

3. ^Anton, H., Bivens, I. C., and Davis, S. (2021). Calculus: Early Transcendentals, 12th edition. Hoboken, New Jersey, US: Wiley.
4. ^Bell, R. J. T. (1938). Coordinate Solid Geometry. London, England, United Kingdom: Macmillan and Co.
5. ^Boas, M. L. (2006). Mathematical Methods in the Physical Sciences (3rd ed.). New York, US: Wiley.
6. ^Byrne, W. E. (1935). Vector analysis and analytic geometry. National Mathematics Magazine 10(2): 44–52.
7. a, bClarke, L. E. (1951). The shortest distance between two skew lines. The Mathematical Gazette 35(312): 120–121.
8. ^Davidson, J. and Pressland A. J. (1926). A Second Geometry. Oxford, England, UK: Oxford University Press.
9. ^https://www.dictionary.com/browse/skew-lines
10. ^Frame, J. S. (1948). Solid Geometry. United Kingdom: McGraw-Hill Book Company.
11. ^Gellert, W., Gottwald, S., Hellwich, M., Kästner, H., and Küstner, H. (2012). The VNR Concise Encyclopedia of Mathematics, Second edition. New York, US: Springer Science & Business Media. https://doi.org/10.1007/978-94-011-6982-0
12. ^Givens, C. (1970). An exercise in vector identities. Mathematics Magazine 43(3): 153–154.
13. ^Green, S. L. (1942). Algebraic Solid Geometry. Cambridge, England, UK: Cambridge University Press.
14. ^Gregory, R. (2011). The art of skew bridges: The technique and its history explored. Published online: 04 Nov 2011.
15. a, bJames, G., and James, R. C. (1943). Mathematics Dictionary. Van Nuys, California, US: The Digest Press.
16. a, bHaertter, L. D. (1930). Solid Geometry. United Kingdom: Century Company.
17. ^Hawkes, H. E., Luby, W. A., Touton, F. C. (1922). Solid Geometry. Boston, Massachusetts, US: Ginn and Company.
18. ^Hedrick, E. R., and Ingold, L. (1914). A set of axioms for line geometry. Transactions of the American Mathematical Society 15(2): 205–214.
19. ^Hobbs, C. A. (1921). Solid Geometry. Cambridge, Massachusetts, US: G. H. Kent.
20. ^Karney, C. F. F. (2013). Algorithms for geodesics. Journal of Geodesy 87: 43–55. https://doi.org/10.1007/s00190-012-0578-z
21. ^Kreyszig, E., Kreyszig, H., & Norminton, E. J. (2011). Advanced Engineering Mathematics (10th ed.). Wiley.
22. ^Larson, R., Boswell, L., Kanold, T. D., and Stiff, L. (2012). Geometry. Boston, Massachusetts, US: McDougal Littell/Houghton Mifflin.
23. ^Marsden, J. and Tromba, A. (2011). Vector Calculus, Sixth edition. New York, US: W. H. Freeman & Company.

24. ^Matthews, P. C. (1998). Vector Calculus. Berlin Heidelberg, Germany: Springer.

25. ^"Skew lines." Merriam-Webster.com Dictionary, Merriam-Webster, https://www.merriam-webster.com/dictionary/skew%20lines. Accessed 21 February 2025.

26. ^Ostermann, A., & Wanner, G. (2012). Geometry by Its History. Springer. https://doi.org/10.1007/978-3-642-29163-0

27. ^Rao, C. V. H. (1920). Imaginaries in geometry, and their interpretation in terms of real elements. The Mathematical Gazette 10(148): 129–133.

28. ^Robinson, G. D. B. (2011). Vector Geometry. Mineola, New York, US: Dover Publication.

29. ^Smith, J., and Henderson, M. (1985). Tetrahedra, skew lines, and volume. The College Mathematics Journal 16(2): 138–140.

30. ^Stewart, J., Clegg, D., and Watson, S. (2020). Calculus: Early Transcendentals, Ninth edition. Boston, Massachusetts, US: Cengage Learning.

31. a, b, cTate, T. T. (1860). Practial Geometry. London, England, UK: Longman, Green, Longman, and Roberts.

32. a, b, cVeblen, O., and Young, J. W. (1910). Projective Geometry. Boston, Massachusetts, US: Ginn and Company.

33. ^Viro, J., & Viro, O. (2006). Configurations of skew lines. arXiv preprint, arXiv:math/0611374 [math.GT]. https://doi.org/10.48550/arXiv.math/0611374

34. ^Wilson, J. M. (1877). The Pennsylvania railroad; Belmont Avenue Bridge, Philadelphia. In Maw, W. H. and Dredge, J. (Eds.) Engineering: An Illustrated Weekly Journal 23(1): 262–264.

35. a, bZhang, S., Yang, Y., Xu, T., Qin, X., & Liu, Y. (2024). Long-range LBL underwater acoustic navigation considering Earth curvature and Doppler effect. Measurement, 115524. https://doi.org/10.1016/j.measurement.2024.115524.

36. ^Zill, D. G., and Wright, W. S. (2011). Calculus: Early Transcendentals, Fourth edition. Burlington, Massachusetts, US: Jones & Bartlett Learning.

Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.