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Research Article

Patterns of Squares Around an Arbitrary Triangle

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H. Ebisui and J.C.G. Notrott studied patterns of squares around respectively a right and an arbitrary triangle, thus generalizing the Pythagorean theorem. We construct a new pattern of squares around an arbitrary triangle, based on the four squares theorem, using simple vector constructions to avoid trigonometric calculations. This gives rise to some known number sequences, with new applications on a geometric pattern of squares.

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Generalizations of the Pythagorean Theorem

The Pythagorean theorem for a right-angled triangle is the most well-known theorem. There are many generalizations, such as Japanese mathematician H. Ebisui's 'Pythagorean fivefold theorem' (see [1]). J.C.G. Notrott had published an even more general result earlier (see [2] and [3]), but he did so in the magazine Pythagoras, in Dutch, so one can assume both discoveries were made independently.

Ebisui considered a right-angled triangle ΔABC (in red on Fig. 1) with right angle in C and squares on the sides $a_1,\ b_1,\ c_1$. The Pythagorean theorem states that $c_1^2 = a_1^2 + b_1^2$. If a ring of (blue) squares with sides $A_1,\ B_1\ ,\ C_1$ is built on the convex hull of these squares, then $5\,C_1^2 = A_1^2 + B_1^2$. Another ring of squares with sides $a_2,\, b_2,\, c_2$ on the convex hull of these squares will again satisfy the Pythagorean theorem, and the next one with sides $A_2, B_2\,, C_2$ will again satisfy $5\,C_2^2 = A_2^2 + B_2^2$. And so on.

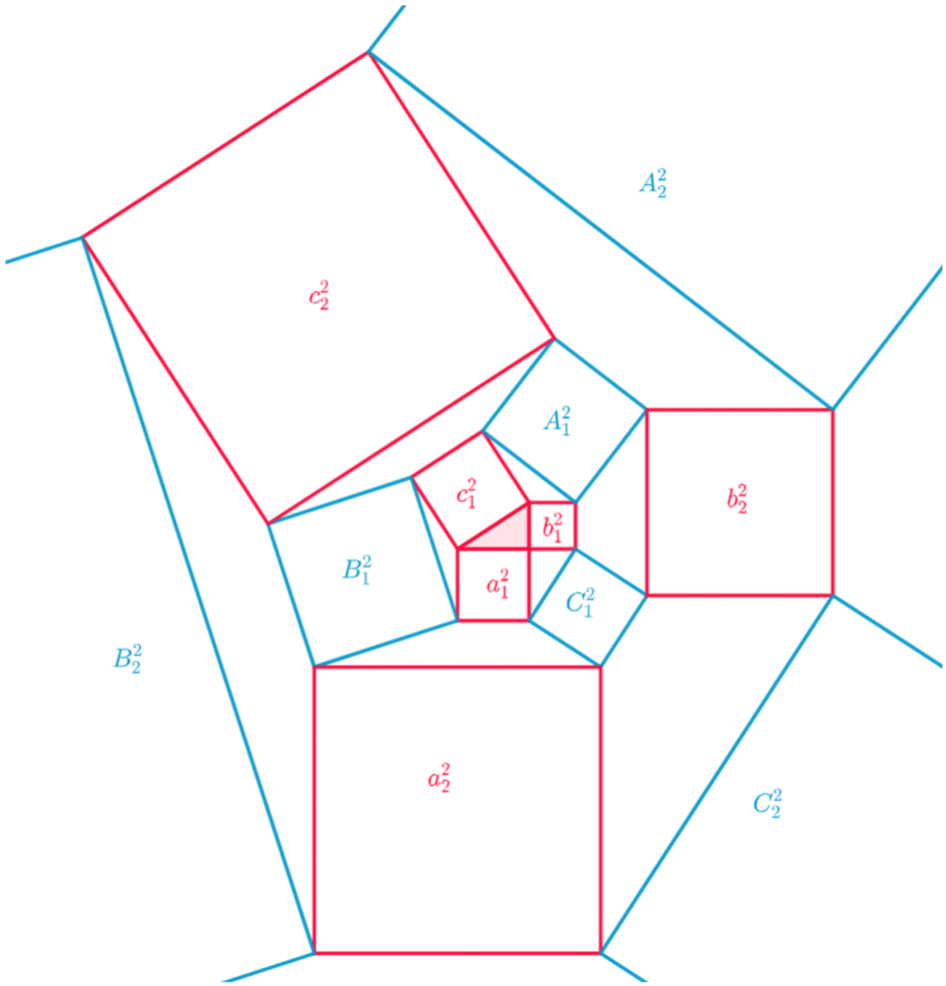

Figure 1. A right-angled triangle (shaded red) and several rings of squares around it, alternately satisfying the Pythagorean theorem $c_i^2 = a_i^2 + b_i^2$ (red squares) or the Pythagorean fivefold theorem $5C_i^2 = A_i^2 + B_i^2$ (blue squares)

Notrott considered an arbitrary triangle and constructed squares around it the same way (see Fig. 2). Using similar notations, the sums of the areas of the squares of the consecutive rings are:

$A_1^2 + B_1^2 + C_1^2 = 3\left(a_1^2 + b_1^2 + c_1^2\right),$

$a_2^2 + b_2^2 + c_2^2 = 16\ \left(a_1^2 + b_1^2 + c_1^2\right),$

$A_2^2 + B_2^2 + C_2^2 = 75\ \left(a_1^2 + b_1^2 + c_1^2\right), \ldots$

which gives rise to the sequence 1, 3, 16, 75, 361, 1728, 8281, ... (A005386 in the Online Encyclopedia of Integer Sequences). Of course, for a right triangle, where $C_1^2 = c_1^2 = a_1^2 + b_1^2$, Notrott's result becomes Ebisui's 'fivefold theorem'.

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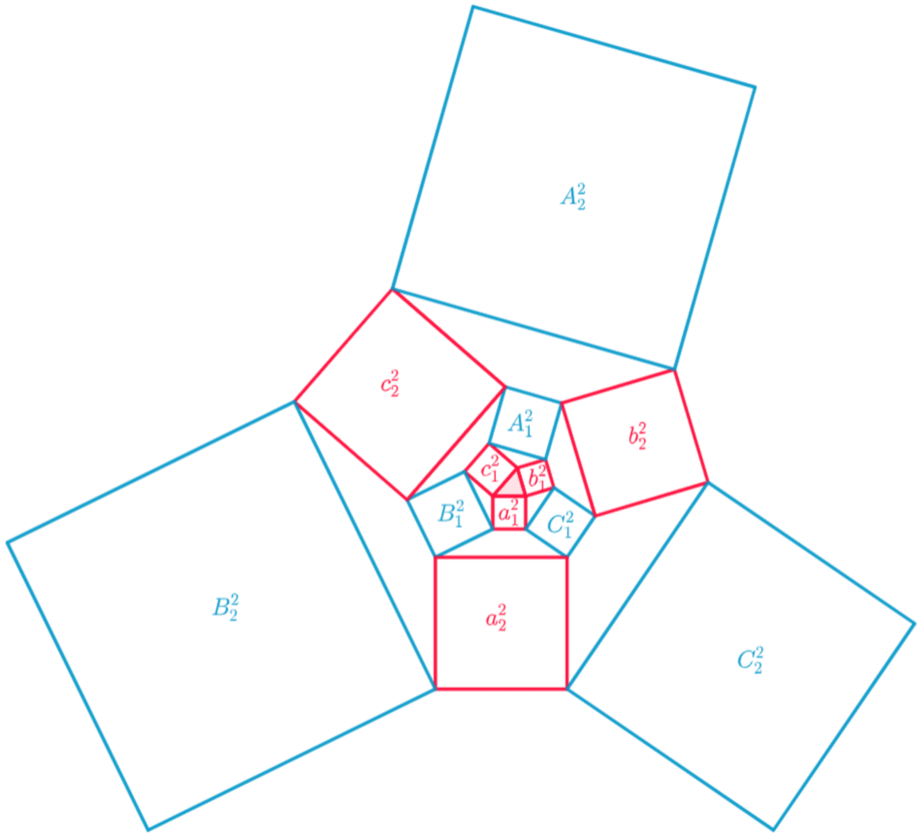

Figure 2. An arbitrary triangle and several rings of squares around it, for which the sum of the areas can be expressed in terms of the sum of areas of the first row by sequence A0005386 of the OEIS

A proof of an even more general theorem for arbitrary triangles was given by Long Huynh Huu (see [1]). He shows that any linear relation that holds between the areas of the squares of the first ring will also be valid for the areas of the squares of the third, fifth, ... ring. And any linear relation that holds for the areas of the squares of the second ring, will also hold for those of the fourth, sixth, ... ring.

To a Pattern of Four Squares around an Arbitrary Triangle

The four squares theorem

There are still other patterns to be discovered in the squares around an arbitrary triangle $\Delta\ ABC$. Let's start with four (red) squares instead of three (see Fig. 4). Denote $\overrightarrow{CB} = \vec{a}$, $\overrightarrow{AC} = \vec{b}$, $\overrightarrow{BA} = \vec{c}$, $\overrightarrow{DG} = \vec{d}$ and the vectors obtained by rotating them 90° clockwise respectively by $\vec{a'}$, $\vec{b'}$, $\vec{c'}$ and $\vec{d'}$. Note this implies that $\left(\vec{a'}\right)' = -\vec{a}$. These notations allow the use of simple vector constructions and avoid trigonometric calculations as used in [4].

Thus, $\vec{c} = -\left(\vec{a} + \vec{b}\right)$, $\vec{d} = \vec{b'} - \vec{a'}$ and $\vec{c'} = -\left(\vec{a'} + \vec{b'}\right)$, $\vec{d'} = \vec{a} - \vec{b}$. Moreover, since the vectors $\vec{a}$ and $\vec{a'}$ have the same length, as well as the vectors $\vec{b}$ and $\vec{b'}$, and the angle between $\vec{a}$ and $\vec{b}$ is the same as the angle between $\vec{a'}$ and $\vec{b'}$, we have that $\vec{a'} \bullet \vec{b'} = \vec{a} \bullet \vec{b}$. From

$$\begin{aligned} \vec{c} \bullet \vec{c'} &= 0 \\ &= \left(\vec{a} + \vec{b}\right) \bullet \left(\vec{a'} + \vec{b'}\right) = \vec{a} \bullet \vec{a'} + \vec{a} \bullet \vec{b'} + \bar{b} \\ &\qquad \bullet \vec{a'} + \vec{b} \bullet \vec{b'} \\ &= \vec{a} \bullet \vec{b'} + \vec{b} \bullet \vec{a'}, \end{aligned}$$

it follows that $\vec{a} \bullet \vec{b'} = -\vec{b} \bullet \vec{a'}$.

The areas of the squares with sides c and d are:

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$c^2 = \vec{c} \bullet \vec{c} = \left(\vec{a} + \vec{b}\right) \bullet \left(\vec{a} + \vec{b}\right) = a^2 + 2\,\vec{a} \bullet \vec{b}$ and $+ b^2$

$d^2 = \vec{d} \bullet \vec{d} = \left(\vec{b'} - \vec{a'}\right) \bullet \left(\vec{b'} - \vec{a'}\right) = b'^2 - 2\,\vec{a'} \bullet \vec{b'} + a'^2 = a^2 - 2\,\vec{a} \bullet \vec{b} + b^2,$

so that: $c^2 + d^2 = 2(a^2 + b^2)$.

Hence, "the sum of the areas of the squares on the non-equal sides of two triangles with two equal sides and supplementary enclosed angles is double the sum of the areas of the squares on the equal sides". This result is known as the four squares theorem.

In Fig. 3, we illustrate the theorem. The triangles ΔABC and ΔBDG have two corresponding sides equal, e.g. $|AB| = |BD|$ and $|BC| = |BG|$ while the enclosed angles in B are supplementary. The theorem states that the sum of the areas of the blue squares is double the sum of the areas of the red squares.

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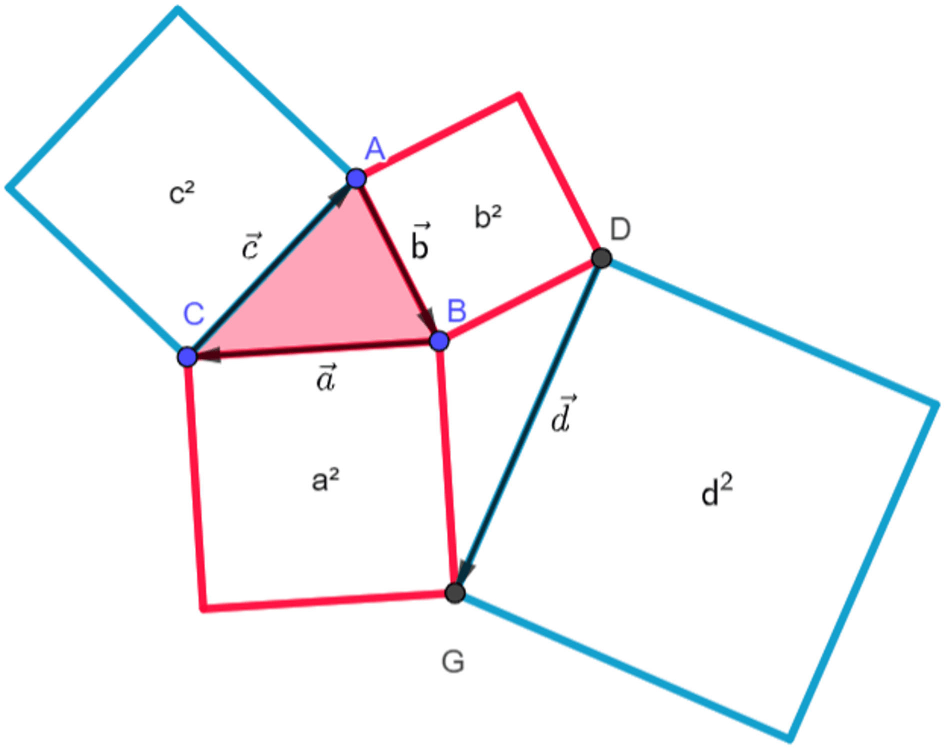

Figure 3. Illustration of the four squares theorem, the sum of the areas of the blue squares is double the sum of the areas of the red squares.

Extending the pattern of the four squares theorem to the next ring of squares

We extend the pattern of the four squares theorem as follows (see Figure 4). We construct four new squares by connecting vertices of the previous set of squares. Then:

$$\overrightarrow{a_1} = \overrightarrow{b'} - \overrightarrow{c'} = \overrightarrow{a'} + 2\,\overrightarrow{b'}$$

$$\overrightarrow{b_1} = \overrightarrow{d'} - \overrightarrow{b} = \left(\overrightarrow{a} - \overrightarrow{b}\right) - \overrightarrow{b} = \overrightarrow{a} - 2\,\overrightarrow{b},$$

$$\overrightarrow{c_1} = \overrightarrow{c'} - \overrightarrow{a'} = -2\,\overrightarrow{a'} - \overrightarrow{b'},$$

$$\overrightarrow{d_1} = -\overrightarrow{a} - \overrightarrow{d'} = -\overrightarrow{a} - \left(\overrightarrow{a} - \overrightarrow{b}\right) = -2\,\overrightarrow{a} + \overrightarrow{b},$$

so that:

$$\overrightarrow{a'_1} = -\overrightarrow{a} - 2\,\overrightarrow{b},$$

$$\overrightarrow{b'_1} = = \overrightarrow{a'} - 2\,\overrightarrow{b'},$$

$$\overrightarrow{c'_1} = 2\,\overrightarrow{a} + \overrightarrow{b},$$

$$\overrightarrow{d'_1} = -2\,\overrightarrow{a'} + \overrightarrow{b'}.$$

Using these expressions, the sums of the areas of opposite (green) squares are equal since:

$$\begin{aligned} a_1^2 + d_1^2 &= \left(\overrightarrow{a'} + 2\,\overrightarrow{b'}\right) \bullet \left(\overrightarrow{a'} + 2\,\overrightarrow{b'}\right) + \left(\overrightarrow{b} - 2\,\overrightarrow{a}\right) \bullet \left(\overrightarrow{b} - 2\,\overrightarrow{a}\right) \\ &= a'^2 + 4\,b'^2 + 4\,\overrightarrow{a'} \bullet \overrightarrow{b'} + 4\,a^2 + b^2 - 4\,\overrightarrow{a} \bullet \overrightarrow{b} = \\ &\quad + 5\,b^2 \end{aligned}$$

and

$$\begin{aligned} b_1^2 + c_1^2 &= \left(\overrightarrow{a} - 2\,\overrightarrow{b}\right) \bullet \left(\overrightarrow{a} - 2\,\overrightarrow{b}\right) + \left(-2\,\overrightarrow{a'} - \overrightarrow{b'}\right) \bullet \left(-2\,\overrightarrow{a'} - \overrightarrow{b'}\right) \\ &= a^2 + 4\,b^2 - 4\,\overrightarrow{a} \bullet \overrightarrow{b} + b'^2 + 4\,a'^2 + 4\,\overrightarrow{a'} \bullet \overrightarrow{b'} = \\ &\quad + 5\,b^2 \end{aligned}$$

Hence, $a_1^2 + d_1^2 = b_1^2 + c_1^2$.

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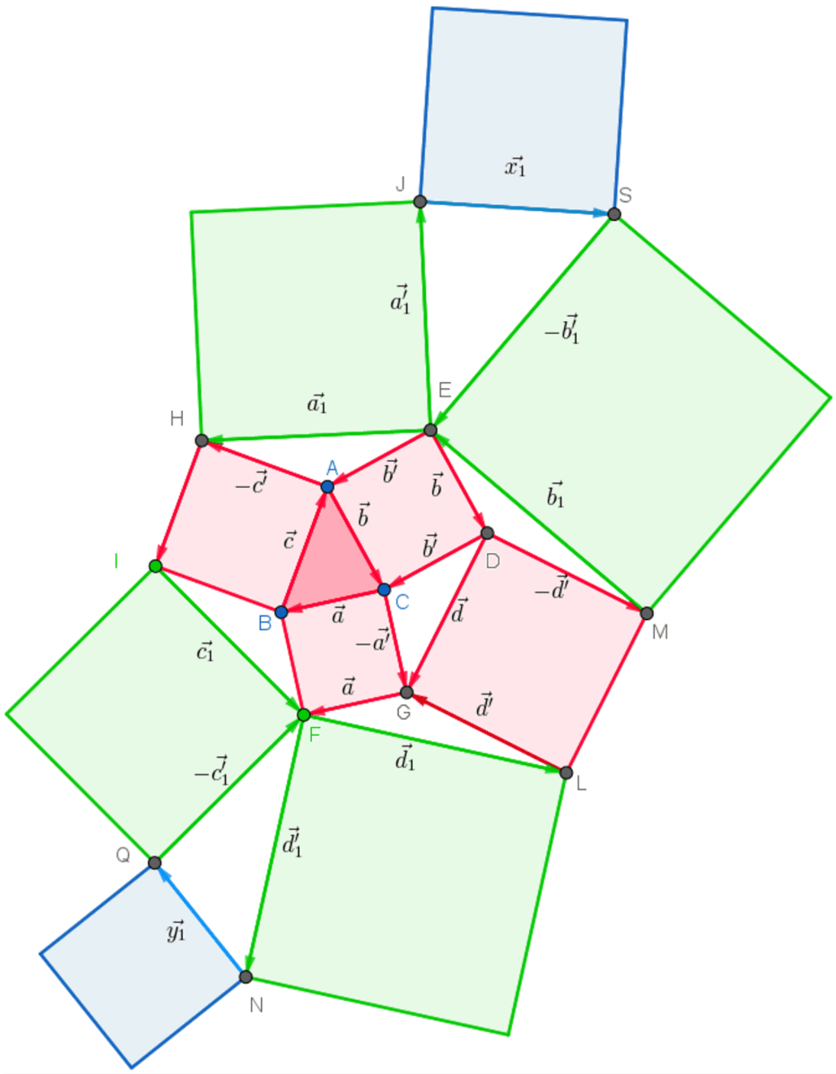

Figure 4. A new pattern of four squares around an arbitrary triangle

The set of the auxiliary squares

Before building the next set of four squares, we first create two auxiliary squares (the blue ones in Figure 4) on the vertices of the squares with sides a_1 and b_1, and on those with sides c_1 and d_1. The vectors associated with its sides can be expressed as:

$$\vec{x}_1 = \overrightarrow{b'_1} - \overrightarrow{a'_1} = \vec{a} + \vec{a'} + 2\,\vec{b} - 2\,\vec{b'} \quad \text{and}$$
$$\vec{y}_1 = \overrightarrow{c'_1} - \overrightarrow{d'_1} = 2\,\vec{a} + 2\,\vec{a'} + \vec{b} - \vec{b'}.$$

Consequently,

$$x_1^2 = \left(\vec{a} + \vec{a'} + 2\,\vec{b} - 2\,\vec{b'}\right) \bullet \left(\vec{a} + \vec{a'} + 2\,\vec{b} - 2\,\vec{b'}\right)$$
$$= \left(\vec{a} + \vec{a'}\right) \bullet \left(\vec{a} + \vec{a'}\right) + 4\,\left(\vec{b} - \vec{b'}\right) \bullet \left(\vec{b} - \vec{b'}\right)$$
$$+ 4\,\left(\vec{a} + \vec{a'}\right) \bullet \left(\vec{b} - \vec{b'}\right)$$
$$= 2\,a^2 + 8\,b^2$$
$$+ 4\,\left(\vec{a} \bullet \vec{b} + \vec{a'} \bullet \vec{b} - \vec{a} \bullet \vec{b'} - \vec{a'} \bullet \vec{b'}\right)$$
$$= 2\,a^2 + 8\,b^2 + 8\,\vec{a'} \bullet \vec{b},$$

and similarly, $y_1^2 = 8\,a^2 + 2\,b^2 + 8\,\vec{a'} \bullet \vec{b}$.

The last part of these expressions can be reformulated in terms of the area $A_{\Delta ABC}$ of the original triangle ΔABC. Note that the (smallest) angle between the vectors $\vec{a'}$ and $\vec{b}$ and the (smallest) angle between the vectors $\vec{a}$ and $\vec{b}$ sum up to

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270° in case of an acute triangle or differ by 90° in case of an obtuse triangle so that:

$\vec{a'} \bullet \vec{b} = ab \cos$, where we $\left(\vec{a'}, \vec{b}\right) = -\,a\,b\,\sin\left(\vec{a}, -\vec{b}\right) = -2\,A_{\Delta ABC}$ denote by $\cos\left(\vec{a'}, \vec{b}\right)$ the cosine of the angle between $\vec{a'}$ and $\vec{b}$ and by $\sin\left(\vec{a}, -\vec{b}\right)$ the sine of the angle between $\vec{a}$ and $-\vec{b}$.

It follows that:

$x_1^2 = 2\,a^2 + 8\,b^2 - 16\,A_{\Delta ABC}$,

$y_1^2 = 8\,a^2 + 2\,b^2 - 16\,A_{\Delta ABC}$,

and so $x_1^2 + y_1^2 = 10\left(a^2 + b^2\right) - 32A_{\Delta ABC}$.

Further extension of the pattern

The next ring of squares is constructed with a vertex of a square of the previous one and a vertex of an auxiliary square. And this procedure can be continued (see Fig. 5). Labelling the vectors associated with the blue auxiliary squares as $\vec{x}_i$ and $\vec{y}_i$, and those associated with the other squares as $\vec{a}_i$, $\vec{b}_i$, $\vec{c}_i$ and $\vec{d}_i$, we find that:

$$\vec{a_2} = \vec{a}_1 + \vec{x'}_1 = 2\,\vec{a'} - \vec{a} + 4\,\vec{b'} + 2\,\vec{b}$$

$$\vec{b_2} = \vec{b}_1 - \vec{x'}_1 = 2\,\vec{a} - \vec{a'} - 4\,\vec{b} - 2\,\vec{b'}$$

$$\vec{x_2} = -\vec{a'}_2 + \vec{x}_1 + \vec{b'}_2 = 4\,\vec{a'} + 4\vec{a} - 8\,\vec{b'} + 8\,\vec{b} = 4\,\vec{x_1}$$

Hence, $x_2^2 = 16\,x_1^2$ and

$$\vec{b'}_2 - \vec{a'}_2 = \vec{b'}_1 - \vec{a'}_1 + 2\,\vec{x}_1 = 3\,\vec{x}_1$$

Similarly,

$$\vec{c_2} = 2\,\vec{a} - 4\vec{a'} - \vec{b} - 2\,\vec{b'}$$

$$\vec{d_2} = 2\,\vec{a'} - 4\,\vec{a} + \vec{b'} + 2\,\vec{b}$$

$$\vec{y_2} = 8\,\vec{a'} + 8\,\vec{a} - 4\,\vec{b'} + 4\,\vec{b} = 4\,\vec{y_1}$$

Calculating the sum of the areas of opposite squares, yields:

$$a_2^2 + d_2^2 = b_2^2 + c_2^2 = 25\left(a^2 + b^2\right) - 64A_{\Delta ABC}$$

In the next paragraph, we will generalize this pattern.

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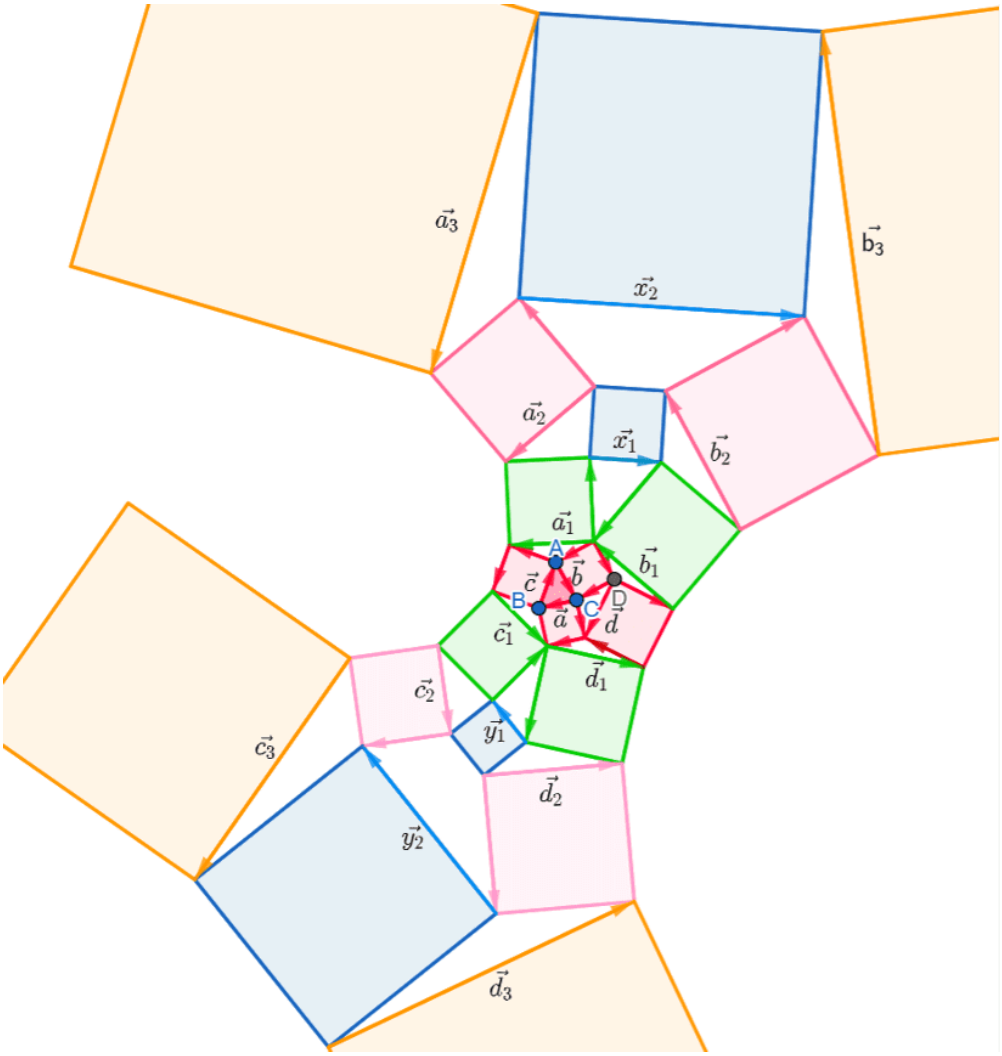

Figure 5. The constructed pattern of four squares around an arbitrary triangle

Generalization of the Pattern of Four Squares around an Arbitrary Triangle

Defining the pattern

With the aid of the set of auxiliary squares, we can generalize the pattern as follows (see Fig. 5).

For $i \geq 2$, we define:

$$\vec{a}_i = \vec{a}_{i-1} + \vec{x'}_{i-1}$$

$$\vec{b}_i = \vec{b}_{i-1} - \vec{x'}_{i-1}$$

$$\vec{x}_i = \vec{x}_{i-1} + \vec{b'}_i - \vec{a'}_i$$

so that

$$\vec{a}_i + \vec{b}_i = \vec{a}_{i-1} + \vec{b}_{i-1} = \vec{a}_1 + \vec{b}_1 = \vec{a} + \vec{a'} - 2 + 2\vec{b'}$$

Rotating 90° clockwise, yields:

$$\vec{a'}_i = \vec{a'}_{i-1} - \vec{x}_{i-1}$$

$$\vec{b'}_i = \vec{b'}_{i-1} + \vec{x}_{i-1}$$

It follows that:

$$\vec{a'}_i + \vec{b'}_i = \vec{a'}_{i-1} + \vec{b'}_{i-1} = \vec{a'}_1 + \vec{b'}_1 = -\vec{a} + \vec{a'} - 2\vec{b} - 2\vec{b'}$$

and

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$$\overrightarrow{b'}_i - \overrightarrow{a'}_i = 2\,\overrightarrow{x}_{i-1} + \left(\overrightarrow{b'}_{i-1} - \overrightarrow{a'}_{i-1}\right)$$

Hence,
$\overrightarrow{x}_i = \overrightarrow{x}_{i-1} + \overrightarrow{b'}_i - \overrightarrow{a'}_i = 3\,\overrightarrow{x}_{i-1} + \left(\overrightarrow{b'}_{i-1} - \overrightarrow{a'}_{i-1}\right).$

The sequence of the areas of the auxiliary squares

Now we turn to the sequence of the $\overrightarrow{x}_i$ and the sequence of the $\overrightarrow{b'}_i - \overrightarrow{a'}_i$ and derive an expression for both in terms of $\overrightarrow{x}_1$.

Theorem 1

For the sequences of the $\overrightarrow{x}_i$ and the $\overrightarrow{b'}_i - \overrightarrow{a'}_i$ defined by (1), we have that

$\overrightarrow{x}_i = k_i\,\overrightarrow{x}_1$ with $k_1 = 1,\ k_2 = 4$ and $k_i = 4\,k_{i-1} - k_{i-2}$ for $i \geq 3$, (2)

$\overrightarrow{b'}_i - \overrightarrow{a'}_i = l_i\,\overrightarrow{x}_1$ with $l_1 = 1,\ l_2 = 3$ and $l_i = 4\,l_{i-1} - l_{i-2}$ for $i \geq 3$. (3)

Proof

From the previous calculations, it follows that the relations hold for $n = 1,\ 2$.

For $n\ \geq 3$, we have by induction on n:

$$\overrightarrow{b'}_i - \overrightarrow{a'}_i = 2\,\overrightarrow{x}_{i-1} + \left(\overrightarrow{b'}_{i-1} - \overrightarrow{a'}_{i-1}\right) = 2\,k_{i-1}\,\overrightarrow{x}_1$$
$$+\,l_{i-1}\,\overrightarrow{x}_1 = (2\,k_{i-1} + l_{i-1})\,\overrightarrow{x}_1 = l_i\,\overrightarrow{x}_1$$

and

$\overrightarrow{x}_i = \overrightarrow{x}_{i-1} + \left(\overrightarrow{b'}_i - \overrightarrow{a'}_i\right) = (\,k_{i-1} + l_i)\,\overrightarrow{x}_1 = k_i\,\overrightarrow{x}_1,$

so that

$$l_i = 2\,k_{i-1} + l_{i-1} \tag{4}$$

and

$$k_i = k_{i-1} + l_i \text{ for } i\ \geq 3 \tag{5}$$

Now, using (4), (5):

$$\begin{aligned} l_{i+1} &= 2\,k_i + l_i && \text{(by 4)}\\ &= 2\,k_{i-1} + 2\,l_i + l_i && \text{(by 5)}\\ &= 3\,l_i + l_i - l_{i-1} && \text{(by 4)}\\ &= 4\,l_i - l_{i-1} \end{aligned}$$

In a similar way, we find that:

$$\begin{aligned} k_{i+1} &= k_i + l_{i+1}\\ &= k_i + l_i + 2\,k_i\\ &= 3\,k_i + l_i\\ &= 3\,k_i + k_i - k_{i-1}\\ &= 4\,k_i - k_{i-1} \end{aligned}$$

which proves the result.

Note that the sequences for the k_i and the l_i correspond to sequences A001353 and A001835 of the Online Encyclopedia of Integer Sequences. Both series are extensively commented, but the geometric applications given here seem new.

The theorem implies that the areas of the auxiliary squares are:

$x_i^2 = k_i^2\,x_1^2 = k_i^2(\,2\,a^2 + 8\,b^2 - 16\,A_{\Delta ABC})$

In an analogous way, we find that:

$$\overrightarrow{c}_i = \overrightarrow{c}_{i-1} - \overrightarrow{y'}_{i-1}$$
$$\overrightarrow{d}_i = \overrightarrow{d}_{i-1} + \overrightarrow{y'}_{i-1}$$
$$\overrightarrow{y}_i = \overrightarrow{y}_{i-1} + \overrightarrow{c'}_i - \overrightarrow{d'}_i$$
$$\overrightarrow{c'}_i - \overrightarrow{d'}_i = l_i\,\overrightarrow{y}_1$$
$$\overrightarrow{y}_i = k_i\,\overrightarrow{y}_1$$
$$y_i^2 = k_i^2\,y_1^2 = k_i^2(\,8\,a^2 + 2\,b^2 - 16\,A_{\Delta ABC})$$

where the sequence of the k_i and the sequence of the l_i are the same as in the case of the $\overrightarrow{x}_i$ and the $\overrightarrow{b'}_i - \overrightarrow{a'}_i$.

The sequences connected to the sum of the areas of the opposite squares

Now let's turn to the sequence of the sums of the areas of opposite squares. We claim that, for $i \geq 2$, $\overrightarrow{a}_i = \overrightarrow{a}_1 + t_{i-1}\overrightarrow{x'}_1$, with $t_i = k_1 + k_2 + \ldots + k_i$.

Obviously, the relation holds for $i = 2$. For $i > 2$, we have, by induction on i that:

$\overrightarrow{a}_i = \overrightarrow{a}_{i-1} + \overrightarrow{x'}_{i-1} = \overrightarrow{a}_1 + t_{i-2}\,\overrightarrow{x'}_1 + k_{i-1}\overrightarrow{x'}_1 = \overrightarrow{a}_1$ since $+\,t_{i-1}\overrightarrow{x'}_1$
$t_{i-1} = t_{i-2} + k_{i-1}$

Similarly,

$$\overrightarrow{b}_i = \overrightarrow{b}_1 - t_{i-1}\overrightarrow{x'}_1$$
$$\overrightarrow{c}_i = \overrightarrow{c}_1 - t_{i-1}\,\overrightarrow{y'}_1$$
$$\overrightarrow{d}_i = \overrightarrow{d}_1 + t_{i-1}\,\overrightarrow{y'}_1$$

It follows that

$$\overrightarrow{a}_i^{\,2} + \overrightarrow{d}_i^{\,2} = a_1^2 + d_1^2 + t_{i-1}^2\,\left(x_1^2 + y_1^2\right)$$
$$+\,2\,t_{i-1}(\overrightarrow{a}_1 \bullet \overrightarrow{x'}_1 + \overrightarrow{d}_1 \bullet \overrightarrow{y'}_1)$$

where

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$$\vec{a}_1 \bullet \vec{x'}_1 + \vec{d}_1 \bullet \vec{y'}_1 = \left(\vec{a'} + 2\,\vec{b'}\right)$$
$$\bullet \left(-\vec{a} + \vec{a'} + 2\,\vec{b} + 2\,\vec{b'}\right) + \left(-2\,\vec{a} + \vec{b}\right)$$
$$\bullet \left(-2\,\vec{a} + 2\,\vec{a'} + \vec{b} + \vec{b'}\right)$$
$$= 5\,a^2 + 5\,b^2 - 16\,A_{\Delta ABC}$$

Using the above expression and those for $a_1^2 + d_1^2$ and $x_1^2 + y_1^2$ derived previously, yields:

$$a_i^2 + d_i^2 = \left[1 + 2\,t_{i-1} + 2\,t_{i-1}^2\right] (5a^2 + 5b^2)$$
$$- \left[t_{i-1} + t_{i-1}^2\right] 32\,A_{\Delta ABC}$$

This expression can be rewritten as:

$$a_i^2 + d_i^2 = 5\,r_i\left(a^2 + b^2\right) - 32\,s_i\,A_{\Delta ABC}$$

where the sequence of the r_i and the sequence of the s_i are given by:

$r_i = 1 + 2\,t_{i-1} + 2\,t_{i-1}^2$ for $i \geq 2$ and $r_1 = 1$

$s_i = t_{i-1} + t_{i-1}^2$ for $i \geq 2$ and $s_1 = 0$.

So, we proved a second theorem.

Theorem 2

The sums of the areas of the opposite squares satisfy the equation:

$$a_i^2 + d_i^2 = 5\,r_i\left(a^2 + b^2\right) - 32\,s_i\,A_{\Delta ABC}$$

where the sequence of the r_i and the sequence of the s_i are given by:

$r_i = 1 + 2\,t_{i-1} + 2\,t_{i-1}^2$ for $i \geq 2$ and $r_1 = 1$

$s_i = t_{i-1} + t_{i-1}^2$ for $i \geq 2$ and $s_1 = 0$.

In a similar way, $b_i^2 + c_i^2 = 5\,r_i\left(a^2 + b^2\right) - 32\,s_i\,A_{\Delta ABC}$.

Note that the proof does not depend on the character of the triangle and that the pattern remains valid for obtuse triangles as well as for acute triangles. Figure 6 illustrates the case of an obtuse triangle. In the construction, the triangles may eventually overlap, but this has no influence on the formula.

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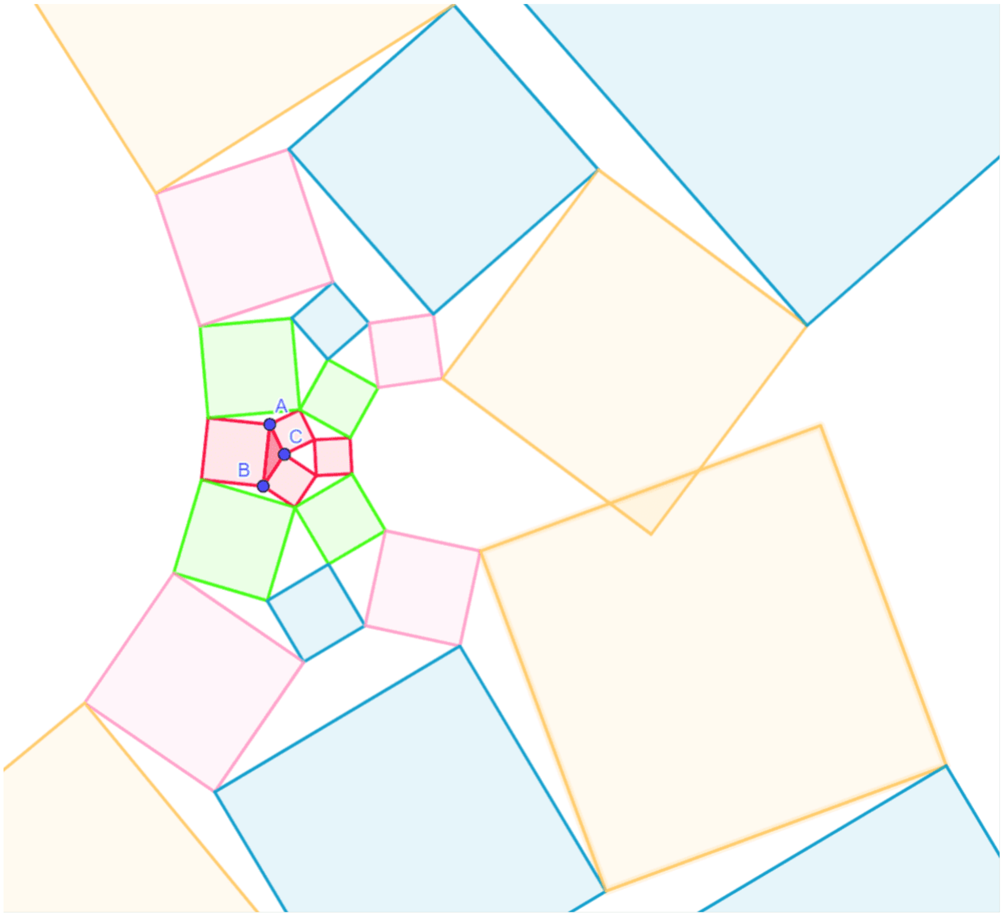

Figure 6. The pattern of four squares around an obtuse triangle

The sequence of the t_i is 1, 5, 20, 76, 285, ... which is known as A061278 in the Online Encyclopedia of Integer Sequences. The sequence of the s_i is 0, 2, 30, 420, 5852, 81510, ... and corresponds to the sequence known as A217855 in the Online Encyclopedia of Integer Sequences. So we found a new geometrical application for both sequences.

The sequence of the r_i is connected to that of the s_i since it follows from the definition that $r_i = 1 + 2\, s_i$. The sequence of the r_i is 1, 5, 61, 841, 11705, 163021, ... which is not yet in the Online Encyclopedia of Integer Sequences but which is connected to the sequence f_i known as A123480, since $f_i = r_{i+1} - 1$. Here again, we give a new geometric application of this sequence.

In the appendix, we show that the sequences correspond indeed to the sequences of the OEIS and derive some new formulas connecting their terms.

Concluding remarks

In this article, we constructed a series of four squares around an arbitrary triangle and found some connections between the areas of these squares and sequences in the Online Encyclopedia of Integer Sequences. In the construction of these series, we made use of the four squares theorem, thus grouping four squares instead of the usual three. It might be interesting to look for other patterns in the squares around a triangle by grouping them in other ways. Maybe there might be some other sequences to be discovered.

Appendix. Formulas for the terms of the sequences

For the sequence of the k_i, it is clear that this sequence corresponds to sequence A001353 since it satisfies the same recurrence relation $k_i = 4\, k_{i-1} - k_{i-2}$ with $k_1 = 1$.

Formulas for the sequence of the $\mathbf{t_i}$

The sequence of the t_i is defined by the partial sums of the sequence of the k_i since

$$t_i = k_1 + k_2 + \ldots + k_i$$

Hence it corresponds to sequence A061278. From the definition, it follows that:

$t_{i+1} = t_i + k_{i+1}$ and $t_i = t_{i+1} - k_{i+1}$

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We derive some other formulas for the terms of this sequence:

$2\ t_i =\ k_{i+1} -\ k_i - 1$ for $i \geq 1$ (6)

$t_i^2 -\ t_i =\ t_{i-1} \bullet t_{i+1}$ for $i \geq 2$ (7)

$t_i = 4\ t_{i-1} -\ t_{i-2} + 1$ for $i \geq 3$ (8)

Proof of the first formula:

We work by induction.

For $i = 1$, we have $k_2 -\ k_1 - 1 = 2 = 2\ t_1$

For $i > 1$, we use the induction hypothesis:

$$2\ t_i = 2\ t_{i-1} + 2\ k_i = 3\ k_i -\ k_{i-1} - 1 =\ k_{i+1} - k_i\ - 1$$

Proof of the second formula:

We use again induction.

For $i = 2, t_2^2 -\ t_2 = 20 =\ t_1 \bullet t_3.$

For $i\ > 2$:

$$t_i^2 -\ t_i = (t_{i-1} +\ k_i)^2 -\ \ (t_{i-1} +\ k_i) =\ t_{i-1}^2 -\ t_{i-1} + 2\ k_i\ t_{i-1} +\ k_i^2 -\ k_i$$

And, by induction,

$t_i^2 -\ t_i =\ t_i\ \ t_{i-2}\ \ + 2\ k_i\ t_{i-1} +\ k_i^2 -\ k_i$

$= (t_{i+1} -\ k_{i+1})\,(t_{i-1} -\ k_{i-1})\ \ + 2\ k_i\ t_{i-1} +\ k_i^2 -\ k_i$

$=\ t_{i\ -1}\ t_{i+1} - (4\ k_i -\ k_{i-1})\,t_{i-1} -\ k_{i-1}\,(t_i +\ \ k_{i+1})$
$+\ k_{i-1}\ k_{i+1} +\ 2\ k_i\ t_{i-1} +\ k_i^2 -\ k_i$

$=\ t_{i\ -1}\ t_{i+1} - 2\ k_i\ t_{i-1} +\ k_{i-1}t_{i-1} -\ k_{i-1}\ t_i\ +\ k_i^2 -\ k_i$

$=\ t_{i\ -1}\ t_{i+1} - 2\ k_i\ t_{i-1} +\ k_{i-1}t_{i-1} -\ k_{i-1}(t_{i-1} + k_i)\ +\ k_i^2$
$-\ k_i$

$=\ t_{i\ -1}\ t_{i+1} +\ k_i\ (-2\ t_{i-1} - k_{i-1} +\ k_i - 1)$

$=\ t_{i\ -1}\ t_{i+1}$ (by (6))

Proof of the third formula:

For $i = 3, 4\ t_2 -\ t_1 + 1 = 20 =\ t_3.$

For $i > 3$, we use induction to find that:

$$4\ t_{i-1} -\ t_{i-2} + 1 = 4\ \ (t_{i-2} +\ k_{i-1}) - (t_{i\ -3} +\ k_{i-2}) + 1 = (4\ t_{i\ -2} -\ t_{i-3} + 1) + (4\ k_{i\ -1} -\ k_{i\ -2})$$

$=\ t_{i-1} +\ k_i\ =\ t_i$

Formula for the sequence of the $\mathbf{s_i}$

The sequence of the s_i is determined by $s_i = t_{i-1} +\ t_{i-1}^2$ for $i \geq 2$ and $s_1 = 0$. We show that it satisfies also the recurrence relation

$s_i = 14\ s_{i-1} -\ s_{i-2} + 2$ for $i\ \geq 3$, (9)

which identifies it as the sequence A217855 of the Online Encyclopedia of Integer Sequences.

Proof:

The formula is valid for $i = 3$ since $14\ s_2 -\ s_1 + 2 = 30 =\ s_3.$

For $i > 3$, we have:

$s_{i+1} =\ t_i +\ t_i^2 =\ \ (4\ t_{i-1} -\ t_{i-2} + 1)$ (by (8))
$+\ \ (4\ t_{i-1} -\ t_{i-2} + 1)^2$

$=\ 16\ t_{i-1}^2 + 12\ t_{i-1} + 2 +\ t_{i-2}^2 - 3\ t_{i-2} - 8\ t_{i-1}t_{i-2}$

$= 14\ s_i\ + 2\ t_{i-1}^2 - 2\ t_{i-1} - s_{i-1} + 2\ t_{i-2}^2 - 2\ t_{i-2} + 2$
$- 8\ t_{i-1}\ t_{i-2}$

$= 14\ \ s_i -\ s_{i-1} + 2$
$+ 2\left(t_{i-1}^2 -\ t_{i-1} +\ t_{i-2}^2 - t_{i-2} - 4\ t_{i-1}\ t_{i-2}\ \right)$

$=\ 14\ \ s_i -\ s_{i-1} + 2 + 2\left(t_{i-2}\ t_i +\ t_{i-2}^2 - t_{i-2} - 4\ t_{i-1}t\ _{i-2}\ \right)$ (by (7))

$=\ 14\ \ s_i -\ s_{i-1} + 2 + 2\ t_{i-2}\,(\ t_i +\ t_{i-2} - 1 - 4\ t_{i-1}\ )$

$=\ 14\ \ s_i -\ s_{i-1} + 2$ (by (8))

Formula for the sequence of the $\mathbf{r_i}$

Since the sequence of the r_i is connected to that of the s_i by the formula $r_i = 1 + 2\ s_i$, the following recurrence relation holds between the r_i, for $i \geq 3$:

$$r_i\ = 14\ r_{i-1} - r_{i-2} - 8$$

Proof:

$$r_i\ =\ 1 + 2\ s_i = 5 + 28\ s_{i-1} - 2\ s_{i-2} = 14\ \ (2\ s_{i-1} + 1) - (2s_{i-\ 1} + 1) - 8 = 14\ r_{i-1} -\ r_{i-2} - 8$$

Statements and Declarations

Author Contributions

- **Conceptualization:** Huylebrouck, D.;
- **Methodology:** De Boeck I.;
- **Formal Analysis:** De Boeck I.;
- **Investigation:** Huylebrouck, D., De Boeck I.;
- **Writing – Original Draft:** De Boeck I.;
- **Writing – Review & Editing:** De Boeck I.

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