Research Article

Some Problems on Quantum Noise in Quantum Field Theories: Filtering, Parameter Estimation, Noisy Feynman Path Integrals, Noisy Scattering, and Noisy Quantum Boltzmann from the Weyl–Moyal–Wigner Transform Perspective

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1. Independent researcher

This paper explores some of the applications of quantum stochastic calculus (qsc) as founded by R.L.Hudson and K.R.Parthasarathy and the theory of the quantum filter established by V.P.Belavkin based on qsc which was simplified considerably by John Gough, Claude Kostler and others to quantum field theory. Specifically, we first derive some general versions of the Belavkin quantum filter for the case when the Hamiltonian and Lindblad operators in the Hudson-Parthasarathy noisy Schrodinger equation (HPSDE) are functions of a classical Markovian time varying parameter process whose statistics is determined by an infinitesimal generator. Using this generalized Belavkin filter, we show how we can estimate on real time basis not only the evolving mixed state of the system but also the probability density of the Markovian parameter. Our generalized filter enables us to obtain a function $\rho(t, \theta)$ that represents an estimate of the state of the quantum system at time t as well as the probability density of the parameter θ at time t. Specifically, when $X(\theta) = \sum_{k=1}^{n} f_k(\theta) X_k$ is a system observable dependent upon the Markovian θ, then, $\int Tr(\rho(t, \theta) X(\theta)) d\theta$ would represent the conditional expectation of $X(t, \theta(t)) = U(t)^* X(\theta(t)) U(t)$ given non-demolition measurements at time tThis paper explores some of the applications of quantum stochastic calculus (qsc) as founded by R.L.Hudson and K.R.Parthasarathy and the theory of the quantum filter established by V.P.Belavkin based on qsc which was simplified considerably by John Gough, Claude Kostler and others to quantum field theory. Specifically, we first

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quantum information and quantum commmunication when we require to measure how much information has been transmitted as a function of time across a quantum channel after filtering out the noise on a real time basis. In the context of quantum communication theory, we illustrate the situation of an optical fibre wherein the information carrying quantum electromagnetic field interacts with a classical electromagnetic field that has been scattered by the phonon lattice and hence contains unknown Markovian parameters. By estimating the Markovian paremeters using our generalized Belvakin filter on a real time basis, we are therefore able to generate a counter Hamiltonian via causing a simulated counter classical field using this estimated parameter to interact with the propagating quantum field and hence mitigate the effect of the interaction of the classical field with the quantum field. One of the novel results of this paper is the exact representation of the Weyl-Moyal-Wigner transform of products of quantum operators that includes Lie brackets between operators to all orders in Planck's constant. We use this idea to obtain a classical representation of the quantum Boltzmann equation in terms of Wigner densities that are complex valued functions of position and momenta representing the system state. Future work on Wigner transforming the quantum Boltzmann equation for all r-particle marginal states of an N-particle system is in progress. Work on interpreting the Belavkin filter and quantum Boltzmann equation in the Wigner domain in terms of corrections to the classical Fujisaki-Kushner-Kallianpur stochastic filter and the classical Boltzmann equation is also in progress.

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1. Real time estimation of states and parameters in the Hudson-Parthasarathy qsde in the presence of only quantum Brownian noise

We assume that $\theta(t)$ is a classical Markov process with infinitesimal generator K. The HPQSDE is

$$dU(t) = -(i(H_0 + V(\theta(t)) + P)dt + LdA(t) - L^* dA(t)^*)U(t), P = LL^* /2$$

[1]. The non-demolition measurements [2][3] are

$$Y_o(t) = U(t)^* Y_i(t)U(t), Y_i(t) = cA(t) + \bar{c}A(t)^*$$

$\eta_0(t)$ is the filtration generated by $Y_o(s), s \leq t$.

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$$J_t(fX) = f(\theta(t))j_t(X), j_t(X) = U(t)^* XU(t)$$

$$dj_t(X) = j_t(\phi_0(X))dt + j_t(\phi_1(X))dA(t) + j_t(\phi_2(X))dA(t)^*$$

where

$$\phi_0(X) = \phi_0(t, X) =$$

$$i[H_0 + V(\theta(t)), X] - PX - XP + LXL^* = i[H_0 + V(\theta(t)), X] - (1/2)(LL^*X + XLL^* - 2LXL^*)$$

$$\phi_1(X) = -LX + XL = [X, L], \phi_2(X) = -XL^* + L^*X = [L^*, X]$$

$$dJ_t(fX) = d(f(\theta(t))j_t(X)) = df(\theta(t)). j_t(X) + f(\theta(t))dj_t(X)$$

$$\mathrm{E}(dJ_t(fX) \Big| \eta_o(t)) = \pi_t((Kf)X)dt + \pi_t(f\phi_0(X))dt$$

Note that we are assuming that the bath is the vacuum coherent state. Further, Here,

$$\pi_t(fX) = \mathrm{E}(f(\theta(t))j_t(X) \Big| \eta_o(t)) = \mathrm{E}(J_t(fX) \Big| \eta_o(t))$$

We may assume that

$$d\pi_t(fX) = F_t(fX)dt + G_t(fX)dY_o(t)$$

Then,

$$dY_o(t) = dY_i(t) - j_t(\bar{c}L + cL^*)dt$$

$$\mathrm{E}(J_t(fX)dY_o(t) \Big| \eta_o(t)) = -\pi_t(fX(\bar{c}L + cL^*))dt,$$

$$\mathrm{E}(dJ_t(fX)dY_o(t) \Big| \eta_o(t)) = \bar{c}\pi_t(f\phi_1(X))dt,$$

$$\mathrm{E}(\pi_t(fX)dY_o(t) \Big| \eta_o(t)) = -\pi(fX)\pi_t(\bar{c}L + cL^*)dt,$$

$$\mathrm{E}(d\pi_t(fX) \Big| \eta_o(t)) = F_t(fX)dt - G_t(fX). \pi_t(\bar{c}L + cL^*)dt,$$

$$\mathrm{E}(d\pi_t(fX). dY_o(t) | \eta_o(t)) = G_t(fX) |c|^2 dt,$$

The basic orthogonality equations are [2]

$$\mathrm{E}((J_t(fX) - \pi_t(fX))dY_o(t) \Big| \eta_o(t)) + \mathrm{E}((dJ_t(fX) - d\pi_t(fX))dY_o(t) \Big| \eta_o(t)) = 0,$$

$$\mathrm{E}(dJ_t(fX) - d\pi_t(fX) \Big| \eta_o(t)) = 0,$$

and these yield

$$-\pi_t(fX(\bar{c}L + cL^*)) + \pi_t(fX). \pi_t(\bar{c}L + cL^*) + \bar{c}\pi_t(f\phi_1(X)) - G_t(fX) |c|^2 = 0,$$

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$$\pi_t((Kf)X) + \pi_t(f\phi_0(X)) - F_t(fX) + G_t(fX)\pi_t(cL + \bar{c}L^*) = 0$$

Assume without loss of generality that c is complex with

$$|c| = 1$$

Then the above equations simplify to give

$$G_t(fX) = \pi_t(fX)\pi_t(cL + \bar{c}L^*) - \pi_t(f(cXL^* + \bar{c}LX))$$

$$F_t(fX) = \pi_t((Kf)X) + \pi_t(f\phi_0(X)) + G_t(fX)\pi_t(cL + \bar{c}L^*)$$

and hence our generalized Belavkin filter is

$$d\pi_t(fX) = (\pi_t((Kf).X)dt + \pi_t(f\phi_0(X)))dt + (\pi_t(fX)\pi_t(cL + \bar{c}L^*) - \pi_t(f(cXL^* + \bar{c}LX))$$

$$\times (dY_o(t) + \pi_t(cL + \bar{c}L^*)dt)$$

Now, write

$$\pi_t(fX) = \int Tr(\rho_t(\theta)X)f(\theta)d\theta$$

Thus, in particular, the state estimated of the quantum system at time t is $\rho(t) = \int \rho(t, \theta)d\theta$ and the pdf of the parameter θ at time t given $\eta_o(t)$ is $p(t, \theta) = Tr(\rho_t(\theta))$. Equivalently, we can write (24) as

$$d\rho_t(\theta) = (K^*(\rho_t)(\theta) + \phi_0^*(\rho_t)(\theta))dt +$$

$$+[Tr(\rho_t(\theta)(cL + \bar{c}L^*))\rho_t(\theta) - (cL^*\rho_t(\theta) + \bar{c}\rho_t(\theta)L)].\,(dY_o(t) + Tr(\rho_t(\theta)(cL + \bar{c}L^*))dt)$$

which is the desired filter that estimates both the quantum state and the classical Markovian parameter on a real time basis from non-demolition measurements.

2. Real time Belavkin estimation parameters and states in the case of generalized Hudson-Parthasarathy quantum noise

The generalized HPQSDE is

$$dU(t) = L_b^a d\Lambda_a^b(t)U(t)$$

Input and output non-demolition measurements are

$$Y_i(t) = c(a, b)\Lambda_b^a(t),\ Y_o(t) = U(t)^* Y_i(t)U(t)$$

Then, quantum Ito's formula [1] gives

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$$dY_o(t) = j_t(M^a_b(1))d\Lambda^b_a(t), dY_o(t)^k = j_t(M^a_b(k))d\Lambda^b_a(t), k \geq 1$$

Let

$$J_t(fX) = f(\theta(t)).j_t(X), j_t(X) = U(t)^* XU(t)$$

$$dj_t(X) = j_t(\phi^a_b(X))d\Lambda^b_a(t),$$

$$dJ_t(fX) = df(\theta(t)).j_t(X) + f(\theta(t)).j_t(\phi^b_a(X))d\Lambda^b_a(t)$$

$$\mathrm{E}(dJ_t(fX)|\eta_o(t)) = [\pi_t((Kf).X) + \pi_t(f\phi^b_a(X))u_b(t)\bar{u}_a(t))]dt$$

The filter stochastic differential equations can be expressed as

$$d\pi_t(fX) = F_t(fX) + \sum_{k\geq 1} G_{k,t}(fX)dY_o(t)^k$$

We compute

$$\mathrm{E}(d\pi_t(fX)\Big|\eta_o(t)) = [F_t(fX) + \sum_k G_{kt}(fX)\pi_t(M^a_b(k))u_b(t)\bar{u}_a(t)]dt,$$

$$\mathrm{E}(J_t(fX)dY_o(t)^m\Big|\eta_o(t)) = \pi_t(fXM^a_b(m))u_b(t)\bar{u}_a(t)dt$$

$$\mathrm{E}(\pi_t(fX).dY_o(t)^m|\eta_o(t)) = \pi_t(fX)\pi_t(M^a_b(m))u_b(t)\bar{u}_a(t)dt$$

$$\mathrm{E}(dJ_t(fX).dY_o(t)^m|\eta_o(t)) = \pi_t(f.\phi^a_b(X)\epsilon^b_c.M^c_d(m))u_d(t)\bar{u}_a(t)$$

$$= \pi_t(f.(\phi(X)\epsilon.M(m))^a_b)u_b(t)\bar{u}_a(t)dt$$

$$\mathrm{E}(d\pi_t(fX)dY_o(t)^m\Big|\eta_o(t)) = \sum_k G_{k,t}(fX)\pi_t(M^a_b(k+m))u_b(t)\bar{u}_a(t)dt$$

Note that $\eta_o(t)$ is an Abelian Von-Neumann algebra generated by $Y_o(s), s \leq t$ and that $\eta_o(t)$ commutes with $J_t(fX)$. The basic orthogonality relation of minimum mean square estimation theory reads:

$$\mathrm{E}[(J_t(fX) - \pi_t(fX))C(t)) = 0$$

for all

$$C(t) \in \eta_o(t) = \sigma(Y_o(s), s \leq t)$$

Choosing $C(t)$ to satisfy

$$dC(t) = \sum_k f_k(t)C(t)dY_o(t)^{\otimes k}, t \geq 0$$

with the complex functions $f_k(t)$ arbitrary, the above orthogonality relations yield after forming time differentials:

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$$\mathrm{E}[(dJ_t(fX) - d\pi_t(fX)) | \eta_o(t)) = 0,$$

and

$$\mathrm{E}[(J_t(fX) - \pi_t(fX))dY_o(t)^m | \eta_o(t)) + \mathrm{E}[(dJ_t(fX) - d\pi_t(fX))dY_o(t)^m | \eta_o(t)) = 0,$$

Substituting the above expressions for the various conditional expectations gives

$$\pi_t((Kf).X) - F_t(fX) + [\pi_t(f\phi_a^b(X)) - \sum_k G_{kt}(fX)\pi_t(M_b^a(k))]u_b(t)\bar{u}_a(t) = 0$$

and

$$[\pi_t(fXM_b^a(m)) - \pi_t(fX)\pi_t(M_b^a(m))$$

$$+\pi_t(f.(\phi(X)\epsilon.M(m))_b^a) - \sum_k G_{k,t}(fX)\pi_t(M_b^a(k+m))]u_b(t)\bar{u}_a(t) = 0$$

Note that $u_0(t) = 1$. (1) gives

$$F_t(fX) = \pi_t((Kf).X) + [\pi_t(f\phi_a^b(X)) - \sum_k G_{kt}(fX)\pi_t(M_b^a(k))]u_b(t)\bar{u}_a(t)$$

and (45) gives us an infinite sequence of linear equations for $G_{k,t}(fX), k \geq 1$:

$$\sum_k G_{k,t}(fX)\pi_t(M_b^a(k+m)).\ u_b(t)\bar{u}_a(t)$$

$$= [\pi_t(fXM_b^a(m)) - \pi_t(fX)\pi_t(M_b^a(m)) + \pi_t(f.(\phi(X)\epsilon.M(m))_b^a)]u_b(t)\bar{u}_a(t),\ m \geq 1$$

Note that (47) can equivalently be expressed as

$$\sum_k G_{k,t}(fX)\pi_t(M_b^a(k+m)).\ u_b(t)\bar{u}_a(t)$$

$$= [-\pi_t(fX)\pi_t(M_b^a(m)) + \pi_t(f.(\phi(X)\epsilon.M(m))_b^a + XM_b^a(m)))]u_b(t)\bar{u}_a(t),\ m \geq 1$$

Define the infinite matrix

$$Z(\pi_t, u(t)) = ((\pi_t(M_b^a(m+k)u_b(t)\bar{u}_a(t)))_{1 \leq m,k \leq \infty}$$

Then define its inverse

$$T(\pi_t, u(t)) = Z(\pi_t, u(t))^{-1} = ((T_{mk}(\pi_t, u(t))))$$

Note that that infinite matrix $T(\pi_t, u(t))$ does not depend on f or X. It is a function of only $u(t)$ and $\eta_o(t)$ and of course $M_b^a(1)$. Note that $M_b^a(k), k \geq 1$ can be expressed in terms of $M_d^c(1), c, d = 0, 1, \ldots, N$. Now, we can write from (5),

$$G_{k,t}(fX) = \sum_m T_{km}(\pi_t, u(t))[-\pi_t(fX)\pi_t(M^a_b(m)) + \pi_t(f.(\phi(X)\epsilon.M(m))^a_b + XM^a_b(m)))]u_b(t)\bar{u}_a(t), m \geq 1$$

or equivalently, defining

$$M(m, u(t)) = M^a_b(m)u_b(t)\bar{u}_a(t),$$

$$(\phi(X)\epsilon.M(m))^a_b u_b(t)\bar{u}_a(t) =$$

$$\phi(X)\epsilon.M(m)(u(t))$$

$$\phi^a_b(X)u_b(t)\bar{u}_a(t) = \phi(X)(u(t))$$

we have

$$G_{k,t}(fX) = \sum_m T_{km}(\pi_t, u(t))[-\pi_t(fX)\pi_t(M(m, u(t))$$

$$+\pi_t(f.(\phi(X)\epsilon.M(m))(u(t)) + XM(m, u(t))))], k \geq 1$$

which we can formally write using infinite dimensional matrix notation as

$$G_t(X) = T(\pi_t, u(t)).[-\pi_t(fX)\pi_t(M(u(t))$$

$$+\pi_t(f.(\phi(X)\epsilon.M)(u(t)) + XM(u(t))))]$$

and hence, we can express our generalized Belavkin filter stochastic differential equations as

$$d\pi_t(fX) = \pi_t((Kf)(X) + f\phi(X)(u(t)))dt$$

$$+\sum_k G_{k,t}(fX)(dY_o(t)^{\otimes k} - \pi_t(M(k, u(t)))$$

$$= \pi_t((Kf)(X) + f\phi(X)(u(t)))dt$$

$$+\sum_k T(\pi_t, u(t)).[-\pi_t(fX)\pi_t(M(u(t))$$

$$+\pi_t(f.(\phi(X)\epsilon.M)(u(t)) + XM(u(t))))][k](dY_o(t)^{\otimes k} - \pi_t(M(k, u(t)))$$

3. Representing observables in quantum mechanics using the Weyl-Moyal-Wigner method based on complex functions

Given a quantum system described by n position and n momentum operators (Q, P) obeying the canonical commutation relations $[Q_a, P_b] = ih\delta_{ab}$. An observable of this system is described by a linear operator X expressible as a function $F(Q, P)$ of the operators Q, P. We denote the eigenvalues of Q, P by $Q', Q'', Q''', P', P'', P'''$ etc and the associated eigenkets by $|Q'>, |Q''>, |Q'''>, |P'>, |P''>, |P'''>$ etc. Thus,

$$Q|Q^{'}> = Q^{'}|Q^{'}>, Q|Q^{''}> = Q^{''}|Q^{''}>, Q|Q^{'''}> = Q^{'''}|Q^{'''}>,$$

$$P|P^{'}> = P^{'}|P^{'}>, P|P^{''}> = P^{''}|P^{''}>, P|P^{'''}> = P^{'''}|P^{'''}>,$$

etc. In the position representation (Dirac), Q is a multiplication operator and $P = -ih\nabla_Q$. Thus,

$$<Q^{'}|P|P^{'}> = P^{'}<Q^{'}\left|P^{'}\right>$$

on the one hand and

$$<Q^{'}|P|P^{'}> = -ih\nabla_{Q^{'}}<Q^{'}\left|P^{'}\right>$$

on the other hand. Therefore, equating these two and solving the resulting differential equation gives us

$$<Q^{'}|P^{'}>= C.\,exp(iP^{'}.\,Q^{'}/h)$$

C is a normalization constant whose magnitude is determined by the condition

$$\delta(P^{'}-P^{''}) = \int <P^{'}\left|Q^{'}\right> dQ^{'} <Q^{'}\left|P^{''}\right> =$$

$$|C|^2\int exp(iP^{''}-P^{'}).\,Q^{'}/h)dQ^{'} = |C|^2(2\pi h)^n\delta(P^{'}-P^{''})$$

so that

$$|C| = (2\pi h)^{-n/2}$$

The phase of C can be an arbitrary constant, which without loss of any generality, can be set equal to zero:

$$C = (2\pi h)^{-n/2}$$

ie,

$$<Q^{'}|P^{'}>= (2\pi h)^{-n/2}exp(iP^{'}.\,Q^{'}/h)$$

For an observable X, we write its position space kernel representation as

$$X(Q^{'}, Q^{''}) = <Q^{'}|X|Q^{''}>$$

This gives for any two observables X, Y,

$$XY(Q^{'}, Q^{''}) = <Q^{'}|XY|Q^{''}> = \int <Q^{'}|X|Q^{'''}> dQ^{'''} <Q^{''}|Y|Q^{''}> = \int X(Q^{'}, Q^{'''}).\,Y(Q^{'''}, Q^{''})dQ^{'''}$$

just like matrix multiplication but in the continuous position domain. We define the Wigner transform of the observable X as a complex function $X_1(Q^{'}, P^{'})$ of the eigenvalues $Q^{'}, P^{'}$ of the operators Q, P respectively by [4]:

$$X_1(Q^{'}, P^{'}) = C_1\int X(Q^{'}+q^{'}/2, Q^{'}-q^{'}/2)exp(iP^{'}.\,q^{'}/h)dq^{'}$$

Fourier inversion gives

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$$X(Q^{'}+q^{'}/2, Q^{'}-q^{'}/2) = C_2\int X_1(Q^{'}, P^{'})exp(-iP^{'}.q^{'}/h)dP^{'}$$

or equivalently,

$$X(Q^{'}, Q^{''}) = C_3\int X_1((Q^{'}+Q^{''})/2, P^{'})exp(-iP^{'}.(Q^{'}-Q^{''})/h)dP^{'}$$

Our aim is to show that for two operators X, Y, $(XY)_1(Q^{'}, P^{'})$ is $X_1(Q^{'}, P^{'})Y_1(Q^{'}, P^{'})$ plus a correction term that can be expressed as a Taylor series in h starting from the first power of h, ie, in the limit $h\to 0$, $(XY)_1$ becomes X_1Y_1. To this end, we have

$$(XY)_1(Q^{'}, P^{'}) = \int(XY)(Q^{'}+q^{'}/2, Q^{'}-q^{'}/2)exp(iP^{'}.q^{'}/h)dq^{'} =$$

$$\int X(Q^{'}+q^{'}/2, Q^{''})Y(Q^{''}, Q^{'}-q^{'}/2)exp(iP^{'}.q^{'}/h)dq^{'}dQ^{''}$$

$$= C_4\int X_1((Q^{'}+Q^{''})/2+q^{'}/4, P^{''})Y_1((Q^{'}+Q^{''})/2-q^{'}/4, P^{'''})exp(-iP^{''}.(Q^{'}-Q^{''}+q^{'}/2)/h).$$

$$\times\, exp(-iP^{'''}.(Q^{''}-Q^{'}+q^{'}/2)/h)exp(iP^{'}.q^{'}/h)dq^{'}dP^{''}dP^{'''}dQ^{''}$$

$$= C_5\int X_1(Q^{'}+x, P^{'}+y)Y_1(Q^{'}+u, P^{'}+v)exp(i2(P^{'}+y).u/h).\, exp(-2i(P^{'}+v).x/h)$$

$$\times\, exp(i2P^{'}.(x-u)/h)dxdydudv$$

where we have made the change of variables

$$x = (Q^{''}-Q^{'})/2+q^{'}/4, u = (Q^{''}-Q^{'})/2-q^{'}/4, P^{''} = P^{'}+y, P^{'''} = P^{'}+v$$

or equivalently,

$$Q^{''} = Q^{'}+x+u, q^{'} = 2(x-u), P^{''} = P^{'}+y, P^{'''} = P^{'}+v$$

This expression simplifies to

$$(XY)_1(Q^{'}, P^{'}) = C_5\int X_1(Q^{'}+x, P^{'}+y)Y_1(Q^{'}+u, P^{'}+v)exp(2i(y.u-x.v)/h)dxdydudv$$

$$= C_5[\int exp(2i(y.u-x.v)/h).\, exp(x.\nabla^X_{Q^{'}}+y.\nabla^X_{P^{'}}+u.\nabla^Y_{Q^{'}}+v.\nabla^Y_{P^{'}}).\, dxdydudv].\,(X_1Y_1)(Q^{'}, P^{'})$$

$$= [C_6\int\delta(u-ih\nabla^X_{P^{'}}/2).\,\delta(v+ih\nabla^X_{Q^{'}}/2).\, exp(u.\nabla^Y_{Q^{'}}).\, exp(v.\nabla^Y_{P^{'}})dudv](X_1Y_1)(Q^{'}, P^{'})$$

$$= exp((ih/2)(\nabla^X_{P^{'}}.\nabla^Y_{Q^{'}}-\nabla^X_{Q^{'}}.\nabla^Y_{P^{'}})).\,(X_1Y_1)(Q^{'}, P^{'})$$

Note that we are using the notation in which the superscript on the ∇ operation denotes the observable (ie X_1 or Y_1) on which this operator acts while the subscript denotes the variable (ie $Q^{'}$ or $P^{'}$) on which it acts. Note that all the four ∇ operations obtained in this way mutually commute.

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4. Calculating the quantum Belavkin filtering equations in the Weyl-Moyal-Wigner representation

With reference to the previous section, we define the partial differential operator acting on the product of two complex functions X_1, Y_1 of Q', P':

$$L_h(X, Y) = exp((ih/2)(\nabla^X_{P'}.\nabla^Y_{Q'} - \nabla^X_{Q'}.\nabla^Y_{P'})) - 1$$

Thus,

$$(XY)_1(Q', P') = X_1 Y_1(Q', P') + L_h(X, Y)(X_1 Y_1)(Q', P')$$

Note that the second term is $O(h)$. By repeated application of this formula, we get for operators $X(k), k = 1, 2, \ldots, N$,

$$[X(1)X(2)\ldots X(N)]_1 = (X(1)\ldots X(N-1))_1 X(N)_1 +$$

$$L_h(X(1)\ldots X(N-1), X(N))((X(1)\ldots X(N-1))_1, Y(N)_1)$$

which can be iterated to express $[X(1)\ldots X(N)]_1(Q', P')$ in terms of $X(k)(Q', P'), k = 1, 2, \ldots, N$ and their partial derivatives. For example, we get

$$[XYZ]_1 = [XY]_1 Z_1 + L_h(XY, Z)([XY]_1 Z_1) = X_1 Y_1 Z_1 +$$

$$L_h(X, Y)(X_1 Y_1).Z_1 + L_h(XY, Z)((X_1 Y_1 + L_h(X, Y)(X_1 Y_1)).Z_1)$$

$$= X_1 Y_1 Z_1 + L_h(X, Y)(X_1 Y_1).Z_1 + L_h(XY, Z)(X_1 Y_1 Z_1) + L_h(XY, Z).L_h(X, Y)(X_1 Y_1 Z_1)$$

Remark:

$$L_h(XY, Z) = exp((ih/2)(\nabla^{XY}_{P'}.\nabla^Z_{Q'} - \nabla^{XY}_{Q'}.\nabla^Z_{P'}))$$

$$= exp((ih/2)((\nabla^X_{P'} + \nabla^Y_{P'})\nabla^Z_{Q'} - (\nabla^X_{Q'} + \nabla^Y_{Q'})\nabla^Z_{P'})))$$

$$= exp((ih/2)(\nabla^X_{P'}\nabla^Z_{Q'} + \nabla^Y_{P'}\nabla^Z_{Q'} - \nabla^X_{Q'}\nabla^Z_{P'} - \nabla^Y_{Q'}\nabla^Z_{P'}))$$

Now we apply this result to the Belavkin filter problem with random classical Markovian parameter:

5. Mitigating classical noise in quantum systems using feedback based on real time estimated parameters

Assume that the field is a quantum electromagnetic field $A_{q\mu}(x)$ with associated electromagnetic fields:

$$E_q(x) = -\nabla A_{q0} - \partial_t Q A_q, B_q(x) = curl A_q$$

where

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$$(A_{q\mu}) = (A_{q0}, A_q), A_q = (A_{qr})_{r=1}^3$$

These quantum field have the modal expansions:

$$A_{q\mu}(x) = \sum_k (c(k)a_{k\mu}(x) + c(k)^* \bar{a}_{k\mu}(x)), [c(k), c(m)] = 0 = [c(k)^*, c(m)^*], [c(k), c(m)^*] = \delta(k, m)$$

$$E_q(x) = \sum_k (c(k)e_k(x) + c(k)^* \bar{e}_k(x)), B_q(x) = \sum_k (c(k)b_k(x) + c(k)^* \bar{b}_k(x))$$

where

$$e_{kr}(x) = -\nabla a_{k0}(x) - \partial t a_{kr}(x), r = 1, 2, 3,$$

$$b_{kr}(x) = \epsilon(rsm)(a_{km,s} - a_{ks,m}), r = 1, 2, 3$$

Let $A_c(x|\theta(t)), E_c(x|\theta(t)), B_c(x|\theta(t))$ denote the incident classical fields. These fields interact with the second quantized phonon lattice thereby generating a quantum noisy electromagnetic field having the form

$$E_N(t, r_x) = \int K_N(t, r_x, r_y, r_z)(A_c(t, r_y \,|\, \theta(t)) \otimes W(t, r_z))d^3r_y d^3r_z,$$

$$B_N(t, r_x) = \int L_N(t, r_x, r_y, r_z)(A_c(t, r_y \,|\, \theta(t)) \otimes W(t, r_z))d^3r_y d^3r_z,$$

Specifically, the mechanism behind these formulas is along the following lines: The initial positions and momenta of the phonon lattice atoms is $(Q(0), P(0))$. These are Heisenberg operators in a phonon bath Hilbert space. Their dynamics under the incident classical field without taking into account the mutual electromagnetic interactions between the phonons, is given by

$$Q_k''(t) = (Ze/M)(E_c(t, Q_k(t)) + Q_k'(t) \times B_c(t, Q_k(t)))$$

$$Q_k'(t) = (P_k(t) + Ze.A_c(t, Q_k(t)))/M$$

$k = 1, 2, \ldots, N$ with initial conditions $Q_k(0), P_k(0)$. Linearizing these equations around $Q(0), P(0)$ gives us the solutions for $\delta Q(t), \delta Q'(t), \delta P(t)$ as linear functionals of $E_c(s, Q_k(0)), B_c(s, Q_k(0)), s \leq 1, k = 1, 2, \ldots, N$. More generally, if we take into account mutual electromagnetic interactions between the phonons, the equations of motion would have the form

$$Q_k''(t) = (Ze/M)(E_c(t, Q_k(t)) + Q_k'(t) \times B_c(t, Q_k(t))) + F_k(Q(t), Q'(t))$$

$$Q_k'(t) = (P_k(t) + Ze.A_c(t, Q_k(t)))/M$$

The unperturbed equations of the phonon lattice are thus,

$$Q_k''(t) = F_k(Q(t), Q'(t))$$

and we denote the solution to this by

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$$Q_{k0}(t), Q_{k0}'(t)$$

Now perturb around $Q_0(t), Q_0'(t)$ to get linearized equations for $\delta Q(t), \delta Q'(t)$ which can be solved when the incident fields are weak, to give $\delta Q(t)$ as linear functions of $E_c(s, Q_0(s) \mid \theta(s)), B_c(s, Q_0(s) \mid \theta(s)), s \le t$. Finally, we write $Q_0(t) = Q_0(0) + W(t)$ with $W(t)$ as quantum noise and expand the solutions upto linear orders in W to obtain the above form for E_N, B_N using standard electrodynamic formulas for fields scattered by charged phonon particles. When the size N of the lattice is large, $Q_0(t), P_0(t)$ execute quantum Brownian motions as result of mutual interactions and the origin of the quantum noise term $W(t)$ is from here. The scattered quantum noisy fields E_N, B_N interact bilinearly with the signal quantum field E_q, B_q to yield the quantum noise perturbation Hamiltonian as

$$H_N = (1/2)\int \epsilon(E_q(t, r_x), E_N(t, r_x)) + (B_q(t, r_x), B_N(t, r_x)))d^3r_x$$

and this can be expressed in the form

$$H_N = \sum_k [L_k(\theta(t)).\, dA_k(t)/dt + L_k(\theta(t))^* .\, dA_k(t)^*/dt]$$

where the Lindblad operator $L_k(\theta(t))$ is bilinear in $(c(m), c(m)^*)$ and $A_c(t, r_x \mid \theta(t)), r_x \in \mathrm{R}^3$. Note that E_N, B_N are quantum noisy fields parameterized by the classical parameters $\theta(t)$ as explained above because A_c, E_c, B_c are parametrized by $\theta(t)$ and $Q_k(t)'s$ satisfy differential equations driven by E_c, B_c. In short, the unperturbed solutions for the $Q_k's$ are functions of quantum noise and the perturbed solutions therefore incorporate this quantum noise as well as the classical parameters. Now, we can formulate using this quantum noisy Hamiltonian, the Hudson-Parthasarathy noisy Schrodinger equation after adding quantum Ito correction terms to guarantee joint unitary evolution of the system and bath and then using the extended Belavkin filter, estimate the evolving quantum state of the system as well as the classical parameter on a real time basis given non-demolition measurements and once the parameter estimate is known, we can apply a feedback control Hamiltonian to mitigatge the noise effects. Specifically, given the parameter estimate at time t, we can apply a counter classical em field by substituting this parameter estimate into the expression for the classical field but with a negative sign and hence generate an estimated counter interaction Hamiltonian by causing this estimated classical field to interact with the propagating quantum em field and thus significantly mitigate the bilinear interaction Hamiltonian between the quantum and classical fields.

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6. Weyl-Moyal-Wigner transform of the Belavkin quantum filter equation

The Belavkin filter is a stochastic Schrodinger equation of the form

$$d\rho(t) = \sum_k A_k(t)\rho(t)B_k(t)dt + \sum_{n(1),\ldots,n(m)\geq 0, s\geq 1}(Tr(\rho(t)C_1)^{n(1)}\ldots Tr(\rho(t)C_m)^{n(m)}D_{t|n(1),\ldots,n(m),s}dV_s(t)$$

where $A_k(t), B_k(t), C_1, \ldots, C_m, D(t|n(1), \ldots, n(m), s)$ are system space matrices while $V_m^{'}s$ are classical independent zero mean Levy processes. This equation can be regarded as a classical matrix stochastic differential equation. We have earlier, defined the operator $L_h(X, Y)$ for two operators (system space matrices) as

$$L_h(X, Y) = exp((ih/2)(\nabla^X_{P'}.\nabla^Y_{Q'} - \nabla^X_{Q'}.\nabla^Y_{P'}))$$

acting on $X_1Y_1(Q^{'}, P^{'})$ and proved that this equals $(XY)_1(Q^{'}, P^{'})$. For brevity of notation, we shall write $X(Q^{'}, P^{'})$ for the Wigner transform $X_1(Q^{'}, P^{'})$ of the operator X. Taking the Wigner transform of the above Belavkin filter equation then gives us

$$\partial_t\rho(t, Q^{'}, P^{'}) = \sum_k L_h(A\rho, C)(L_h(A, \rho)(A_k(t, Q^{'}, P^{'})\rho(t, Q^{'}, P^{'}))C(t, Q^{'}, P^{'})))$$

plus terms involving the Levy processes. Now, we observe that with $X(Q^{''}, Q^{'}) = \langle Q^{''}|X|Q^{'}\rangle$, we have that

$$X_1(Q^{'}, P^{'}) = c.\int X(Q^{'} + q/2, Q^{'} - q/2)exp(iP^{'}.q/h)dq$$

and so

$$\int X_1(Q^{'}, P^{'})dQ^{'}dP^{'} = \int X(Q^{'} + q/2, Q^{'} - q/2)\delta(q)dqdQ^{'} = \int X(Q^{'}, Q^{'})dQ^{'} = Tr(X)$$

Therefore,

$$Tr(\rho(t)C) = \int(\rho(t)C)_1(Q^{'}, P^{'})dQ^{'}dP^{'} = \int L_h(\rho, C)(\rho_1(Q^{'}, P^{'})C_1(Q^{'}, P^{'}))dQ^{'}dP^{'}$$

$$= \int\rho_1(Q^{'}, P^{'})C_1(Q^{'}, P^{'})dQ^{'}dP^{'}$$

because for any function $F(Q^{'}, P^{'})$, we have

$$\int\nabla^X_{P'},\nabla^Y_{Q'} - \nabla^X_{Q'}.\nabla^Y_{P'})(F(Q^{'}, P^{'}))dQ^{'}dP^{'} = 0$$

This gives us finally the Wigner transformed Belavkin filter equation:

$$\partial_t\rho(t, Q^{'}, P^{'}) = \sum_k L_h(A\rho, C)(L_h(A, \rho)(A_k(t, Q^{'}, P^{'})\rho(t, Q^{'}, P^{'}))C(t, Q^{'}, P^{'})))$$

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$$+\sum_{n(1),\ldots,n(m),s}\Pi_{k=1}^{m}\left(\int\rho(t,Q',P')C_k(Q',P')dQ'dP'\right)^{n(k)}.\,D(t,Q',P'\Big|n(1),\ldots,n(m),s)dV_s(t)$$

We can expand this equation in powers of Planck's constant h and observe that the zeroth order term is the classical Fujisaki-Kushner-Kallianpur filter with $\rho(t,Q',P')$ interpreted as the probability density of $(Q(t),P(t))$ assuming commutativity. It should however be noted that the complex function $\rho(t,Q',P')$ is not a probability density, its marginals, however, are the probability densities of $Q(t)$ and $P(t)$.

7. Boson-Fermion noise in quantum systems

Here, we give a quantum field theoretic description of how quantum noise can arise in physical theories.

L_b^a is an odd system operator for $\sigma(a,b)=1(mod2)$ and it is an even operator for $\sigma(a,b)=0(mod2)$. Here, $\sigma(a,b)=\sigma(a)+\sigma(b)$ where $a,b=0,1,\ldots,N$. Supersymmetric quantum noise processes [5] are defined by

$$d\xi_b^a(t)=G(t)^{\sigma(a,b)}d\Lambda_b^a(t)$$

where

$$G(t)=(-1)^{\Lambda(t)},\ \Lambda(t)=\sum_{c=r+1}^{N}\Lambda_c^c(t)$$

and $\sigma(a)=0, a=0,1,\ldots,r$, $\sigma(a)=1$ for $a=r+1,\ldots,N$. It follows that $\sigma(a,b)=0$ modulo 2 when either $0\le a,b\le r$ or else $r+1\le a,b\le N$ while $\sigma(a,b)=1$ when either $0\le a\le r, r+1\le b\le N$ or $r+1\le a\le N, 0\le b\le r$. The HP qsde is given by

$$dU(t)=(L_b^a d\xi_a^b(t))U(t)$$

where $L_b^a d\xi_d^c(t)=L_b^a\otimes d\xi_d^c(t)$ with $\otimes$ being the Z_2 graded tensor product between system and noise operators. This means that for example if L is a system operator and W a noise operator of define parities, then

$$L\otimes X=p(L)p(X)X\otimes L,\ p(L)=(-1)^{\sigma(L)},\ p(W)=(-1)^{\sigma(W)}$$

so that we have

$$(L\otimes X)^*=X^*\otimes L^*=p(L)p(X)L^*\otimes X^*$$

Note that

$$\sigma(\xi_b^a)=\sigma(a,b)$$

and hence $L_b^a d\xi_a^b$ is always a Bosonic operator since L_b^a and ξ_b^a have the same parities, namely $\sigma(a,b)$.

Example: Consider the second quantized Schrodinger wave field $\psi(x), x = (t, r)$ in the presence of an electromagnetic field

$$A(t, r) = A_c(t, r) + dW(t, r)/dt, A_0(t, r) = A_{0c}(t, r) + dW_0(t, r)/dt$$

where A_c, A_0 are classical fields while W, W_0 are quantum Bosonic noise fields. The Lagrangian density is

$$L(\psi, \psi^*, \partial_t\psi) = \psi^*(i\partial_t - (-i\nabla + eA)^2/2m + eA_0)\psi$$

ψ is taken as the canonical position field with its corresponding momentum field given by

$$P = \partial L/\partial\partial_t\psi = i\psi^*$$

Hence, the Hamiltonian density is

$$\mathcal{H} = P\partial_t\psi - L = \psi^*[(-i\nabla + eA)^2/2m - eA_0]\psi = -iP[(-i\nabla + eA)^2/2m - eA_0]\psi$$

as expected. The Hamiltonian equations are

$$\partial_t\psi = \delta\mathcal{H}/\delta P = -i[(-i\nabla + eA)^2/2m - eA_0]\psi$$

and its complex conjugate. So the first quantized unitary evolution equation is

$$dU(t) = -i[((p + eA_c)^2/2m - eA_{0c})dt + (e^2/2m)(dW/dt)^2dt +$$

$$(e/2m)((dW/dt, p + eA_c) + (p + eA_c, dW/dt))dt - edW_0]U(t)$$

where $p = -i\nabla$ is the first quantized momentum operator. Now, we write

$$dW = L^a_b dd\xi^b_a, dW_0 = M^a_b d\xi^b_a$$

so that

$$(dW)^2 = L^a_b d\xi^b_a . L^c_d d\xi^d_c = p(L^c_d).\, p(\xi^b_a) L^a_b L^c_d \epsilon^b_c d\xi^d_a$$

$$= (-1)^{\sigma(c,d)\sigma(a,b)}(L\epsilon L)^a_d d\xi^d_a$$

and hence,

$$(dW/dt)^2 dt = (-1)^{\sigma(c,d)\sigma(a,b)}(L\epsilon L)^a_d d\xi^d_a/dt = F^a_d(t, r)d\xi^d_a(t)/dt$$

where

$$F^a_d(t, r) = (-1)^{\sigma(c,d)\sigma(a,b)}(L\epsilon L)^a_d$$

We write $dt \approx \Delta^{-1}$ and redefine F^a_d as $e^2F^a_d/2m\Delta$ and and then obtain the noisy Schrodinger evolution equation as

$$dU(t) = -i[(((p + eA_c)^2/2m - eA_{0c})dt + F^a_b d\xi^b_a(t) + (e/2m)((L^a_b, p + eA_c) + (p + eA_c, L^a_b)) - eM^a_b)d\xi^b_a(t)]U(t)$$

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Note: The approximation of replacing dt by Δ in the term $(e^2/2m)(dW)^2/dt$ can be justified because $e^2/2m$ is very small and hence $e^2/2m.\, dt$ can be assumed to be a finite quantity as compared with the $O(e)$ terms. Writing therefore

$$Q_b^a = F_b^a + (e/2m)((L_b^a, p + eA_c) + (p + eA_c, L_b^a)) - eM_b^a)$$

and

$$H_0(t) = (p + eA_c(t, r))^2/2m - eA_{0c}(t, r),$$

it follows that $U(t)$ satisfies the following Hudson-Parthasarathy qsde:

$$dU(t) = -i[H_0(t)dt + Q_b^a(t)d\xi_a^b(t)].\, U(t)$$

or equivalently,

$$dU(t) = P_b^a(t)\xi_a^b(t).\, U(t)$$

where

$$P_b^a(t) = -iH_0(t)\delta_{a0}\delta_{b0} - iQ_b^a(t)$$

Note that $P_b^a(t)$ is a system operator having the parity $\sigma(a, b)$

8. Some remarks regarding Z2-grading of system and noise operators

Taking into account the Z_2 grading of system operators as well as of the noise operators, we have the supersymmetric HPQSDE

$$dU(t) = L_b^a d\xi_a^b(t).\, U(t)$$

where both L_b^a and ξ_b^a have the Z_2-grading $\sigma(a, b)$. So

$$dj_t(X) = d(U(t)^* XU(t)) =$$

$$dU(t)^* XU(t) + U(t)^* XdU(t) + dU(t)^* X.\, dU(t)$$

$$= U(t)^* d\xi_b^a(t).\, (L_b^a)^* XU(t) + U(t)^* XL_b^a d\xi_a^b(t)U(t) + U(t)^* d\xi_b^a(t)(L_b^a)^* XL_d^c d\xi_c^d(t)U(t)$$

$$= (-1)^{\sigma(a,b)\sigma(X)} U(t)^* (L_b^a)^* XU(t)d\xi_b^a(t) + U(t)^* XL_b^a U(t)d\xi_a^b(t)$$

$$+(-1)^{\sigma(X)(\sigma(a,b)+\sigma(c,d))}.\, U(t)^* (L_a^b)^* XL_d^c.\, U(t).\, \epsilon_c^a d\xi_b^d(t) =$$

$$(-1)^{\sigma(a,b)\sigma(X)} U(t)^* (L_b^a)^* XU(t)d\xi_b^a(t) + U(t)^* XL_b^a U(t)d\xi_a^b(t)$$

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$$+(-1)^{\sigma(X)\sigma(b,d)}.\,U(t)^*(L_a^b)^*XL_d^c.\,U(t).\,\epsilon_c^a d\xi_b^d(t)$$

$$= j_t(\phi_b^a(X))d\xi_a^b(t)$$

where

$$\phi_b^a(X) = (-1)^{\sigma(a,b)\sigma(X)}(L_a^b)^*X + XL_b^a + (-1)^{\sigma(X).\sigma(a,b)}(L_d^a)^*XL_b^c\epsilon_c^d$$

$$\mathrm{E}(dj_t(X).\,dY_o(t)^m \mid \eta_o(t)) =$$

$$\mathrm{E}(j_t(\phi_b^a(X))d\xi_a^b(t).\,j_t(M_d^c(m))d\xi_c^d(t) \mid \eta_o(t)) =$$

$$= (-1)^{\sigma(a,b).\sigma(c,d)}\pi_t(G(t)^{\sigma(a,d)}\phi_b^a(X)M_d^c(m))\epsilon_c^b u_d(t)\bar{u}_a(t)dt$$

For $m \geq 1$,

$$\mathrm{E}(d\pi_t(X).\,dY_o(t)^m \mid \eta_o(t)) =$$

$$\mathrm{E}(G_{k,t}(X)dY_o(t)^{k+m} \Big| \eta_o(t)) = G_{k,t}(X)\pi_t(M_b^a(k+m))u_b(t)\bar{u}_a(t)dt$$

Here, we do not see the presence of the grading effect. Note that the output measurement noise $Y_o(t)$ is always Bosonic, because it is a superposition of Bosonic quantum Brownian noise Bosonic Poisson noise and Fermionic counting noise. Fermionic quantum Brownian noise is not included in the measurement because it would destroy the non-demolition property.

9. Quantum noise in filtering theory for quantum fields

Consider the field Lagrangian

$$L(\phi(x), \phi_{,\mu}(x), W^\mu(x)) = L_0(\phi(x), \phi_{,\mu}(x)) + L_{1\mu}(\phi(x))W^\mu(x)$$

where L_0 is the unperturbed Lagrangian density of the quantum field ϕ and the second term is the noisy perturbation. $W^\mu(x)$ are quantum noise fields. The Euler-Lagrange equations are

$$\partial_\mu \frac{\partial L_0}{\partial \phi_{,\mu}(x)} - \frac{\partial L_0}{\partial \phi(x)} = \frac{\partial L_{1\nu}}{\partial \phi(x)}W^\nu(x)$$

For example, suppose

$$L_0(\phi(x)), \phi_{,\mu}(x)) = (1/2)\phi_{,\mu}(x)A_{\mu\nu}(\phi(x))\phi_{,\nu}(x) - V(\phi(x))$$

Then the equations of motion are

$$(A_{\mu\nu}(\phi)\phi_{,\nu})_{,\mu} - (1/2)\phi_{,\mu}A'_{\mu\nu}(\phi)\phi_{,\nu} + V'(\phi) = L'_{1\mu}(\phi)W^\mu$$

which can be expanded as

$$A_{\mu\nu}(\phi)\phi_{,\mu\nu} + A'_{\mu\nu}(\phi)\phi_{,\mu}\phi_{,\nu} - (1/2)A'_{\mu\nu}(\phi)\phi_{,\mu}\phi_{,\nu} + V'(\phi) - L'_{1\mu}(\phi)W^\mu$$

or equivalently,

$$A_{\mu\nu}(\phi)\phi_{,\mu\nu} + \Gamma_{\mu\nu}(\phi)\phi_{,\mu}\phi_{,\nu} + V'(\phi) - L'_{1\mu}(\phi)W^\mu = 0$$

where

$$\Gamma_{\mu\nu}(\phi) = A'_{\mu\nu}(\phi) - (1/2)A'_{\mu\nu}(\phi)$$

We now look at the Hamiltonian formulation of this noisy field theory: $\phi(x)$ is the canonical position field and its corresponding canonical momentum field is

$$P(x) = \partial L/\partial\phi_{,0} = \partial L_0/\partial\phi_{,0} =$$

$$A_{0\nu}(\phi)\phi_{,\nu} = A_{00}(\phi)\phi_{,0} + A_{0r}(\phi)\phi_{,r}$$

so

$$\phi_{,0} = A_{00}(\phi)^{-1}[P - A_{0r}(\phi)\phi_{,r}]$$

and hence, the Legendre transformation gives the Hamiltonian density as

$$\mathcal{H} = P\phi_{,0} - L = (1/2)[A_{00}(\phi)\phi^2_{,0} + A_{rs}(\phi)\phi_{,r}\phi_{,s}] + V(\phi) - L_{1\mu}(\phi)W^\mu$$

$$= (1/2A_{00}(\phi)).\,(P - A_{0r}(\phi)\phi_{,r})^2 + (1/2)A_{rs}(\phi)\phi_{,r}\phi_{,s} + V(\phi) - L_{1\mu}(\phi)W^\mu$$

$$= \mathcal{H}_0(P, \phi, \nabla\phi) - L_{1\mu}(\phi)W^\mu$$

where $\mathcal{H}_0$, the unperturbed Hamiltonian, is given by [6]

$$\mathcal{H}_0 =$$

$$(P - A_{0r}(\phi)\phi_{,r})^2/2A_{00}(\phi) + (1/2)A_{rs}(\phi)\phi_{,r}\phi_{,s} + V(\phi)$$

$$= (P - \mathbf{a}(\phi)^T\nabla\phi)^2/2a_0(\phi) + (1/2)(\nabla\phi)^T\mathbf{A}(\phi)\nabla\phi + V(\phi)$$

with

$$\mathbf{a}(\phi) = ((A_{0r}(\phi)))_{r=1}^3,\ a_0(\phi) = A_{00}(\phi),\ \mathbf{A}(\phi) = ((A_{rs}(\phi)))_{1\le r,s\le 3}$$

More generally, we can consider a situation in which $L_{1\mu} = L_{1\mu}(\phi, \nabla\phi)$ depends on $\phi, \nabla\phi = (\phi_{,r})$ and in that case the equations of motion are

$$\partial_\mu \frac{\partial L_0}{\partial\phi_{,\mu}} - \frac{\partial L_0}{\partial\phi} + V'(\phi) - \frac{\partial L_{1\mu}}{\partial\phi}W^\mu$$

$$+\partial_r[\frac{\partial L_{1\mu}}{\partial\phi_{,r}}W^\mu] = 0$$

Note that the time derivative of the noise W^μ does not appear anywhere. Therefore, we can assume W^μ to be a white Gaussian or white Poisson or more generally, a white Levy noise field. When L_0 has the special form suggested above, the Hamiltonian density, in this case, has the same form as above with $L_{1\mu}$ being a function of $\phi, \phi_{,r}$. Let us now look at the Feynman path integral method to this Lagrgangian density/Hamiltonian density for calculating the evolution of the wave function: Define the noisy kernel

$$K(T, \phi(T) | 0, \phi(0), W) = \int exp(i\int_{D \times [0,T]} L_0(\phi(x), \phi_{,\mu}(x))d^4x +$$

$$i\int_{D \times [0,T]} L_{1\mu}(\phi(x), \phi_{,r}(x))W^\mu(x)d^4x)D\phi(D \times (0, T))$$

Then modulo a multiplicative constant, this kernel describes the correct Schrodinger evolution of the wave functional of the field ϕ. Here, D is the spatial region, usually, a connected open subset of R^3, where the field is defined.

Schrodinger's equation for the wave functional can be expressed as

$$i\partial_t\psi(t, \phi) = (\int \mathcal{H}(\phi(r), \nabla\phi(r), -i\delta/\delta\phi(r) | W(t, r), \nabla W(t, r))d^3r)\psi(t, \phi)$$

which in the above mentioned special case, assumes the form

$$i\partial_t\psi(t, \phi) = [\int [(i\delta/\delta\phi(r) + \mathbf{a}(\phi(r))^T\nabla\phi(r))^2/2a_0(\phi(r))$$

$$+(1/2)(\nabla\phi(r))^T\mathbf{A}(\phi(r))\nabla\phi(r) + V(\phi(r))]d^3r]\psi(t, \phi)$$

$$-[\int L_{1\mu}(\phi(r), \nabla\phi(r))W^\mu(t, r)d^3r]\psi(t, \phi)$$

Defining the noise field

$$V^\mu(t, r) = \int_0^t W^\mu(s, r)ds$$

and adding an Ito correction term to ensure unitarity of the evolution, this equation can be expressed as a stochastic partial differential equation:

$$d_t\psi(t, \phi) = -i[\int [(i\delta/\delta\phi(r) + \mathbf{a}(\phi(r))^T\nabla\phi(r))^2/2a_0(\phi(r))$$

$$+(1/2)(\nabla\phi(r))^T\mathbf{A}(\phi(r))\nabla\phi(r) + V(\phi(r))]d^3r]\psi(t, \phi)dt$$

$$+i[\int L_{1\mu}(\phi(r), \nabla\phi(r))dV^\mu(t, r)d^3r]\psi(t, \phi)$$

$$-(1/2)[\int L_{1\mu}(\phi(r), \nabla\phi(r)). L_{1\nu}(\phi(r'), \nabla\phi(r'))dV^\mu(t, r). dV^\nu(t, r')d^3rd^3r']\psi(t, \phi)$$

For example, if

$$V^\mu(t, r) = B^\mu(t, r) + \int f^\mu(r, s)N(t, ds)$$

with B^μ a Brownian field and N a Poisson field, so that

$$dV^\mu(t,r).dV^\nu(t,r') = \delta^{\mu\nu}dt.\delta^3(r-r') + \int f^\mu(r,s)f^\nu(r',s)dN(t,ds),$$

the Ito correction term simplifies to

$$\int L_{1\mu}(\phi(r),\nabla\phi(r)).L_{1\nu}(\phi(r'),\nabla\phi(r'))dV^\mu(t,r).dV^\nu(t,r')d^3rd^3r' =$$

$$dt\int(\sum_\mu L_{1\mu}(\phi(r),\nabla\phi(r))^2)d^3r +$$

$$\int L_{1\mu}(\phi(r),\nabla\phi(r)).L_{1\nu}(\phi(r'),\nabla\phi(r'))f^\mu(r,s)f^\nu(r',s)dN(t,ds)d^3rd^3r' =$$

$$dt\parallel L_1(\phi,\nabla\phi)\parallel^2 + \int <f(.,s),L_1(\phi,\nabla\phi)>^2 dN(t,ds)$$

where

$$\|L_1(\phi,\nabla\phi)\|^2 = \int(\sum_\mu L_{1\mu}(\phi(r),\nabla\phi(r))^2)d^3r$$

and

$$<f(.,s),L_1(\phi,\nabla\phi)> = \int f^\mu(r,s)L_{1\mu}(\phi(r),\nabla\phi(r))d^3r$$

In deriving this, we have used Ito's formula for space-time Brownian and Poisson fields in the form

$$dB^\mu(t,r).dB^\nu(t,r') = dt.\delta^{\mu\nu}\delta^3(r-r')$$

and

$$dN(t,ds).dN(t,ds') = \delta_{s,s'}dN(t,ds)$$

10. How to add quantum noise to a supersymmetric field theory

First consider the case of quantum electrodynamics, ie, a cavity enclosing photons, electrons and positrons. There are two quantum fields here, one, the Maxwell photon four potential field $A_\mu(x)$ and two, the Dirac electron-positron wave field $\psi(x)$. The total Lagrangian density for these two fields is

$$L = (-1/4)F_{\mu\nu}(x)F^{\mu\nu}(x) + \psi(x)^*[\alpha^\mu.(i\partial_\mu + eA_\mu(x)) - \beta m]\psi(x)$$

where

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

is the electromagnetic antisymmetric field tensor. We can add noise to this in two ways: One add a noisy electromagnetic field $A_{N\mu}(x)$ to the interaction term between the Dirac current and the Maxwell photon, ie, modify the interaction term $e\psi(x)^*\alpha^\mu\psi(x)A_\mu(x)$ to $e\psi(x)^*\alpha^\mu\psi(x)(A_\mu(x)+A_{N\mu}(x))$. This can be done by inserting a laser into the cavity. Two, by inserting a wire carrying noisy current J_N^μ into the cavity, the total current density becomes $-e\psi(x)^*\alpha^\mu\psi(x) + J_N^\mu(x)$ and the interaction term gets modified from $e\psi(x)^*\alpha^\mu\psi(x)A_\mu(x)$ to

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$(e\psi(x)^* \alpha^\mu \psi(x) - J_N^\mu(x))A_\mu(x)$. We note that both the two kinds of noises that we have added, namely $A_{N\mu}$ and J_N^μ are noisy Bosonic fields. The question therefore is how to add a noisy Fermionic field to the interaction ? The clue is provided by supergravity wherein, we have a metric superfield $H^\mu(x,\theta)$ comprising gravitons, gravitinos and auxiliary fields, and a current superfield $\Theta_\mu(x)$ that is derived from Chiral matter superfields $\Phi(x,\theta)$. The interaction Lagrangian density between gravity and matter in supergravity theories (See Steven Weinberg, "The quantum theory of fields vol.III") is given by

$$[H^\mu \Theta_\mu]_D = \int H^\mu \Theta_\mu . d^4\theta$$

and the interaction action is of course

$$\int [H^\mu \Theta_\mu]_D d^4x = \int H^\mu \Theta_\mu . d^4x . d^4\theta$$

In weak supergravity theories, this interaction term contains both the energy-momentum/graviton interaction term $\int T^{\mu\nu} h_{\mu\nu} f^4 x$ as well as the supercurrent-gravitino interaction. Note that both of these interaction Lagrangians are Bosonic, the former being trivially true and the latter being true because both the supercurrent and gravitino are Fermionic fields. Now, we can add a noise current super-field $\Theta_{N\mu}$ to the already existing supercurrent Θ_{mu} thereby generating the additional noisy interaction action $\int H^\mu \Theta_{N\mu} d^4x d^4\theta$ and then proceed with setting up the corresponding Hudson-Parthasarathy noisy Schrodinger equation. Just as we added a random current to the interaction term in quantum electrodynamics, here we add a random current superfield to the existing interactions in super-gravity. Now we would also like to add noisy gauge superfields to the interactions existing in supersymmetric field theories. In a supersymmetric field theory not involving gravity, the matter-gauge interaction is given by the Lagrangian density $[\Phi^* exp(-V).\Phi]_D$ where $V = V^A(x,\theta)t_A$ is the gauge superfield and $\Phi(x,\theta)$ is the left chiral matter superfield (Steven Weinberg, "The quantum theory of fields vol.III). We add to $V = V^A t_A$, a noisy gauge superfield $V_N = V_N^A t_A$ so that the under this noisy interaction, the matter-gauge interactions are given by

$$[\Phi^* . exp(-V - V_N). \Phi]_D$$

and hence the noise component of this interaction Lagrangian density is given by

$$[\Phi^* . exp(-V - V_N). \Phi]_D - [\Phi^* . exp(-V). \Phi]_D$$

$$= [\Phi^* . (exp(-V - V_N) - exp(-V)). \Phi]_D$$

Note that the gauge superfields V, V_N contain only $\theta^k, k = 2,3,4$ terms, ie, quadratic, cubic and biquadratic terms in the Grassmannian parameters $\theta_m, m = 1,2,3,4$. It can be shown that by using a supergauge transformation, any super field can be brought into this form. A superfield in this form is said to be in the Wess-Zumino gauge.

11. Quantum Boltzmann equation in the Weyl-Moyal-Wigner transform domain

The one particle quantum Boltzmann equation for a system of N identical particles is

$$\rho'(t) = -i[H_0, \rho(t)] - i(N-1)Tr_2[V \otimes \rho(t) \otimes \rho(t)]$$

We've already noted that the Wigner transform of the product of two observables X, Y is given in terms of the Wigner transform of the individual operators by the formula

$$(XY)_1(Q', P') = L_h(X, Y)(X_1 Y_1)(Q', P')$$

where

$$L_h(X, Y) = exp((ih/2)(\nabla^X_{Q'}.\nabla^Y_{P'} - \nabla^X_{P'}.\nabla^X_{Q'}))(X_1(Q', P').Y_1(Q', P'))$$

Equivalently, denoting the classical Poisson bracket opeartor by $\mathcal{P}(X, Y)$, we have

$$\mathcal{P}(X, Y)(X_1.Y_1) = (\nabla^X_{Q'}.\nabla^Y_{P'} - \nabla^Y_{Q'}.\nabla^X_{P'})(X_1 Y_1)(Q', P')$$
$$= \nabla_{Q'}X_1.\nabla_{P'}Y_1 - \nabla_{Q'}Y_1.\nabla_{P'}X_1 = \{X_1, Y_1\}\}$$

and we can write

$$(XY)_1 = exp((ih/2)\mathcal{P}(X, Y))(X_1 Y_1)$$

and hence,

$$[X, Y]_1 = (XY)_1 - (YX)_1 = 2i.sin((h/2)\mathcal{P}(X, Y))(X_1 Y_1) = h\{X_1, Y_1\} + O(h^3) =$$
$$h\mathcal{P}(X, Y)(X_1 Y_1) + O(h^3)$$

Also V is a two particle multiplication operator:

$$< Q_1' Q_2' | V | Q_1'', Q_2'' > = V(Q_1', Q_2').\delta(Q_1' - Q_1'').\delta(Q_2' - Q_2'') = V(Q_1', Q_2'; Q_1'', Q_2'')$$

and hence its Wigner transform is

$$V(Q_1', Q_2', P_1', P_2') = \int V(Q_1' + q_1'/2, Q_2' + q_2'/2; Q_1' - q_1'/2, Q_2' - q_2'/2)exp(i(P_1'.q_1'/h + P_2'.q_2'/h))dq_1'.dq_2'$$
$$= \int V(Q_1' + q_1'/2, Q_2' + q_2'/2)\delta(q_1').\delta(q_2')exp(i(P_1'.q_1'/h + P_2'.q_2'/h))dq_1'.dq_2'$$
$$= V(Q_1', Q_2')$$

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ie the Wigner transform of this two particle potential is independent of the momenta. We have also observed that for two operators X, Y,

$$Tr(XY) = \int X_1(Q', P')Y_1(Q', P')dQ'dP'$$

and hence

$$(Tr_2[V, \rho(t) \otimes \rho(t)])_1(Q_1', P_1') = 2i\int sin((h/2)\mathcal{P}(V, \rho \otimes \rho))[V_1(Q_1', Q_2', P_1', P_2').\rho_1(t, Q_1', P_1')\rho_1(t, Q_2', P_2')]dQ_2'dP_2'$$

$$= 2i\int sin((h/2)\mathcal{P}(V, \rho \otimes \rho))[V_1(Q_1', Q_2')\rho_1(t, Q_1', P_1')\rho_1(t, Q_2', P_2')]dQ_2'dP_2'$$

Note that

$$\mathcal{P}(V, \rho \otimes \rho) = \nabla^V_{Q_1', Q_2'}.\nabla^{\rho\otimes\rho}_{P_1', P_2'} - \nabla^{\rho\otimes\rho}_{Q_1', Q_2'}.\nabla^V_{P_1', P_2'}$$

$$= (\nabla^V_{Q_1'} + \nabla^V_{Q_2'}).(\nabla^\rho_{P_1'} + \nabla^\rho_{P_2'})$$

$$-(\nabla^\rho_{Q_1'} + \nabla^\rho_{Q_2'}).(\nabla^V_{P_1'} + \nabla^V_{P_2'})$$

$$= \nabla^V_{Q_1'}.\nabla^\rho_{P_1'} + \nabla^V_{Q_2'}.\nabla_{P_2'} - \nabla^\rho_{Q_1'}.\nabla^V_{P_1'} - \nabla^\rho_{Q_2'}.\nabla^V_{P_2'}$$

$$= (\nabla^V_{Q_1'} + \nabla^V_{Q_2'}).(\nabla^\rho_{P_1'} + \nabla^\rho_{P_2'})$$

$$= \nabla_{Q_1'}.\nabla^\rho_{P_1'} + \nabla_{Q_2'}.\nabla_{P_2'}$$

where in the penultimate step, we have used the fact that V_1 is independent of P_1', P_2'. The integral of the gradient w.r.t P_2' over P_2' cancels out in the above evaluation of the partial trace using the Wigner transform and we get the $O(h)$ approximation as

$$(Tr_2[V, \rho(t) \otimes \rho(t)])_1(Q_1', P_1') =$$

$$2i\int sin((h/2)\mathcal{P}(V, \rho \otimes \rho))[V_1(Q_1', Q_2')\rho_1(t, Q_1', P_1')\rho_1(t, Q_2', P_2')]dQ_2'dP_2'$$

$$= ih\int \mathcal{P}(V, \rho \otimes \rho)[V_1(Q_1', Q_2')\rho_1(t, Q_1', P_1')\rho_1(t, Q_2', P_2')]dQ_2'dP_2' + O(h^3)$$

$$= ih\int ((\nabla^V_{Q_1'}.\nabla^\rho_{P_1'}).[V_1(Q_1', Q_2')\rho_1(t, Q_1', P_1')\rho_1(t, Q_2', P_2')]dQ_2'dP_2' + O(h^3)$$

$$= ih\int [(\nabla_{Q_1'}V_1(Q_1', Q_2')).(\nabla_{P_1'}\rho_1(t, Q_1', P_1'))]\rho_1(t, Q_2')dQ_2' + O(h^3)$$

where

$$\rho_1(t, Q_2') = \int \rho_1(t, Q_2', P_2')dP_2'$$

is the single particle probability density of the position of one particle at time t. In fact, it is now easy to see that we have the following exact formula:

$$(Tr_2[V, \rho(t) \otimes \rho(t)])_1(Q_1', P_1')$$

$$= 2i\int sin((h/2)(\nabla^{V}_{Q_1'}.\nabla^{\rho}_{P_1'}))(V_1(Q_1', Q_2')\rho_1(t, Q_1', P_1'))\rho_1(t, Q_2')dQ_2'$$

$$= 2i\sum_{n\geq 0}((h/2)^{2n+1}/(2n+1)!)(-1)^n\int(\nabla^{V}_{Q_1'}.\nabla^{\rho}_{P_1'})^{2n+1}(V_1(Q_1', Q_2')\rho_1(t, Q_1', P_1')).\rho_1(t, Q_2')dQ_2'$$

$$= 2i\sum_{n\geq 0}((h/2)^{2n+1}/(2n+1)!)(-1)^n\int[(\nabla_{Q_1'})^{\otimes 2n+1}V_1(Q_1', Q_2')]^T[(\nabla_{P_1'})^{\otimes 2n+1}\rho_1(t, Q_1', P_1')].\rho_1(t, Q_2')dQ_2'$$

$$= 2i\sum_{n\geq 0}((h/2)^{2n+1}/(2n+1)!)(-1)^n[(\nabla_{Q_1'})^{\otimes 2n+1}(\int V_1(Q_1', Q_2')\rho(t, Q_2')dQ_2')]^T[(\nabla_{P_1'})^{\otimes 2n+1}\rho_1(t, Q_1', P_1')]$$

12. Weyl-Moyal-Wigner domain representation of the Hudson-Parthasarathy noisy Schrodinger-Heisenberg equation

12.1. Gisin-Percival stochastic Schrodinger equation from HPQSDE

The differential equation (qsde) for the unitary evolution operator $U(t)$ is given by

$$dU(t) = L^a_b(Q, P)d\Lambda^b_a(t).U(t)$$

This leads immediately to the Evans-Hudson flow also called the quantum noisy Heisenberg equation for an initial system observable X:

$$dj_t(X) = j_t(\phi^a_b(X))d\Lambda^b_a(t)$$

where j_t is the quantum noisy Heisenberg evolution of a system observable:

$$j_t(X) = U(t)^* XU(t)$$

In particular, if we assume that the joint state of the system and bath is pure and is given by

$$|\psi(t, u)> \otimes |e(u)> <e(u)| = U(t)(|\psi(0, u)> \otimes |e(u)>)$$

where u is a coherent parameter, then

$$d[|\psi(t, u)> \otimes |e(u)>] =$$

$$(L^a_b(Q, P)|\psi(t, u)>) \otimes d\Lambda^b_a(t)|e(u)>$$

and since

$$d\Lambda^b_a(t)|e(u)> = (dA_a(t)^* dA_b(t)/dt)|e(u)> = u_b(t).dA_a(t)^*|e(u)>$$

$$= u_b(t)(dA_a(t)^* + dA_a(t) - u_a(t)dt)|e(u)> = u_b(t)(dB_a(t) - u_a(t)dt)|e(u)>$$

where

$$B_a(t) = A_a(t)^* + A_a(t), t \geq 0, a = 1, 2, \ldots, N$$

are commutative independent classical Brownian motions with respective drifts $\int_0^t u_a(s)ds$, our HPQSDE simplifies to

$$d|\psi(t,u)>=u_b(t)(dB_a(t)-u_a(t)dt)L_b^a(Q,P))|\psi(t,u)>$$

Note that in this expression, it is to be understood that

$$A_0(t)=A_0(t)^*=t$$

so that

$$B_0(t)=2t$$

and hence, since $u_0(t)=1$,

$$dB_0(t)-u_0(t)dt=dt$$

Thus, we can express the above equation s

$$d|\psi(t,u)>=[L_0^0(Q,P)dt+\theta(a\geq 1)u_b(t)(dB_a(t)-u_a(t)dt)L_b^a(Q,P))]|\psi(t,u)>$$

We note that for $a\geq 1$,

$$V_a(t)=B_a(t)-\int_0^t u_a(s)ds$$

$a=1,2,\ldots,N$ are independent, classical commutative standard Brownian motion processes (without drift) in the coherent state $|e(u)>$ and it terms of these processes, we can write the above differential equation as

$$d|\psi(t,u)>=[L_0^0(Q,P)dt+\theta(a\geq 1)u_b(t)dV_a(t)L_b^a(Q,P))]|\psi(t,u)>$$

The general solution for the mixed state of system and bath is then given by a path integral expression of the form

$$\rho(t)=\int(|\psi(t,u)><\psi(t,u)|\otimes|e(u)><e(u)|)f(u,\bar{u})Du D\bar{u}$$

$$=\int(|\psi(t,u)>\otimes|e(u)>).(<\psi(t,u)|\otimes<e(u)|)f(u,\bar{u})Du.D\bar{u}$$

Remark: If we regard system space operators, ie, functions of Q,P only not involving noise as matrices, then the above differential equation can be interpreted as classical matrix stochastic differential equations driven by a classical commutative standard Brownian motion vector processes

$$V(t)=[V_1(t),\ldots,V_N(t)]$$

and this equation has the form

$$dx(t) = [C_0(t)dt + \sum_{k=1}^{N} C_k(t)dV_k(t)]x(t)$$

This equation takes into account the fact that the operators $L_b^a(Q,P)$ can in general, depend explicitly on t, ie, $L_b^a(t,Q,P)$. The matrix $C_0(t)$ represents the operator $L_0^0(t,Q,P)$ while the matrix $C_a(t)$ represents the operator $\sum_{b=0}^{N} u_b(t)L_b^a(t,Q,P)$ for $a = 1,2,\ldots,N$.

Remark: In the literature, the above classical matrix stochastic differential equation derived from the HPQSDE when the bath is always in a fixed coherent state is known as the Gisin-Percival equation and also termed as a stochastic Schrodinger equation. The Belavkin filter equation is another example of a stochastic Schrodinger equation. When noise in the Schrodinger equation is classical, the equation is generally termed as a stochastic Schrodinger equation in contrast to the HPQSDE whjch is a quantum noisy Schrodinger equation, by quantum noisy, it is understood that the noise is non-commutative.

12.2. Weyl-Moyal-Wigner transform of the HP quantum noisy Heisenberg equation

The quantum noisy Heisenberg qsde for an observable X are generally described by an Evans-Hudson flow of the form:

$$dj_t(X) = j_t(\theta_b^a(X))d\Lambda_a^b(t)$$

where θ_b^a are the structure maps acting on the space of system observables linearly. This equation can be put into the form

$$dX(t) = \theta_b^a(X)(t)d\Lambda_a^b(t)$$

where we write $X(t)$ for $j_t(X)$. In the position space-representation for system operators, this as the form

$$dX(t,Q',Q'') = \theta_b^a(X)(t,Q',Q'')d\Lambda_a^b(t)$$

and taking the Wigner-transform of this equation, we get

$$dX_1(t,Q',P') = \theta_b^a(X)_1(t,Q',P')d\Lambda_a^b(t)$$

We can express the structure maps as

$$\theta_b^a(X) = \sum_k A(k,a,b)XB(k,a,b)$$

where $A(k,a,b), B(k,a,b)$ are system operators. We thus have from the algebra homomorphism property of th3e map j_t,

$$j_t(\theta_b^a(X)) = \sum_k j_t(A(k,a,b))j_t(X)j_t(B(k,a,b))$$

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which can be expressed as

$$\theta_b^a(X)(t) = \sum_k A(t,k,a,b)X(t)B(t,k,a,b)$$

Note that for any system operator X, $X(t) = j_t(X)$ belongs the algebra $\mathcal{L}(\mathfrak{h} \otimes \Gamma_s(\mathcal{H}_{t]})$ of linear operators on the tensor product of the system Hilbert space $\mathfrak{h}$ and the Boson Fock space upto time t. In short, such an operator can be expressed as a functional of X and of the noise processes $\Lambda_b^a(s), s \le t, a, b = 0, 1, \ldots, N$ upto time t. It will be linear in X but highly nonlinear in the noise processes. Application of the Wigner transform to this equation gives

$$\theta_b^a(X)_1(t, Q', P') = \sum_k exp((ih/2)P(AX, B)(A_1(t, Q', P', k, a, b).$$

$$\times\, exp((ih/2)P(X, B))(X_1(t, Q', P')B_1(t, Q', P', k, a, b)))$$

where

$$P(X, Y) = exp((ih/2)(\nabla_{P'}^X . \nabla_{Q'}^Y - \nabla_{Q'}^X . \nabla_{P'}^Y))$$

Note that we can express

$$exp((ih/2)P(X, Y)) = \sum_{n,m\ge 0} ((ih/2)^{n+m}/n!m!)(-1)^m[(\nabla_{P'}^{X\otimes n})^T . (\nabla_{Q'}^Y)^{\otimes n} . (\nabla_{Q'}^{X\otimes m})^T . (\nabla_{P'}^Y)^{\otimes m}]$$

so that specifically, we have

$$exp((ih/2)P(X, Y))(X_1(Q', P')Y_1(Q', P')) =$$

$$\sum_{n,m\ge 0} ((ih/2)^{n+m}/n!m!)(-1)^m[((\nabla_{P'}^{\otimes n}X_1(Q', P'))^T . (\nabla_{Q'}^{\otimes n}Y_1(Q', P')). (\nabla_{Q'}^{\otimes m}X_1(Q', P'))^T . (\nabla_{P'}^{\otimes m}Y_1(Q', P'))]$$

13. Appendix 1. Feynman path integral approach to the Hudson-Parthasarathy noisy Schrodinger equation

The coherent state parameter at time t is given by

$$u_t = (u_t(s))_{s\ge 0}$$

The HPS equation is

$$dU(t) = (L_b^a(t, Q, P)d\Lambda_a^b(t)). U(t)$$

We are assuming here that the system space is defined by n canonical position operators Q and n canonical momentum operators P satisfying the canonical commutation relations. The system space operators L_b^a are time dependent functions of these canonical position and momentum operators and

$\Lambda_b^a(t)$ are the usual quantum noise processes of the Hudson-Parthasarathy theory. Quantum Ito corrections are used to ensure that the $L_b^{a'}s$ are constrained so that $U(t)$ is unitary for all t. The infinitesimal unitary evolution of the system and bath during the time interval $(t, t+dt)$ is given by

$$U(t, t+dt) = U(t+dt).\, U(t)^{-1} = (U(t) + dU(t)).\, U(t)^{-1} = 1 + dU(t).\, U(t)^{-1}$$

$$= 1 + L_b^a(t, Q, P)d\Lambda_a^b(t)$$

The matrix elements of this infinitesimal evolution in position-coherent state representation are given by

$$< Q(t+dt), \phi(u_{t+dt}) | U(t, t+dt) | Q(t), \phi(u_t) >$$

$$= \ < Q(t+dt), \phi(u_{t+dt}) | 1 + L_b^a(t, Q, P)d\Lambda_a^b(t) | Q(t), \phi(u_t) >$$

$$=< Q(t+dt), \phi(u_{t+dt}) | Q(t), \phi(u_t) > \ + \int < Q_(t+dt), \phi(u_{t+dt}) | P(t), \phi(v_t) > dP(t)dv_t$$

$$< P(t), \phi(v_t) | L_b^a(t, Q, P)d\Lambda_a^b(t) | Q(t), u_t >$$

where we have made use of the Glauber-Sudarshan non-orthogonal resolution of the identity

$$\int | \phi(v_t) > dv_t < \phi(v_t) | \ = 1$$

It should be noted that $| \phi(u) >$ are normalized coherent pure states in the Boson Fock space, ie,

$$| \phi(u) > \ = exp(-|u|^2/2) | e(u) >$$

where

$$|u|^2 = \int_0^\infty |u(t)|^2 dt$$

and

$$< e(u) | e(v) > \ = exo(< u | v >), \ < u | v > \ = \int_0^\infty < u(t) | v(t) > dt$$

Assuming now that the $P's$ are to the left of all the $Q's$ in the operator $L_b^a(t, Q, P)$, it follows from the above that

$$< Q(t+dt), \phi(u_{t+dt}) | U(t, t+dt) | Q(t), \phi(u_t) >$$

$$= \ < Q(t+dt), \phi(u_{t+dt}) \Big| P(t), \phi(v_t) > dP(t)dv_t < P(t), \phi(v_t) \Big| Q(t), \phi(u_t) > \ +$$

$$\int < Q_(t+dt), \phi(u_{t+dt}) | P(t), \phi(v_t) > dP(t)dv_t < P(t), \phi(v_t) | L_b^a(t, Q, P)d\Lambda_a^b(t) | Q(t), u_t) >$$

$$= \int < \phi(u_{t+dt}) | \phi(v_t) >< \phi(v_t) | \phi(u_t) > .\, exp(iP(t).\, (Q(t+dt) - Q(t))).$$

$$[1 + L_b^a(t, Q(t), P(t))u_{tb}(t)\bar{v}_{ta}(t)dt]dv_t dP(t)$$

Now observe that

$$< \phi(u_{t+dt}|\phi(v_t) >< \phi(v_t)|\phi(u_t) >=$$

$$= exp(-|u_{t+dt}|^2/2 - |v_t|^2 - |u_t|^2/2 + < u_{t+dt}\Big|v_t > + < v_t\Big|u_t >)$$

$$= exp(-|u_t|^2 - |v_t|^2 - Re < u_t^{'}\Big|u_t > dt + 2Re < u_t\Big|v_t > + < u_t^{'}\Big|v_t > dt)$$

$$= exp(-|u_t - v_t|^2 - (Re < u_t^{'}\Big|u_t - v_t > - iIm < u_t^{'}\Big|v_t >)dt)$$

Hence, we get

$$< Q(t+dt), \phi(u_{t+dt})|U(t, t+dt)|Q(t), \phi(u_t) >$$

$$= \int exp(iP(t).\, Q^{'}(t)dt + \bar{L}_b^a(t, Q(t), P(t))u_{tb}(t)v_{ta}(t)dt + (- Re < u_t^{'}|u_t - v_t > + iIm < u_t^{'}|v_t >)dt).$$

$$exp(-|u_t - v_t|^2)dv_t dP(t)$$

From this expression, by comoosing the amplitudes with intermediate integrations, we immediately deduce the quantum stochastic generalization of the Feynman path integral formula:

$$< Q(T), \phi(u_T)|U(T)|Q(0), \phi(u_0) > =$$

$$\int exp(\int_0^T [iP(t).\, Q^{'}(t) + \bar{L}_b^a(t, Q(t), P(t))u_{tb}(t)v_{ta}(t) + (- Re < u_t^{'}|u_t - v_t > + iIm < u_t^{'}|v_t >)]dt).$$

$$\times exp(-\sum_{t \le T} |u_t - v_t|^2)Du_{(0,T)}Dv_{[0,T]}DQ(0, T)DP[0, T]$$

14. Entropy rate computation for the stochastic Schrodinger equation driven by Levy differential processes

14.1. An algorithm for computing the perturbation in the logarithm of a matrix based on differentials of the matrix exponential

This stochastic Schrodinger equation appears in quantum stochastic filtering theory, ie, the Belavkin filter. The evolution of the filtered state $\rho(t)$ has the form

$$d\rho(t) = F_0(\rho(t)) + \sum_k G_k(\rho(t))dV_k(t)$$

where $V_k^{'}s$ are joint zero mean Levy processes. From generalized Ito formula we have

$$\mathrm{E}(dV_{k_1}(t_1)\ldots dV_{k_n}(t_n)) = \delta(t_1, \ldots, t_n)\mu_n(k_1, \ldots, k_n)dt$$

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where $\delta(t_1, \ldots, t_n)$ equals one if $t_1 = \ldots = t_n$ and zero otherwise. The aim is to calculate the rate of change of the entropy [7]

$$S(t) = -\mathrm{E}Tr(\rho(t).\, ln(\rho(t))) = -Tr(\mathrm{E}(\rho(t).\, ln(\rho(t))))$$

We calculate

$$d(\rho.\, ln(\rho)) = d\rho.\, ln(\rho) + \rho.\, dln(\rho) + d\rho.\, dln(\rho)$$

Now, we can write using the formula for the differential of the exponential map

$$d.\, ln(\rho) = ln(\rho + d\rho) - ln(\rho) = \sum_{n \geq 1, k \geq 1} X_{n,1,k}(\rho) d\rho.\, X_{n,2,k}(\rho).\, d\rho \ldots .\, d\rho.\, X_{n,n+1,k}(\rho)$$

where the $X_k(\rho)$'s are matrix valued nonlinear functions of ρ. In fact, to arrive at such an expansion, we apply the following method: Let

$$ln(A + tB) = Z(t)$$

Then

$$A + tB = exp(Z(t))$$

so that

$$B = (A + tB).\, g(ad(Z(t))(Z'(t))$$

$$Z'(t) = g(adZ(t))^{-1}((A + tB)^{-1}B) = g(adZ(t))((1 + tA^{-1}B)^{-1}A^{-1}B)$$

Here,

$$g(z) = (1 - exp(-z))/z$$

$$g(z)^{-1} = z/(1 - exp(-z)) = \sum_{n \geq 0} c(n) z^n,\ c(0) = 1$$

Writing

$$Z(t) = \sum_{n \geq 0} t^n Z_n$$

we derive by substituting this into the above differential equation,

$$\sum_{n \geq 0} (n + 1) t^n Z_{n+1} = \sum_{n \geq 0, m \geq 0, k_1, \ldots, k_n \geq 0} (-1)^m c(n) t^{k_1 + \ldots + k_n + m} ad(Z_{k_1}) \ldots ad(Z_{k_n})((A^{-1}B)^{m+1})$$

from which, we arrive at the recursion (on equating coefficients of t^n on both sides)

$$(n + 1) Z_{n+1} = \sum_{r, k_1, \ldots, k_r, m \geq 0, r + k_1 + \ldots + k_r + m = n} c(r) ad(Z_{k_1}) \ldots ad(Z_{k_r})((A^{-1}B)^{m+1}),\ n \geq 0$$

This equation can be used to recursively compute Z_n, $n = 1, 2, \ldots$ with of course

$$Z_0 = ln(A)$$

In particular, we see that

$$Z_1 = A^{-1}B$$

We apply this formula to the equation

$$dln(\rho) = ln(\rho + d\rho) - ln(\rho) = \sum_{n \geq 1} Z_n(\rho, d\rho)$$

with $A = \rho, B = d\rho, t = 1$. We find that the recursion implies that $Z_n(\rho, d\rho)$ is a homogeneous matrix polynomial of degree n in $d\rho$ and hence can be expressed as

$$Z_n(\rho, d\rho) = \sum_{n,k \geq 1} X_{n,1,k}(\rho).\, d\rho.\, X_{n,2,k}(\rho)\ldots.\, d\rho.\, X_{n,n+1,k}(\rho)$$

Then,

$$\mathrm{E}(d\rho.\, ln(\rho)) = dt.\, \mathrm{E}(F_0(\rho).\, ln(\rho)),$$

$$\mathrm{E}(\rho.\, dln(\rho)) = \sum_{n \geq 2} \mathrm{E}(\rho.\, Z_n(\rho, d\rho)) =$$

$$dt. \sum_{n \geq 2, k \geq 1, m_1, \ldots, m_n \geq 1} \mathrm{E}(\rho.\, X_{n,1,k}(\rho) G_{m_1}(\rho).\, X_{n,2,k}(\rho) G_{m_2}(\rho) \ldots X_{n,n,k}(\rho) G_{m_n}(\rho).\, X_{n,n+1,k}(\rho)) \mu_n(m_1, \ldots, m_n)$$

and finally,

$$\mathrm{E}(d\rho.\, dln(\rho)) =$$

$$dt. \sum_{n \geq 1, k \geq 1, m_0, m_1, \ldots, m_n \geq 1} \mathrm{E}(G_{m_0}(\rho).\, X_{n,1,k}(\rho) G_{m_1}(\rho).\, X_{n,2,k}(\rho) G_{m_2}(\rho) \ldots X_{n,n,k}(\rho) G_{m_n}(\rho).\, X_{n,n+1,k}(\rho))$$

$$\times \mu_n(m_0, m_1, \ldots, m_n)$$

14.2. Accurate computation of the rate of entropy change for the Belavkin quantum filter

$$d\rho(t) = \theta(t, \rho(t)) + \sum_k G_k(t, \rho(t)) dN_k(t)$$

where $N_k^{'}s$ are η_0-martingales. Specifically, the measurement process is

$$Y_o(t) = U(t)^* Y_i(t) U(t),\ Y_i(t) = c(a, b) \Lambda_b^a(t)$$

$$dU(t) = (L_b^a d\Lambda_a^b(t)) U(t)$$

$$dY_o(t) = j_t(M_b^a(1)) d\Lambda_a^b(t)$$

$$dY_o(t)^m = j_t(M_b^a[m]) d\Lambda_a^b(t),\ m \geq 1$$

$$d\rho(t) = \theta(t, \rho(t)) + \sum_{m \geq 1} G_m(t, \rho(t)) dY_o(t)^m$$

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$$\mathrm{E}(dY_o(t)^m \,\Big|\, \eta_0(t)) = \pi_t(M_b^a[m])\bar{u}_b(t)u_a(t)dt = Tr(\rho(t)M_b^a[m])\bar{u}_b(t)u_a(t)dt,\ u_0(t) = 1$$

Thus,

$$dN_m(t) = dY_o(t)^m - Tr(\rho(t)M_b^a[m])\bar{u}_b(t)u_a(t)dt,\ m \geq 1$$

defines a sequence of η_o-Martingales $N_m, m = 1, 2, \ldots$. For any matrix valued function F of a matrix,

$$dF(\rho) = F(\rho + d\rho) - F(\rho)$$

$$= \sum_{m\geq 1}\sum_k X_{km1}(\rho)d\rho.\, X_{km2}(\rho).\, d\rho\ldots d\rho.\, X_{kmm+1}(\rho)$$

$$= \sum_k X_{k11}(\rho).\, d\rho.\, X_{k11}(\rho) + \sum_{m\geq 2}\sum_k X_{km1}(\rho)d\rho.\, X_{km2}(\rho).\, d\rho\ldots d\rho.\, X_{kmm+1}(\rho)$$

$$= \sum_k X_{k11}(\rho).\, \theta(\rho).\, X_{k11}(\rho)dt + \sum_{m\geq 1, l_1, \ldots, l_m}\sum_k X_{km1}(\rho)G_{l_1}(\rho).\, X_{km2}(\rho).\, G_{l_2}(\rho)\ldots G_{l_m}(\rho).\, X_{kmm+1}(\rho)$$

$$\times\, dY_o(t)^{l_1+\ldots+l_m}$$

Thus,

$$\mathrm{E}(dF(\rho(t)) \,\Big|\, \eta_o(t)) = \sum_k X_{k11}(\rho).\, \theta(\rho).\, X_{k11}(\rho)dt$$

$$+ \sum_{m\geq 1, l_1, \ldots, l_m}\sum_k X_{km1}(\rho)G_{l_1}(\rho).\, X_{km2}(\rho).\, G_{l_2}(\rho)\ldots G_{l_m}(\rho).\, X_{kmm+1}(\rho)\mathrm{E}(dY_o(t)^{l_1+\ldots+l_m} \,\Big|\, \eta_o(t))$$

or equivalently,

$$\mathrm{E}(dF(\rho(t)) \,\Big|\, \eta_o(t)) = \sum_k X_{k11}(t,\rho).\, \theta(\rho).\, X_{k11}(\rho)dt$$

$$+ \sum_{m\geq 1, l_1, \ldots, l_m}\sum_k X_{km1}(\rho)G_{l_1}(t,\rho).\, X_{km2}(\rho).\, G_{l_2}(t,\rho)\ldots G_{l_m}(t,\rho).\, X_{kmm+1}(\rho)$$

$$\times\, Tr(\rho.\, M_b^a[l_1 + \ldots + l_m])\bar{u}_b(t)u_a(t)dt$$

Note that we could recast the filter equation as

$$d\rho(t) = (\theta(t,\rho(t)) + \sum_{m\geq 1} G_m(t,\rho(t))Tr(\rho(t)M_b^a[m])\bar{u}_b(t)u_a(t))dt + \sum_{m\geq 1} G_m(t,\rho(t)).\, dN_m(t)$$

In the special case when the bath is in a vacuum coherent state, this simplifies to

$$d\rho(t) = (\theta(t,\rho(t)) + Tr(\rho(t)M_0^0[m])dt + \sum_{m\geq 1} G_m(t,\rho(t)).\, dN_m(t)$$

where

$$dN_m(t) = dY_o(t)^m - Tr(\rho(t)M_0^0[m])dt$$

and the above formula for the conditional expectation of the differential of a function of the filtered state also simplifies to

$$\mathrm{E}(dF(\rho(t))\,\Big|\,\eta_o(t)) = \sum_k X_{k11}(t,\rho).\,\theta(\rho).\,X_{k11}(\rho)dt$$

$$\sum_{m\geq 1,l_1,\ldots,l_m}\sum_k X_{km1}(\rho)G_{l_1}(t,\rho).\,X_{km2}(\rho).\,G_{l_2}(t,\rho)\ldots G_{l_m}(t,\rho).\,X_{kmm+1}(\rho)Tr(\rho.\,M_0^0[m])dt$$

We've seen that

$$d\rho = (\theta(t,\rho) + \sum_k G_k(t,\rho)\chi_k(t,\rho))dt + \sum_k G_k(t,\rho)(dY_o^k - \chi_k(t,\rho))dt)$$

where

$$\chi_k(t,\rho) = Tr(\rho(t)M_b^a[k])u_b(t)\bar{u}_a(t)$$

We could write this as

$$d\rho(t) = \theta_1(t,\rho(t))dt + \sum_k G_k(t,\rho(t))dN_k(t)$$

where

$$\theta_1(t,\rho) = \theta(t,\rho) + \sum_k G_k(t,\rho)\chi_k(t,\rho),\ N_k(t) = \int_0^t dY_o(s)^k - \int_0^t \chi_k(s,\rho(s))ds$$

with the $N_k^{'}s$ being η_o-martingales. We also observe that for $p \geq 2$,

$$\mathrm{E}(dN_{k_1}(t)\ldots dN_{k_p}(t)\,\Big|\,\eta_o(t)) = \mathrm{E}(dY_o(t)^{k_1+\ldots+k_p}\,\Big|\,\eta_o(t))$$

$$= Tr(\rho(t)M_b^a[k_1+\ldots+k_p])u_b(t)\bar{u}_a(t)dt = \chi_{k_1+\ldots+k_p}(t,\rho(t))$$

15. Appendix 2. Scattering theory in the context of quantum stochastic differential equations

The unitary evolution is

$$dU_1(t) = L_b^a d\Lambda_a^b(t).\,U_1(t),\ dU_0(t) = -iH_0U_0(t)dt$$

The wave operator is given by

$$\Omega_+ = lim_{t\to\infty}U_1(t)^{-1}U_0(t)$$

We shall instead study

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$$\Omega = lim_{t\to\infty} U_0(t)^{-1} U_1(t)$$

Let

$$W(t) = U_0(t)^{-1} U_1(t)$$

Then,

$$\Omega = lim_{t\to\infty} W(t)$$

We now have

$$dW(t) = U_0(t)^{-1} (L^a_b d\Lambda^b_a(t) + iH_0 dt) U_1(t) = U_0(t)^{-1} (L^a_b d\Lambda^b_a(t) + iH_0 dt) U_0(t) W(t)$$

$$= [L^a_b(t) d\Lambda^b_a(t) + iH_0 dt] W(t)$$

where

$$L^a_b(t) = U_0(t)^{-1}. L^a_b. U_0(t)$$

We write

$$M^a_b(t) = L^a_b(t), a + b \geq 1, M^0_0(t) = L^0_0(t) + iH_0$$

Note that

$$M^a_b(t) = U_0(t)^{-1} M^a_b. U_0(t), M^a_b = L^a_b + iH_0 \delta_{a0} \delta_{b0}$$

and then we have that

$$dW(t) = M^a_b(t) d\Lambda^b_a(t). W(t)$$

yielding thereby

$$\Omega = W(\infty) = 1 + \int_0^\infty M^a_b(t) d\Lambda^b_a(t). W(t)$$

Iterating the equation

$$W(t) = 1 + \int_0^t M^a_b(s) d\Lambda^b_a(s) W(s)$$

gives us the quantum stochastic Dyson series

$$W(T) = 1 + \sum_{n\geq 1} \int_{0 < t_n < \ldots < t_1 < T} \Pi^n_{k=1} M^{a(k)}_{b(k)}(t_k) \Pi^n_{k=1} d\Lambda^{b(k)}_{a(k)}(t_k) dt_1 \ldots dt_n$$

where for operators $A_k, k = 1, 2, \ldots$

$$\Pi^n_{k=1} A_k = A_1 A_2 \ldots A_n$$

We easily find that

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$$< fe(u)|W(T)|ge(v) >=< e(u)|e(v) > [1 + \sum_{n\geq 1}\int_{0<t_n<\ldots<t_1<T} < f|\Pi_{k=1}^{n} M_{b(k)}^{a(k)}(t_k)|g >$$

$$\times (\Pi_{k=1}^{n} \bar{v}_{b(k)}(t_k) u_{a(k)}(t_k)) dt_1 \ldots dt_n$$

For weak convergence of $W(T)$ as $T \to \infty$, it is sufficient therefore that for all f, g, u, v,

$$lim_{T\to\infty}\sum_{n\geq 1}\int_{0<t_n<\ldots<t_1<T} | < f|\Pi_{k=1}^{n} M_{b(k)}^{a(k)}(t_k)|g > | \, |\Pi_{k=1}^{n} | \bar{v}_{b(k)}(t_k) u_{a(k)}(t_k)) | dt_1 \ldots dt_n < \infty$$

and for this, it is in turn sufficient that

$$lim_{T\to\infty}\sum_{n\geq 1}\sum_{n\geq 1}\int_{0<t_n<\ldots<t_1<T} \| \Pi_{k=1}^{n}(M_{b(k)}^{a(k)} U_0(t_k))|g >\| \, . \, |\Pi_{k=1}^{n} | \bar{v}_{b(k)}(t_k) u_{a(k)}(t_k)) | dt_1 \ldots dt_n < \infty$$

and for this, it is in turn sufficient that for all g, u, v

$$lim_{T\to\infty}\sum_{n\geq 1}\sum_{n\geq 1}\int_{0<t_n<\ldots<t_1<T} \sum_{\{a(k), b(k)\geq 0, k=1,2,\ldots,n\}} \| \Pi_{k=1}^{n}(M_{b(k)}^{a(k)} U_0(t_k))|g >\| \, . \, sup_{0\leq t\leq T}[\| v(t) \|^n . \| u(t) \|^n]$$

$$\times dt_1 \ldots dt_n < \infty$$

Alternate conditions can be derived for the existence of the wave operator such as:

$$dW(t) = M_b^a(t) d\Lambda_a^b(t). \, W(t) = U_0(t)^{-1} M_b^a . \, U_1(t) d\Lambda_a^b(t)$$

so that

$$W(T) - 1 = \int_0^T U_0(t)^{-1} M_b^a . \, U_1(t) d\Lambda_a^b(t)$$

which gives

$$< fe(u)|W(T) - 1|ge(v) > \; = \int_0^T < fe(u)|U_0(-t) M_b^a U_1(t)|ge(v) > \bar{v}_b(t) u_a(t) dt$$

so that a sufficient condition for $W(T)$ to be weakly convergent as $T \to \infty$ is that

$$\sum_{a,b}\int_0^\infty \|M_b^a U_1(t)|ge(v) > \|. \, \|v(t)\|. \, \|u(t)\|. \, dt < \infty$$

for all g, u, v. Recall that

$$M_0^0 = L_0^0 + iH_0, M_b^a = L_b^a, a + b \geq 1, a, b \geq 0$$

16. A more general formula for the existence of the wave operator when both the Hamiltonians are noisy

$U_0(t)$ and $U_1(t)$ are both quantum stochastic unitary evolutions on the tensor product of the system and bath Hilbert space. Both of these satisfy the Hudson-Parthasarathy noisy Schrodinger equation:

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$$dU_k(t) = L_b^a(k)d\Lambda_a^b(t)U_k(t),\ k = 0, 1$$

The wave operator is defined as the limit of

$$W(t) = U_0(t)^{-1}U_1(t)$$

as $t \to \infty$ whenever it exists [8]. We can write

$$dW = U_0^{-1}dU_1 + d(U_0^{-1}).\ U_1 + d(U_0^{-1}).\ dU_1$$

$$= U_0^{-1}dN_1.\ U_1 + ((U_0 + dU_0)^{-1} - U_0^{-1}).\ U_1 + ((U_0 + dU_0)^{-1} - U_0^{-1}).\ dU_1 =$$

$$U_0^{-1}[dN_1 + ((1 + dN_0)^{-1} - 1) + ((1 + dN_0)^{-1} - 1).\ dN_1].\ U_1$$

where

$$dN_k(t) = L_b^a(k)d\Lambda_a^b(t),\ k = 0, 1$$

because the quantum stochastic evolutions can be expressed in the form

$$dU_k(t) = dN_k(t).\ U_k(t),\ k = 0, 1$$

A condition for the strong limit of $W(T)$ as $T \to \infty$ to exist is thus

$$\int_0^\infty \| [dN_1(t) + ((1 + dN_0(t))^{-1} - 1) + ((1 + dN_0(t))^{-1} - 1).\ dN_1(t)].\ U_1(t)g|e(v)>\| \ .\ dt < \infty$$

A condition for the weak limit of the same to exist is that

$$\int_0^\infty | <fe(u)|U_0(t)^{-1}[dN_1(t) + ((1 + dN_0(t))^{-1} - 1) + ((1 + dN_0(t))^{-1} - 1).\ dN_1(t)].\ U_1(t)g|e(v)> |.\ dt < \infty$$

Using the geometric series

$$(1 + dN_0)^{-1} = 1 + \sum_{n \geq 1}(-1)^n(dN_0)^n$$

and repeated applications of the quantum Ito formula, we can express

$$dN_1(t) + ((1 + dN_0(t))^{-1} - 1) + ((1 + dN_0(t))^{-1} - 1).\ dN_1(t)$$

$$= P_b^a d\Lambda_a^b(t)$$

where P_b^a are system operators built out of the $L_d^c(k), c, d \geq 0, k = 0, 1$. Then, the above condition for the existence of the wave operator gives the following sufficient condition for the weak limit of $W(T)$ to exist:

$$\sum_{a,b}\int_0^\infty \|P_b^a U_1(t)g|e(v)>\|.\ \|v(t)\|.\ \|u(t)\|.\ dt < \infty$$

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17. Conclusions

In this paper, we have explored some applications of quantum stochastic calculus as originally developed by R.L.Hudson and K.R.Parthasarathy and the quantum filter based on this calculus as originally developed by V.P.Belavkin and perfected by J.Gough and others. The applications explored in this paper include simultaneous real time state and parameter estimation in noisy quantum systems with further applications to real time classical-quantum noise mitigation in quantum communcations, development of a Feynman path integral approach to solving the Hudson-Parthasarathy noisy Schrodinger equation, using the Weyl-Moyal-Wigner representation of states and observables in terms of complex functions on classical phase space to deduce similarities between quantum mechanical problems and corresponding classical problems such as stochastic filtering, Boltzmann equation and noisy Schrodinger equations. We also discuss herein a problem involving computing the rate of change of the Von-Neumann entropy of a noisy quantum system after generalized Belavkin filtering and stating potential applications of this to the computation of the rate of transmission of quantum information in the form of a quantum electromagnetic field through an optical fibre after mitigation of noise using the Belavkin filter. Finally, we discuss the important problem of computing the quantum scattering wave operator after deriving conditions for its existence for a pair of noisy unitary evolutions described by the Hudson-Parthasarathy noisy Schrodinger equation.

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