Research Article

A Dynamic Model for an Optimal Consumption Tax Rate

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Following Ramsey, the existing literature on optimal taxation only compares the pre- and post-tax market equilibriums in order to account for the efficiency losses. However, when the government imposes an ad valorem tax on the consumer, the buyer's price jumps to the pre-tax equilibrium price plus the amount of the tax, and the supply and demand of the taxed commodity then adjust over time to bring about the new post-tax market equilibrium. The existing literature does not take into account the efficiency losses during the adjustment process while computing the optimal ad valorem taxes. This paper shows how the adjustment process influences the optimal ad valorem taxes, minimizing the efficiency losses (output and/or consumption lost) during the dynamic adjustment process as well as at the post-tax market equilibrium. (JEL H20, H21, H22)

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1. Introduction

A tax on consumption is a consumer tax levied through imposing a tax on the purchase of goods and services. It takes the form of a direct or an indirect tax, such as a value-added tax (VAT), sales tax, excise tax on consumption, tariffs at the import stage, or an income tax where savings are tax-free. The tax base is the price of consumption goods before tax.

A value-added tax is a consumption tax that is levied and collected at each stage in the supply chain on the difference between sales and purchases of all agents, such as importer, manufacturer, wholesaler, distributor, and retailer, etc. A simple VAT is proportional to consumption and is regressive in nature, as with an increase in income, the proportion of consumption in total income falls. Investment and savings do not get taxed under the ambit of consumption tax; however, as soon as they get converted to consumption, they are taxed. In some countries, such as the European Union, it is common to exclude certain goods from VAT to make it less regressive. Consumption tax can have different nomenclature in different countries, e.g., it is called a "Goods and Services Tax" in New Zealand, Australia, Singapore, India, and Canada (in Canada, also known as the Harmonized Sales Tax, when combined with a provincial sales tax). It is also known as a sales tax in some countries, as it is applied at the final point of sale in a supply chain and is applicable on the sale of goods and/or services. It is an ad valorem tax, i.e., a percentage of the final price of goods and services. It is also called a use tax when consumers are liable to deposit the tax directly into the government treasury.

Laws may exempt certain items from consumption tax. An excise tax is also a sales tax and is applied at times to reduce the consumption of certain goods, such as tobacco, alcohol, etc. However, it is also applied for revenue generation, e.g., gasoline (petrol), tourism, etc.

Consumption tax can also be in the form of a direct tax as an expenditure tax, which is an income tax after the deduction of savings and investment, such as the Hall-Rabushka flat tax. In the form of a direct tax, it is generally called an expenditure tax, a cash-flow tax, or a consumed-income tax and can be either flat or progressive. In the past, some countries had implemented a direct consumption tax, such as India and Sri Lanka. The base of this kind of tax is income minus savings. If the direct consumption tax rate is flat, it is regressive with respect to income; however, it can be made progressive by applying progressive tax rates, i.e., an increase in the tax rate with an increase in personal consumption.

An optimal taxation is one that minimizes efficiency losses and distortion in the market as a result of deviation from the pre-policy efficient market equilibrium, given the economic constraints when a tax is imposed. The first contribution to the theory of optimal taxation was made by Ramsey (1927), who developed a theory for optimal commodity taxes and proposed a theoretical solution that the consumption tax on each good should be "proportional to the sum of the reciprocals of its supply and demand elasticities." Suits and Musgrave (1953) find that ad valorem taxation yields a larger total surplus than unit taxes provided they give the same yield. Diamond and Mirrlees (1971) consider commodity taxation along with other kinds of taxes. Mirrlees (1975) modified the standard problem by considering simultaneously excise taxes and a poll tax. Diamond (1975) examines the Ramsey rule for a many-person economy with excise taxes and a poll tax. Atkinson and Stiglitz (1976) show that with an optimal nonlinear income tax, discriminatory commodity taxes are only necessary to the extent that individual commodities are not weakly separable from leisure. In Deaton (1981), rules for optimal differential commodity taxes have been derived for the three different cases usually studied in the literature: the one-consumer economy, the unidimensional continuum of consumers economy, and the finite number of discrete consumers economy. Lucas and Stokey (1983) derive a time-consistent optimal fiscal policy in an economy without capital, maximizing consumer welfare subject to the condition that a competitive equilibrium holds in each time period.

In Judd (1985), the government taxes capital income net of depreciation at a proportional rate, which is assumed to be constant. Chamley (1986) analyzes the optimal tax on capital income in general equilibrium models of the second best. Deaton and Stern (1986) show that optimal commodity taxes for an economy with many households should be at a uniform proportional rate under certain conditions. Cremer and Gahvari (1993) incorporate tax evasion into Ramsey's optimal taxation problem. Skeath and Trandel (1994) show that ad valorem taxes Pareto dominate specific taxes. Cremer and Gahvari (1995) prove that optimal taxation requires a mix of differential commodity taxes and a uniform lump-sum tax. Naito (1999) shows that imposing a non-uniform commodity tax can Pareto-improve welfare even under nonlinear income taxation if the production side of an economy is taken into consideration. Saez (2002b) shows that a small tax on a given commodity is desirable if high-income earners have a relatively higher taste for this commodity or if consumption of this commodity increases with leisure.

Nordhaus (1993) proposes an optimal carbon tax (tax per ton of carbon). Chari, Christiano, and Kehoe (1994) deal with the labor and capital income taxes instead of an ad valorem tax as in our model. Ekins (1996) takes into account the secondary benefits of carbon dioxide abatement for an optimal carbon tax. Coleman (2000) derives the optimal dynamic taxation of consumption, income from labor, and income from capital, and estimates the welfare gain that the US could attain by switching from its current income tax policy to an optimal dynamic tax policy. Pizer (2002) explores the possibility of a hybrid permit system and a dynamic optimal policy path in order to accommodate growth and not because of the adjustment over time to equalize the marginal benefit and cost. It is implicitly assumed that the marginal cost equals the marginal benefit in each time period. Jensen and Schjelderup (2011) study how a change in specific and ad valorem taxes under nonlinear pricing affects tax incidence. Aiura and Ogawa (2013) examine the choice of tax method between an ad valorem tax and a specific tax. Blackorby and Murty (2013) employ a general equilibrium framework with a monopoly, hundred percent profit taxation, and uniform lump-sum transfers. Nawaz (2017) discusses the optimal quantity taxes in a dynamic setting.

In existing literature, while deriving an optimal tax, the efficiency loss in the post-tax policy equilibrium/dead weight loss is minimized; however, there are additional efficiency losses during the market adjustment toward the final post-tax policy equilibrium. Without minimizing the total efficiency loss, i.e., the one during market adjustment as well as the dead weight loss in the final equilibrium, the derived tax policy cannot be optimal in the true sense and can be improved upon. When a consumption tax is imposed, the buyer's price jumps to the initial price plus the amount of tax. The price adjusts over time to bring the final post-tax market equilibrium with some dead weight loss. Supply and demand also adjust along with the price, including tax, until the new equilibrium arrives. Existing literature ignores efficiency losses on the adjustment path of the market to the final equilibrium after the imposition of a tax to derive an optimal consumption tax. The quantum of efficiency loss during market adjustment is contingent upon market parameters, as shown in the later part (section 4) of this article; however, theoretically, ideally, the total efficiency loss, i.e., during the adjustment of the market as well as that in the final post-tax policy equilibrium, must be minimized to derive an optimal tax. This paper considers total efficiency loss (output and/or consumption lost), i.e., during market adjustment as well as the dead weight loss in the post-tax policy equilibrium, as an objective function to be minimized subject to a tax revenue constraint to derive an optimal consumption tax.

The remainder of this paper is organized as follows: Section 2 explains how individual components of the market system are joined together to form a dynamic market model. Section 3 provides the solution of the model with a consumption tax. Section 4 derives an optimal consumption tax minimizing efficiency losses subject to a tax revenue constraint. Section 5 summarizes findings and concludes. Section 6 constitutes the appendix.

2. The Model

Suppose there is a perfectly competitive market for a homogeneous good, and the market is in equilibrium, i.e., initial conditions of a market equilibrium apply. Four types of infinitely-lived market agents are present, i.e., a representative -or a unit mass of- producer, who demands capital and labor to produce goods; a middleman who purchases goods from the producer, holds an inventory of goods, i.e., stores them to be subsequently sold, and sells those to the consumer; a representative -or a unit mass of- consumer who supplies labor inelastically, accumulates capital through investment, and buys goods from the middleman; and the government. The middleman's role is instrumental in capturing the adjustment of the market to the final equilibrium, as the producer is a price taker and cannot change the price. The middleman is shown to have an incentive to change the price only during market adjustment and is not better off deviating from the market price once equilibrium is achieved. The middleman sells goods to the consumer at the market price p, which is chosen by maximizing the difference between the revenue generated by selling goods to the consumer and the cost of holding/storing goods after buying from the producer, i.e., the cost of holding an inventory of goods. The middleman pays a fixed fraction of the initial market price to the producer, i.e., αp with $\alpha < 1$, and with fixed α and p, the producer is a price taker.

The price adjustment mechanism is based on the lack of coordination among buyers and suppliers at current prices when a shock puts the market out of equilibrium, and is illustrated as given below: Suppose the market is in an initial equilibrium, and the middleman holds an equilibrium level of inventory due to supply and demand rates being the same. Inventory is the stock variable and reflects the difference between supply and demand rates accumulated over time. A change in inventory happens when either supply or demand, or both, rates change by different magnitudes. Supply and demand rates are flow variables, i.e., the quantity supplied/demanded per unit time. If an exogenous shock leading to a demand contraction happens to the market, the stock of inventory will pile up at the existing price as the supply from the producer continues to be the same as before. The middleman will reduce the price, which will increase demand along the demand schedule, and the producer will find it optimal to produce a lower quantity than before. A new equilibrium with both lower price and output will be reached. The equilibrium is defined as follows:

i. The middleman and the producer maximize their profits, and the consumer maximizes utility subject to their respective constraints (see Section 2).

ii. The quantity consumed by the consumer equals the quantity supplied by the producer (the inventory remains the same when the market is in equilibrium).

The equilibrium conditions, i.e., the Routh-Hurwitz stability criterion, which provides a necessary and sufficient condition for the stability of a linear dynamical system, are mentioned in Section 3.

When the market is in equilibrium, the middleman is a price taker and sells the goods to the consumer at the given market price on account of the fact that the setup is for a perfectly competitive market. The middleman can change the price along the dynamic adjustment path when the market is out of equilibrium, until the middleman again becomes a price taker when the new equilibrium arrives. An ad valorem tax is announced and implemented at the same time by the government (the agents' expectations will be taken into account in a future research project when the announcement and implementation dates of the tax could be different). The market does not suddenly jump to the post-tax market equilibrium after the imposition of an ad valorem tax; rather, the adjustment of price takes place over time to bring about the new equilibrium. The adjustment of price involves endogenous decision-making (on the basis of self-interest) by all the agents in the market, i.e., producer, consumer, and the middleman. Suppose a producer produces a perishable good and sells it to a middleman who subsequently sells it to a consumer living in a community. The middleman and the producer sell a quantity equal to the quantity produced by the producer in each time period, and the market stays in equilibrium. Suppose that the government announces and imposes an ad valorem tax on the consumer, which decreases the demand for the good; part of the production sold to the middleman by the producer will remain unsold to the consumer by the end of the time period in which the tax was imposed and be wasted. If we assume that the middleman and the producer can change the price and the production, respectively, immediately, had they known the pattern of new demand, they would immediately pick the price (by the middleman) and quantity (by the producer) to maximize their profits and clear the market without any waste of production. However, this information is lacking, so the middleman could decrease the price based on the best guess he could have about the new demand (based on the amount of unsold production), which would drive the market close to the new equilibrium. The producer produces a lower quantity at the lower price. If the producer's production is fully sold out to the consumer by the middleman in the following time period, he will not be changing the production anymore, knowing that the new equilibrium has arrived; however, if some of his production still remains unsold, the middleman will choose to reduce the price further (and the producer, the production accordingly) to bring the market closer to the new equilibrium. The market eventually settles at a new equilibrium after some efficiency losses. The resources that went into the unsold production in each time period due to the imposition of the tax are wasted. A new equilibrium will finally be arrived at, with a deadweight loss due to the ad valorem tax. As a result of an ad valorem tax, the efficiency loss is the waste of resources during the adjustment period plus the loss in the final equilibrium.

In mathematical terms, the objective function of all the market agents is maximized through the first-order conditions, and the equations representing their individual actions are solved simultaneously to capture the collective result of their individual actions. We assume for simplification that the new equilibrium is not too far off from the initial equilibrium after the imposition of the ad valorem tax. This makes the linearization of demand and supply curves a reasonable approach. In Figure 1, linearization seems to be a good approximation when moving from point a to b, whereas it does not seem to be a good approximation in the movement from point a to c. For the movement of the market equilibrium from point a to c, we need to model a non-linear dynamical system (not covered under the scope of this article).

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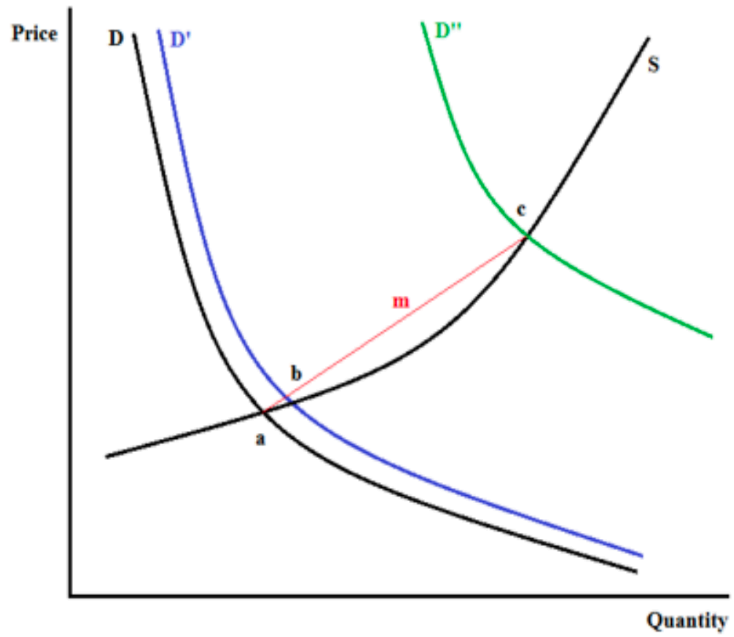

Figure 1. When is Linearity a Reasonable Assumption?

2.1. Middleman

The middleman buys goods from the producer and sells them to the consumer for profit. The middleman does not purchase and sell the same quantity at all points in time, and hence holds an inventory of the goods purchased to be sold subsequently. Inventory is an intermediary stage between demand and supply, which reflects the quantum of difference between demand and supply of the goods in the market. If there is no change in inventory, it implies that supply and demand rates are the same. A decrease or increase in inventory implies a change in demand, supply, or both at different rates.

Figure 2 illustrates the link between supply, demand, inventory, and prices. When the supply shifts to the right (while demand does not change), the inventory increases at the initial price, and the new equilibrium brings the price down. Similarly, when the demand shifts to the right (while supply remains the same), the inventory depletes from the market at the initial price, and the new equilibrium brings the price up. This implies that there is an inverse relationship between an inventory change and a price change (ceteris paribus). If both the demand and supply curves shift by the same magnitude such that the inventory remains constant, then the price will also not change. Inventory unifies the demand and supply shocks in the sense that they are both affecting the same factor, i.e., inventory. Therefore, each kind of shock is just an inventory shock. According to the above discussion, there is an inverse relationship between an inventory change and a price change; let us discuss the mechanism which brings about such a change. Consider a market of homogeneous goods where the middlemen, such as wholesalers, retailers, etc., hold inventories of goods, incur some cost for holding those, and sell the goods to the consumers to make profits. The cost

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of holding an inventory is a positive function of the size of the inventory, i.e., a larger inventory is more costly to hold as compared to a smaller inventory. In the absence of an exogenous shock, if the demand and supply rates are equal, then the market is in equilibrium and the price does not change with time. Suppose that a technological advancement decreases the marginal cost of production and increases the supply rate, whereas the demand remains the same. As the supply and demand rates are no longer equal, the difference will appear somewhere in the economy in the form of piled-up inventories. As the production flows from producers to consumers through the middlemen, it is reasonable to assume that the middlemen will be holding the net difference (Note: The piled-up inventories can also appear as producers' finished goods; however, the key point is that a difference in demand and supply rates directly affects the inventories in the economy). The economy cannot sustain this situation for an indefinite period of time, and the middlemen have to think of some means of getting rid of piled-up inventories. The only resort they have is to decrease the price to bring the demand up along the demand curve.

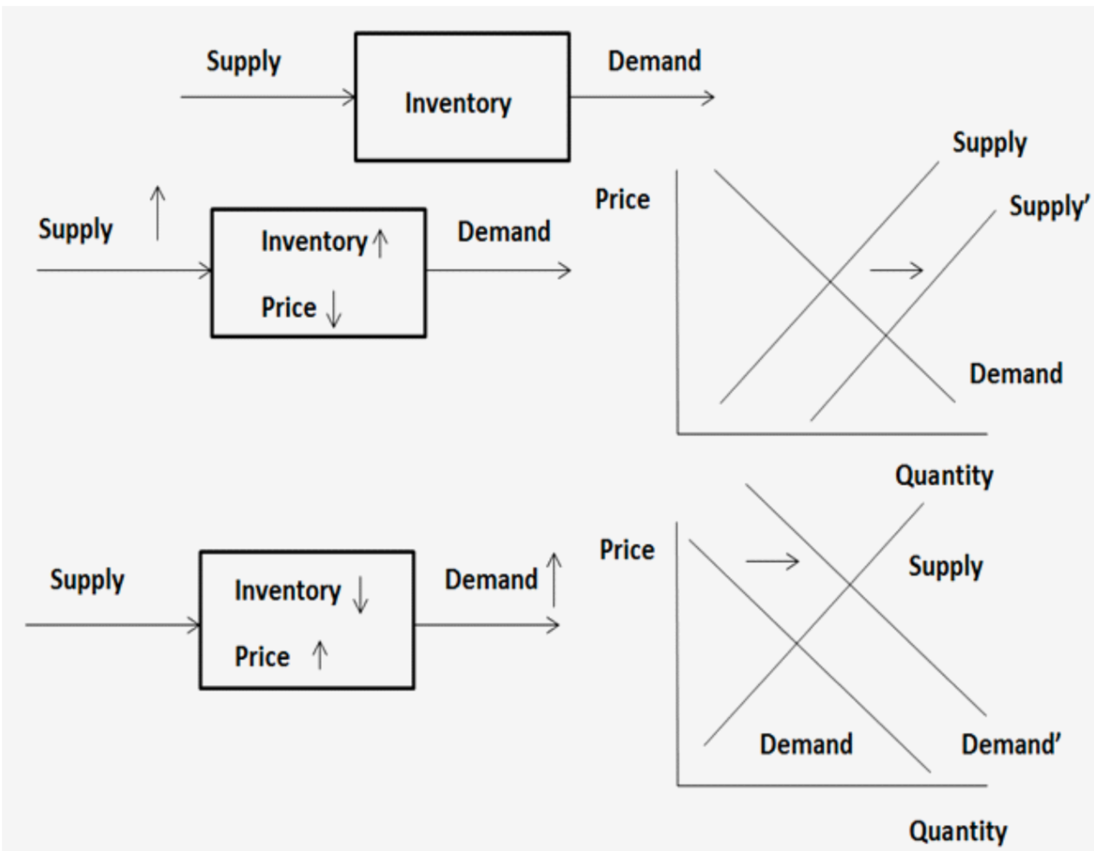

Figure 2. Movement of Price with Inventory.

The price, in a perfectly competitive market, will eventually come down to equalize the new marginal cost; however, the adjustment path depends on the actions of the middlemen, i.e., how they react to the change in their inventories. Notice that although the marginal cost of production for the producer has decreased, the marginal cost of holding an

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extra unit of inventory for the middleman has increased. This intuitive explanation is theoretically consistent with the supply, demand, profit, and utility maximization by a producer and a consumer, respectively. In the real world, examples of this kind of behavior by middlemen are as follows: we enjoy the *end of year sales* as consumers, and there are offers such as *buy one get one free, gift offers* if consumers buy above a certain quantity threshold, etc. For a mathematical picture, let us consider the profit maximization problem of the *middleman* as follows:

2.1.1. Short-run Problem

Let us first consider the short-run problem (Note: The middleman has a myopic objective, rather than doing dynamic optimization. This is a one-period analysis with a discrete analog, and is presented for an intuitive purpose for clarity of the more complicated dynamic problem in section 2.1.2) of the middleman as follows:

$$\Pi = pq(p) - \varsigma(m(p,e)), \tag{1}$$

where

Π = profit,

p = market price,

$q(p)$ = quantity sold at price p,

m = inventory (total number of goods held by the *middleman*),

e = other factors which influence inventory other than the market price, including the middleman's purchase price from the producer,

$\varsigma(m(p,e))$ = cost as a function of inventory (increasing in inventory).

The first-order condition (with respect to price) is as follows:

$$pq'(p) + q(p) - \varsigma'(m(p,e))m_1'(p,e) = 0, \tag{2}$$

The middleman's incentive to change the price is only during the adjustment process and will incur losses by deviating from the price equal to the marginal cost when the market is in equilibrium. The supply does not equal the demand during the adjustment process, and the market drifts toward the new equilibrium. However, the price cannot change automatically and has to be changed by some economic agent for his/her own benefit; therefore, a change in price by the middleman in the direction of bringing the new equilibrium is not against the market forces, so he/she does not lose business by changing the price on the dynamic adjustment path to the new equilibrium, unlike when the market is already in equilibrium and the middleman faces an infinitely elastic demand as given below:

$$pq'(p) + q(p) = \varsigma'(m(p,e))m_1'(p,e),$$

$$p\left[1 + \frac{1}{demand\ elasticity}\right] = \varsigma'(m(p,e))\frac{m_1'(p,e)}{q'(p)}.$$

The expression on the right-hand side is the marginal cost, which equals the price when the middleman faces an infinitely elastic demand. Suppose a supply shock shifts the supply curve downward due to a reduced marginal cost of production, e.g., due to a technological innovation. The competitive market is no longer in equilibrium as supply does

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not equal demand after the supply shock at the initial equilibrium price. As supply has expanded, the price will come down in the final equilibrium; however, there cannot be a sudden jump in price from one equilibrium to the other, and rather the middleman will continue charging the same price as before, i.e., higher than the new lower marginal cost, until his inventory piles up enough and market forces make him realize that his profit maximization condition has changed due to an expansion in market supply and he needs to reduce the price to meet his new profit-maximizing condition after the supply shock. The same reasoning goes for a reverse supply shock, i.e., in case of a shrinkage in market supply, the price will increase in the final equilibrium, and the middleman will not change the price and continue charging a price lower than the new higher marginal cost until the inventory level goes down substantially to make the middleman choose a higher selling price than before. In this scenario, the consumer will be the short-term beneficiary for paying a price lower than the marginal cost. In the first scenario, the middleman was a short-term beneficiary as he charged a price higher than the marginal cost during the adjustment period of the market. The final equilibrium price is equal to the marginal cost of the producer plus the marginal cost of the middleman for storing goods, i.e., the total marginal cost, in the absence of a tax/subsidy, so neither does the middleman nor does the consumer get any economic rent or economic benefit respectively by charging and paying a price respectively different from the marginal cost when the market is in equilibrium.

To put it in mathematical terms, suppose due to a positive supply shock while demand stays the same, such as a technological advancement that brings down the marginal cost of production and shifts supply downward, for the middleman to have another unit of inventory, the marginal cost, i.e., $\varsigma'(m(p,e))\frac{m_1'(p,e)}{q'(p)}$ is higher at the existing price due to the term $\varsigma'(m(p,e))$ being higher at the current price. This could be due to higher storage charges as a result of increased demand for storage places after the positive supply shock. The second term, i.e., $\frac{m_1'(p,e)}{q'(p)}$, being a function of price, is the same as before until the price changes. The middleman's purchase price is the same as before due to the producer being a price taker during the adjustment of the market too, and charging a fixed fraction of the market price to the middleman. To understand the concept intuitively, in a discrete analog of the above scenario, the middleman maximizes profits in each time period, taking the purchase price from the producer as given and choosing the selling market price without considering future time periods. The middleman faces the following inequality at the existing price:

$$\frac{\partial \Pi}{\partial p} = pq'(p) + q(p) - \varsigma'(m(p,e))m_1'(p,e) < 0, \tag{3}$$

implying that the middleman decreases the price after the supply shock to have another unit of inventory to maximize profits after the shock. In the above scenario, the producer is the short-term beneficiary due to a reduced marginal cost but receives the same price as before from the middleman. If profit-maximizing combinations of inventories and respective prices for the middleman are plotted together with price on the y-axis and inventory on the x-axis, a downward-sloping *inventory curve* results, which is analogous to traditional *supply* and *demand curves* for profit-maximizing producers and utility-maximizing consumers respectively.

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2.1.2. Dynamic Problem

In this section, the dynamic problem of the middleman is discussed. The middleman maximizes the present discounted value of a series of future profits with zero time value as given below:

$$V(0)=\int_{0}^{\infty}\left[pq(p)-\varsigma(m(p,e))\right]e^{-rt}dt, \tag{4}$$

with the following description of variables in the above expression: r as the discount rate, $p(t)$ as the *control variable*, and $m(t)$ as the *state variable*. The middleman's maximization problem is as given below:

$$\underset{\{p(t)\}}{Max}V(0)=\int_{0}^{\infty}\left[pq(p)-\varsigma(m(p,e))\right]e^{-rt}dt,$$

subject to the following constraints:

$\dot{m}(t)=m_1'(p(t),e(p(t),z))\dot{p}(t)+m_2'(p(t),e(p(t),z))e_1'(p(t),z)\dot{p}(t)$ (state equation, which describes the change in the state variable with respect to time; and z being exogenous inputs in the model),

$m(0)=m_s$ (initial condition),

$m(t)\geq 0$ (non-negativity constraint on the state variable),

$m(\infty)$ free (terminal condition).

For this case, the current-value Hamiltonian can be expressed as given below:

$$\widetilde{H}=p(t)q(p(t))-\varsigma(m(p(t),e(p(t),z)))+\mu(t)\dot{p}(t)\begin{bmatrix}m_1'(p(t),e(p(t),z))+m_2'(p(t),e(p(t),z))*\\ e_1'(p(t),z)\end{bmatrix}. \tag{5}$$

Maximizing conditions can be written as follows:

(i) $p^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial\widetilde{H}}{\partial p}=0$,

(ii) $\dot{\mu}-r\mu=-\frac{\partial\widetilde{H}}{\partial m}$,

(iii) $\dot{m}^*=\frac{\partial\widetilde{H}}{\partial\mu}$ (this just gives back the state equation),

(iv) $\lim\limits_{t\to\infty}\mu(t)m(t)e^{-rt}=0$ (the transversality condition).

The first two maximizing conditions can be expressed as given below:

$$\frac{\partial\widetilde{H}}{\partial p}=0, \tag{6}$$

and

$$\dot{\mu}-r\mu=-\frac{\partial\widetilde{H}}{\partial m}=\varsigma'(m(p(t),e(p(t),z))). \tag{7}$$

In equilibrium, $\dot{p}(t)=0$, and $\frac{\partial\widetilde{H}}{\partial p}$ reduces to the following expression (see appendix):

$$p(t)\left[1+\frac{1}{demand\ elasticity}\right]=\zeta'(m(p(t),e(p(t),z)))\left\{\frac{m_1'(p(t),e(p(t),z))}{q'(p(t))}+\frac{m_2'(p(t),e(p(t),z))e_1'(p(t),z)}{q'(p(t))}\right\}.$$

In a dynamic setting, the right side of the above expression is the marginal cost, which is quite different from that in the short-term/myopic problem, due to the fact that for a dynamic problem, the middleman also considers the impact of the price he/she chooses on the future purchase price from the producer. Price equals marginal cost for an infinitely elastic demand. Suppose a positive supply shock hits the market, and the middleman wants to increase the size of the inventory. To have an extra unit in inventory, the middleman's marginal cost is higher at the existing price due to the term $\zeta'(m(p(t), e(p(t), z)))$, which is higher at the previous price at that time. The term in parentheses in the above expression, i.e., $\frac{m_1'(p(t),e(p(t),z))}{q'(p(t))} + \frac{m_2'(p(t),e(p(t),z))e_1'(p(t),z)}{q'(p(t))}$ being a function of price, is the same as before until the price gets changed by the middleman. Therefore, at the existing price, the middleman's profit-maximizing expression changes to the following:

$$\frac{\partial \widetilde{H}}{\partial p} < 0.$$

This implies that at the previous price after the supply shock, the middleman's profit-maximizing condition is not being satisfied if he wants to have an extra unit of inventory; therefore, after the supply shock, the middleman must decrease the price to have another unit and to maximize profits. To increase inventory, the price must be decreased; therefore, there is a negative relationship between price and inventory change. Inventory is the state between supply and demand, and hence unifies both kinds of shocks, i.e., each kind of shock influences the inventory size; therefore, each kind of shock is just an inventory shock. If supply equals demand, the market is in equilibrium; however, if any kind of shock happens and either supply or demand or both rates get changed, and the economic agents do not respond to the shock, the price will be changing continuously until the system saturates, e.g., if a positive exogenous supply shock happens, and the producer and consumer do not modify their responses with a change in price, the market will get flooded with supply till the point of saturation. This response can be depicted by the following mathematical expression:

$$\begin{aligned}
Price\ change &\propto change\ in\ market\ inventory. \\
P &= price\ change. \\
M &= m - m_s = change\ in\ inventory\ in\ the\ market, \\
m &= inventory\ at\ time\ t, \\
m_s &= inventory\ in\ steady\ state\ equilibrium. \\
Input - output &= \frac{dm}{dt} = \frac{d(m - m_s)}{dt} = \frac{dM}{dt}, \\
\text{or } M &= \int (input - output)\, dt. \\
Price\ change &\propto \int (supply\ rate - demand\ rate)\, dt, \text{ or} \\
P &= -K_m \int (supply\ rate - demand\ rate)\, dt,
\end{aligned}$$

where K_m is the constant of proportionality. *Supply* and *demand rates* are flow variables and reflect the flow of supply and demand respectively per unit time in the market. When (*supply rate* $-$ *demand rate*) is positive, P is negative, i.e., excessive supply compared to demand leads to a decrease in market price and vice versa. The above equation can also be written as follows:

$$\int (supply\ rate - demand\ rate)\, dt = -\frac{P}{K_m},\ \text{or}$$

$$\int (w_i - w_0)\, dt = -\frac{P}{K_m}, \tag{8}$$

$$\begin{aligned} w_i &= supply\ rate, \\ w_0 &= \text{demand rate}, \\ K_m &= dimensional\ constant. \end{aligned}$$

At $t = 0$, the market is in a steady state equilibrium, and the *supply rate* equals the *demand rate*. Putting initial conditions in eq. (8), it can be expressed as given below:

$$\int (w_{is} - w_{0s})\, dt = 0. \tag{9}$$

Subscript s stands for steady state equilibrium, the state which reflects initial values of the market, and $P = 0$, when the market is in a steady state equilibrium. Subtracting eq. (9) from (8) results in the following expression:

$$\int (w_i - w_{is})\, dt - \int (w_0 - w_{0s})\, dt = -\frac{P}{K_m},\ \text{or}$$

$$\int (W_i - W_0)\, dt = -\frac{P}{K_m}, \tag{10}$$

$$\begin{aligned} \text{where } w_i - w_{is} &= W_i = change\ in\ supply\ rate, \\ w_0 - w_{0s} &= W_0 = change\ in\ demand\ rate. \end{aligned}$$

P,W_i and W_0 reflect deviation from initial equilibrium values, and hence have initial values equal to zero. Eq. (10) can also be expressed as given below:

$$P = -K_m \int W dt = -K_m M, \tag{11}$$

$$\text{where } W = W_i - W_0.$$

If the price gets a jump due to an input other than a change in inventory, the output is just a sum of the impact of various inputs for a linear dynamical system; therefore, P in eq. (11) can be expressed as given below:

$$P = -K_m \int W dt + J = -K_m M + J. \tag{11a}$$

Inventory can also get an exogenous shock, such as fire, etc.

2.2. Producer

The producer maximizes the present discounted value of a series of future profits with zero time value as given below:

$$V(0) = \int_0^\infty \left[\alpha p(t) F(K(t), L(t)) - w(t) L(t) - \Re(t) I(t)\right] e^{-rt} dt, \tag{12}$$

with the following description of variables in the above expression: α as the fraction of the market price charged by the producer to the middleman, $L(t)$ (labor) and $I(t)$ (level of investment) as *control variables*, and $K(t)$ as *state variable*. The producer's maximization problem is as given below:

$$\underset{\{L(t),I(t)\}}{Max} V(0) = \int_0^\infty \left[\alpha p(t)F\left(K\left(t\right),L\left(t\right)\right) - w(t)L\left(t\right) - \Re(t)I(t)\right]e^{-rt}dt,$$

subject to the following constraints:

$\dot{K}(t) = I(t) - \delta K(t)$ (state equation, which describes change in state variable with respect to time),

$K(0) = K_0$ (initial condition),

$K(t) \geq 0$ (non-negativity constraint on state variable),

$K(\infty)$ free (terminal condition).

For this case, the current-value Hamiltonian can be expressed as given below:

$$\widetilde{H} = \alpha p(t)F\left(K\left(t\right),L\left(t\right)\right) - w(t)L\left(t\right) - \Re(t)I(t) + \mu(t)\left[I(t) - \delta K(t)\right]. \tag{13}$$

Maximizing conditions can be written as follows:

(i) $L^*(t)$ and $I^*(t)$ maximize $\widetilde{H}$ for all t: $\frac{\partial\widetilde{H}}{\partial L} = 0$ and $\frac{\partial\widetilde{H}}{\partial I} = 0$,

(ii) $\dot{\mu} - r\mu = -\frac{\partial\widetilde{H}}{\partial K}$,

(iii) $\dot{K}^* = \frac{\partial\widetilde{H}}{\partial\mu}$ (this just gives back the state equation),

(iv) $\lim_{t\to\infty}\mu(t)K(t)e^{-rt} = 0$ (the transversality condition).

The first two maximizing conditions can be expressed as given below:

$$\frac{\partial\widetilde{H}}{\partial L} = 0, \tag{14}$$

$$\frac{\partial\widetilde{H}}{\partial I} = 0, \tag{15}$$

and

$$\dot{\mu} - r\mu = -\frac{\partial\widetilde{H}}{\partial K}. \tag{16}$$

After a price increase, the producer needs to increase production to satisfy the new dynamic optimization condition (see appendix). Let $p =$ market price, $c =$ a reference price (e.g., retail price including production cost, and profits of producer and middleman), i.e., a reference point parameter with respect to which variation in p is the basis of the producer's decision-making regarding production.

$$W_m = \textit{Change in production due to change in price},$$

A higher value of $(p - c)$ provides the producer an incentive to produce more. Therefore,

$$W_m \propto \alpha(p - c), \text{ or}$$

$$W_m = K_s(p - c). \tag{17}$$

During market equilibrium, $W_m = 0$, which implies that

$$0 = K_s(p_s - c_s). \tag{18}$$

K_s is the constant of proportionality; p_s and c_s are values in steady-state equilibrium. Subtracting eq. (18) from (17) gives the following expression:

$$W_m = K_s\left[(p - p_s) - (c - c_s)\right] = -K_s\left(C - P\right) = -K_s\varepsilon, \tag{19}$$

where W_m, C and P reflect deviation from initial equilibrium values, and hence have initial values equal to zero.

2.3. Consumer

The consumer maximizes the present discounted value of a series of future utilities with zero time value as given below:

$$V(0) = \int_0^\infty U(x(t))e^{-\rho t}\,dt, \tag{20}$$

with the following description of variables in the above expression: ρ as the discount rate and $x(t)$ as the control variable. The maximization problem can be written as

$$\underset{\{x(t)\}}{Max} V(0) = \int_0^\infty U(x(t))e^{-\rho t}\,dt,$$

subject to the following constraints:

$\dot{a}(t) = R(t)a(t) + w(t) - p(t)x(t)$ (state equation, which describes change in state variable with respect to time). $a(t)$ as asset holdings is a *state variable*, and $w(t)$ and $R(t)$ are exogenous time paths of wages and return on assets.

$a(0) = a_s$ (initial condition),

$a(t) \geq 0$ (non-negativity constraint on state variable),

$a(\infty)$ free (terminal condition).

For this case, the current-value Hamiltonian can be expressed as given below:

$$\widetilde{H} = U(x(t)) + \mu(t)\left[R\left(t\right)a\left(t\right) + w\left(t\right) - p\left(t\right)x\left(t\right)\right]. \tag{21}$$

Maximizing conditions can be written as follows:

(i) $x^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial x} = 0$,

(ii) $\dot{\mu} - \rho\mu = -\frac{\partial \widetilde{H}}{\partial a}$,

(iii) $\dot{a}^* = \frac{\partial \widetilde{H}}{\partial \mu}$ (this just gives back the state equation),

(iv) $\lim\limits_{t\to\infty} \mu(t)a(t)e^{-\rho t} = 0$ (the transversality condition).

The first two maximizing conditions can be expressed as given below:

$$\frac{\partial \widetilde{H}}{\partial x} = U'(x\left(t\right)) - \mu(t)p(t) = 0, \tag{22}$$

and

$$\dot{\mu} - \rho\mu = -\frac{\partial \widetilde{H}}{\partial a} = -\mu(t)R(t). \tag{23}$$

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For an increase in the price of the consumption good, the consumer faces the following expression at the existing level of consumption:

$$\frac{\partial\widetilde{H}}{\partial x} = U'(x\,(t)) - \mu(t)p(t) < 0.$$

To satisfy the modified dynamic optimization condition after a price increase, the consumer must reduce consumption. If the change in consumption is proportional to the change in price, the following formulation results:

$$\textit{Change in demand} \propto P, \text{ or}$$

$$W_d = -K_d P. \tag{24}$$

W_d is the change in demand due to the change in price, i.e., when P is positive, W_d is negative.

3. Solution of the Model with a Consumption Tax

The model has been solved (see appendix), resulting in the following expression:

$$\frac{dP(t)}{dt} = -K_m\left[-K_s\varepsilon(t) + K_d P(t)\right]. \tag{25}$$

If an ad valorem consumption tax T is imposed on the buyer, the market price the buyer will be paying will be inclusive of the consumption tax; however, the price consideration for the producer's decision-making regarding how much to produce will be the one before tax, i.e.,

$$\varepsilon(t) = Tp(t) - P(t). \tag{26}$$

This implies that,

$$\frac{dP(t)}{dt} + K_m\left\{K_s(1-T) + K_d\right\}P(t) = K_m K_s T p_s. \tag{27}$$

For the above differential equation, the stability criterion, i.e., Routh-Hurwitz, is as follows: $K_m\left\{K_s(1-T)+K_d\right\} > 0$. If this criterion is met, the market will always adjust on its own to another equilibrium after a shock. Solving the differential equation with initial conditions, $t = 0$, $P(0) = Tp_s$, we obtain the following expression:

$$P(t) = C_1 + C_2 e^{-[K_m\{K_s(1-T)+K_d\}]t}. \tag{28}$$

After putting values of C_1 and C_2 in eq. (28), we obtain the following expression:

$$P(t) = \frac{K_s T p_s}{\{K_s(1-T)+K_d\}} + \frac{(K_d - K_s T)Tp_s}{\{K_s(1-T)+K_d\}}e^{-[K_m\{K_s(1-T)+K_d\}]t}. \tag{29}$$

Initial conditions, i.e., $t = 0$, $P(0) = Tp_s$ are being satisfied. When $t = \infty$, $P(\infty) = \frac{K_s T p_s}{\{K_s(1-T)+K_d\}}$, which is the final steady-state equilibrium value. In final equilibrium, quantity demanded equals quantity supplied (see appendix).

4. An Optimal Ad Valorem Tax

Deadweight loss due to the imposition of a tax in post-tax equilibrium is the only efficiency loss taken into consideration in the existing literature for the mathematical derivation of an optimal tax; however, there is also a loss of efficiency during the adjustment of the market to final equilibrium after the imposition of tax. When a tax is imposed on goods, the price jumps to the initial price plus the amount of tax, and gradually adjusts to bring the final post-tax equilibrium in which the price is higher than the initial price and less than the one existing when the tax was imposed, depending on the elasticities of demand and supply schedules. If inventory grows in size, it indicates that supply is higher than demand, and vice versa. When supply becomes equal to demand after the adjustment of the market, the final equilibrium has arrived. During adjustment, due to supply and demand not being equal, there is a loss of output/consumption. Also, there is a lower level of production in post-tax equilibrium, which implies that not all resources are fully employed in the final equilibrium and hence some efficiency loss. By summing up the total consumption or production lost, total efficiency loss can be computed and is as follows:

$$EL = -\int_0^\infty W_d(t)dt = M(t) - \int_0^\infty W_m(t)dt. \tag{30}$$

For each unit of consumption/production in the economy, efficiency gain is as follows:

$$\begin{aligned} \textit{Efficiency gain} &= (\textit{willingness to pay} - \textit{consumer price}) + (\textit{tax}) + \\ &(\textit{producer price} - \textit{factors of production cost}) + (\textit{factors of production cost} - \textit{natural resources}). \\ &= \textit{willingness to pay} - \textit{natural resources}. \\ &\simeq \textit{willingness to pay}. \end{aligned}$$

Using the above concept of efficiency, loss in efficiency in value terms due to the imposition of tax is as follows:

$$\begin{aligned} EL &= -\int_0^\infty \left[\frac{1}{2}\{\text{new price}(t) - \text{old equilibrium price}(t)\} + \{\text{old equilibrium price}(t)\}\right] W_d(t)dt \\ &= -\int_0^\infty \left[\frac{1}{2}P(t) + p_s\right] W_d(t)dt \end{aligned}$$

The above expression can be written as given below:

$$EL = \int_0^\infty K_d P(t)\left[\frac{1}{2}P(t) + p_s\right] dt, \tag{31}$$

p_s is the price in initial equilibrium. Efficiency loss in value terms gets minimized when it is minimized in terms of quantity. The initial value of $-W_d(t)$ is as follows: $-W_d(0) = K_d T p_s$ (i.e., decrease in demand due to tax at $t = 0$ from eq. (24)). Figure 3 illustrates that the consumption change jumps to $K_d T p_s$, i.e., there is a decrease in demand due to the imposition of tax at $t = 0$. Demand is not equal to supply any longer, and market forces come into play. Price, along with demand, adjusts over time until the final equilibrium arrives, i.e., $W_d(\infty)$. The shaded area in the figure is the efficiency loss, i.e., consumption lost during the adjustment of the market to the final equilibrium. The area between lines $-W_d(t) = 0$, and $-W_d(t) = -W_d(\infty)$ is the efficiency loss due to a shift in equilibrium caused by the tax. The tax revenue expression is as given below:

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$$TR = Tp(t)\left[w_{id}(0) - K_d P(t)\right]. \tag{32}$$

The problem of minimizing efficiency loss subject to a revenue constraint, i.e., tax revenue generated is greater than or equal to G in a given time period, is as given below:

$$\min_T EL \quad \text{s.t.} \quad TR \geq G.$$

The tax rate is the control variable with the constraint being binding. No closed-form solution exists; hence, a numerical solution is found as given below:

Suppose the tax revenue target is \$1000. With $K_m = K_d = K_s = 1, w_{id}(0) = 100$, and $p_s = 10$, the tax revenue expression can be written as given below:

$$T\left[P(t) + 10\right]\left[100 - P(t)\right] = 1000.$$

For $t = 0$, $P(0) = Tp_s = 10T$, we have the above expression as follows:

$$T^3 - 9T^2 - 10T + 10 = 0.$$

This implies that

$$T = -1.556098,\ 0.648635, 9.907463.$$

As the price has to be positive, $T = -1.556098$ gets overruled. Among the other two values, expression (31) gets minimized for $T = 0.648635$. For $t = \infty$, the expression for tax revenue is as given below:

$$11T^2 - 25T + 10 = 0.$$

This implies that

$$T = 0.518115,\ 1.754612.$$

Expression (31) for efficiency loss is minimized for $T = 0.518115$. The optimal consumption tax is that the government initially imposes a tax $T = 0.648635$, and then gradually decreases it to a final value of $T = 0.518115$.

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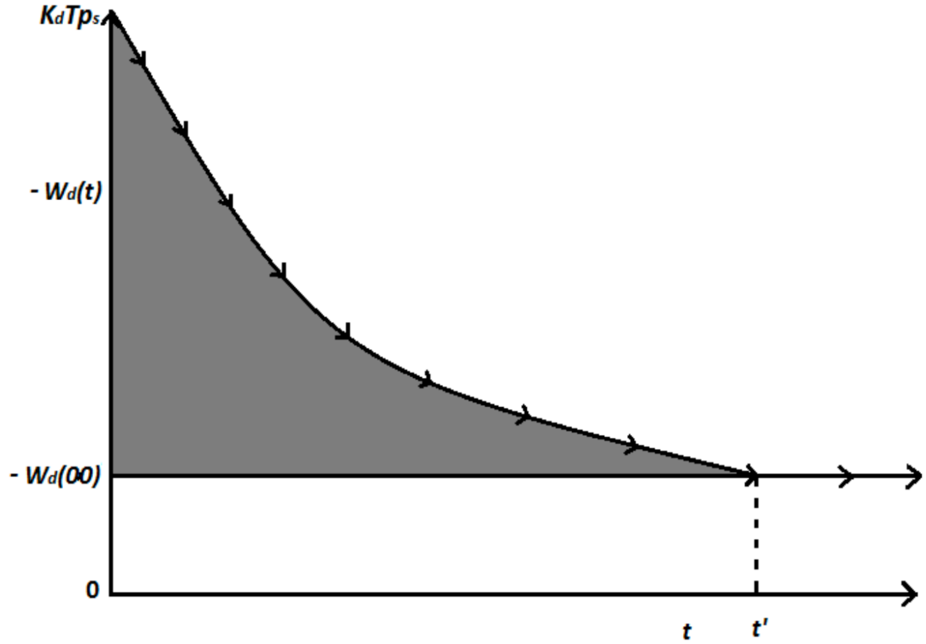

Figure 3. Dynamic Efficiency Loss because of an Ad Valorem Tax.

5. Conclusion

When an ad valorem consumption tax is imposed on a buyer, the price of goods jumps to the initial price plus tax. Market forces come into play, making supply and demand adjust to result in a final equilibrium. Traditionally, it is just the efficiency loss in the final equilibrium that is considered for the derivation of an optimal tax; however, when the market adjusts to arrive at a new equilibrium, there are extra efficiency losses as supply and demand are not equal during market adjustment. If efficiency losses during market adjustment are ignored, the tax schedule derived is not optimal. The last section dealt with an optimal tax schedule derivation generating a target amount of tax revenue considering the adjustment of market supply and demand. For the estimation of an optimal consumption tax schedule, data to estimate the slopes of the supply, demand, and inventory curves; price; and quantity in the initial equilibrium is required.

6. Appendix

6.1. Dynamic Problem of the Middleman

In this section, the dynamic problem of the middleman is discussed. The middleman maximizes the present discounted value of a series of future profits with zero time value as given below:

$$V(0)=\int_{0}^{\infty}\left[pq(p)-\varsigma(m(p,e))\right]e^{-rt}dt, \tag{33}$$

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with the following description of variables in the above expression: r as the discount rate, $p(t)$ as the *control variable*, and $m(t)$ as the *state variable*. The middleman's maximization problem is as given below:

$$\underset{p(t)}{\operatorname{Max}} V(0) = \int_0^\infty [p(t)q(p) - \varsigma(m(p,e))]e^{-rt}\,dt,$$

subject to the following constraints: $\dot{m}(t) = m'_1(p(t), e(p(t), z))\dot{p}(t) + m'_2(p(t), e(p(t), z))e'_1(p(t), z)\dot{p}(t)$ (state equation, which describes the change in the state variable with respect to time; and z being exogenous inputs in the model),

$m(0) = m_s$ (initial condition),

$m(t) \geq 0$ (non-negativity constraint on the state variable),

$m(\infty)$ free (terminal condition).

For this case, the current-value Hamiltonian can be expressed as given below:

$$\begin{aligned}\widetilde{H} &= p(t)q(p(t)) - \varsigma(m(p(t), e(p(t), z))) + \mu(t)\dot{p}(t) \\ &[m'_1(p(t), e(p(t), z)) + m'_2(p(t), e(p(t), z)) * e'_1(p(t), z)]\end{aligned} \tag{34}$$

Maximizing conditions can be written as follows:

(i) $p^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial p} = 0$,

(ii) $\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial m}$,

(iii) $\dot{m}^* = \frac{\partial \widetilde{H}}{\partial \mu}$ (this just gives back the state equation),

(iv) $\lim_{t\to\infty} \mu(t)m(t)e^{-rt} = 0$ (the transversality condition).

The first two maximizing conditions can be expressed as given below:

$$\begin{aligned}\frac{\partial \widetilde{H}}{\partial p} =& q(p(t)) + p(t)q'(p(t)) \\ &- \varsigma'(m(p(t), e(p(t), z)))\,[m'_1(p(t), e(p(t), z)) + m'_2(p(t), e(p(t), z))e'_1(p(t), z)] \\ &+ \mu(t)\dot{p}(t)\,[m''_{11}(p(t), e(p(t), z)) + m''_{12}(p(t), e(p(t), z))e'_1(p(t), z) + m''_{21}(p(t), e(p(t), z))e'_1(p(t), z)] \\ &+ m''_{22}(p(t), e(p(t), z))e'^2_1(p(t), z) + m'_2(p(t), e(p(t), z))e''_{11}(p(t), z) = 0\end{aligned} \tag{35}$$

and

$$\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial m} = \varsigma'(m(p(t), e(p(t), z))). \tag{36}$$

In equilibrium, $\dot{p}(t) = 0$, and $\frac{\partial \widetilde{H}}{\partial p}$ reduces to the following expression:

$$q(p(t)) + p(t)q'(p(t)) - \varsigma'(m(p(t), e(p(t), z)))\,[m'_1(p(t), e(p(t), z)) + m'_2(p(t), e(p(t), z))e'_1(p(t), z)] = 0,$$

$$\begin{aligned}p(t)q'(p(t)) + q(p(t)) &= \varsigma'(m(p(t), e(p(t), z)))\,\{m'_1(p(t), e(p(t), z)) \\ &\quad + m'_2(p(t), e(p(t), z)) * e'_1(p(t), z)\}, \\ p(t)\left[1 + \frac{1}{\text{demand elasticity}}\right] &= \varsigma'(m(p(t), e(p(t), z)))\left\{\frac{m'_1(p(t), e(p(t), z))}{q'(p(t))}\right. \\ &\quad \left. + \frac{m'_2(p(t), e(p(t), z))e'_1(p(t), z)}{q'(p(t))}\right\}.\end{aligned}$$

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In a dynamic setting, the right side of the above expression is the marginal cost, which is quite different from that in the short-term/myopic problem, due to the fact that for a dynamic problem, the middleman also considers the impact of the price he/she chooses on the future purchase price from the producer. Price equals marginal cost for an infinitely elastic demand. Suppose a positive supply shock hits the market, and the middleman wants to increase the size of the inventory. To have an extra unit in inventory, the middleman's marginal cost is higher at the existing price due to the term $\varsigma'(m(p(t), e(p(t), z)))$, which is higher at the previous price at that time. The term in parentheses in the above expression, i.e., $\frac{m_1'(p(t),e(p(t),z))}{q'(p(t))} + \frac{m_2'(p(t),e(p(t),z))e_1'(p(t),z}{q'(p(t))}$ being a function of price, is the same as before until the price gets changed by the middleman. Therefore, at the existing price, the middleman's profit-maximizing expression changes to the following:

$$\frac{\partial \widetilde{H}}{\partial p} = q(p(t)) + p(t)q'(p(t)) - \varsigma'(m(p(t), e(p(t), z))) \begin{Bmatrix} m_1'(p(t), e(p(t), z)) + m_2'(p(t), e(p(t), z)) * \\ e_1'(p(t), z) \end{Bmatrix}$$

$$+\mu(t)\dot{p}(t) * \begin{bmatrix} m_{11}''(p(t), e(p(t), z)) + m_{12}''(p(t), e(p(t), z))e_1'(p(t), z) + \\ m_{21}''(p(t), e(p(t), z))e_1'(p(t), z) + m_{22}''(p(t), e(p(t), z))e_1'^2(p(t), z) + \\ m_2'(p(t), e(p(t), z))e_{11}''(p(t), z) \end{bmatrix}$$

$$< 0.$$

This implies that at the previous price after a supply shock, the middleman's profit-maximizing condition is not being satisfied if he wants to have an extra unit of inventory; therefore, after a supply shock, the middleman must decrease the price to have another unit and to maximize profits. To increase inventory, the price must be decreased; therefore, there is a negative relationship between price and inventory change. Inventory is the state between supply and demand, and hence unifies both kinds of shocks, i.e., each kind of shock influences the inventory size; therefore, each kind of shock is just an inventory shock. If supply equals demand, the market is in equilibrium; however, if any kind of shock happens and either supply or demand or both rates get changed, and the economic agents do not respond to the shock, the price will be changing continuously until the system saturates, e.g., if a positive exogenous supply shock happens, and the producer and consumer do not modify their responses with a change in price, the market will get flooded with supply till the point of saturation. This response can be depicted by the following mathematical expression:

$$\begin{aligned}
Price\ change &\propto change\ in\ market\ inventory. \\
P &= price\ change. \\
M &= m - m_s = change\ in\ inventory\ in\ the\ market, \\
m &= inventory\ at\ time\ t, \\
m_s &= inventory\ in\ steady\ state\ equilibrium. \\
Input - output &= \frac{dm}{dt} = \frac{d(m - m_s)}{dt} = \frac{dM}{dt}, \\
\text{or } M &= \int (input - output)\, dt. \\
Price\ change &\propto \int (supply\ rate - demand\ rate)\, dt, \text{ or} \\
P &= -K_m \int (supply\ rate - demand\ rate)\, dt,
\end{aligned}$$

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where K_m is the constant of proportionality. *Supply* and *demand rates* are flow variables and reflect the flow of supply and demand respectively per unit time in the market. When (*supply rate* $-$ *demand rate*) is positive, P is negative, i.e., excessive supply compared to demand leads to a decrease in market price and vice versa. The above equation can also be written as follows:

$$\int (supply\ rate - demand\ rate)\, dt = -\frac{P}{K_m},\ \text{or}$$

$$\int (w_i - w_0)\, dt = -\frac{P}{K_m}, \tag{37}$$

$$\begin{aligned} w_i &= supply\ rate, \\ w_0 &= demand\ rate, \\ K_m &= dimensional\ constant. \end{aligned}$$

At $t = 0$, the market is in a steady state equilibrium, and *supply rate* equals *demand rate*. Putting initial conditions in eq. (37), it can be expressed as given below:

$$\int (w_{is} - w_{0s})\, dt = 0. \tag{38}$$

Subscript s stands for steady state equilibrium, the state which reflects initial values of the market, and $P = 0$, when the market is in a steady state equilibrium. Subtracting eq. (38) from (37) results in the following expression:

$$\int (w_i - w_{is})\, dt - \int (w_0 - w_{0s})\, dt = -\frac{P}{K_m},\ \text{or}$$

$$\int (W_i - W_0)\, dt = -\frac{P}{K_m}, \tag{39}$$

$$\begin{aligned} \text{where } w_i - w_{is} = W_i &= change\ in\ supply\ rate, \\ w_0 - w_{0s} = W_0 &= change\ in\ demand\ rate. \end{aligned}$$

P, W_i and W_0 reflect deviation from initial equilibrium values, and hence have initial values equal to zero. Eq. (39) can also be expressed as given below:

$$P = -K_m \int W dt = -K_m M, \tag{40}$$

$$\text{where } W = W_i - W_0.$$

If the price gets a jump due to an input other than change in inventory, the output is just a sum of the impact of various inputs for a linear dynamical system, therefore P in eq. (40) can be expressed as given below:

$$P = -K_m \int W dt + J = -K_m M + J. \tag{43a}$$

6.2. Producer

The producer maximizes the present discounted value of a series of future profits with zero time value as given below:

$$V(0) = \int_0^\infty \left[\alpha p(t) F\left(K\left(t\right), L\left(t\right)\right) - w(t) L\left(t\right) - \Re(t) I(t)\right] e^{-rt}\, dt, \tag{41}$$

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with the following description of variables in the above expression: α as the fraction of the market price charged by the producer to the middleman, $L(t)$ (labor) and $I(t)$ (level of investment) as *control variables*, and $K(t)$ as *state variable*. The producer's maximization problem is as given below:

$$\underset{\{L(t),I(t)\}}{Max} V(0) = \int_0^\infty \left[\alpha p(t) F\left(K\left(t\right), L\left(t\right)\right) - w(t) L\left(t\right) - \Re(t) I(t)\right] e^{-rt} dt,$$

subject to the following constraints:

$\dot{K}(t) = I(t) - \delta K(t)$ (state equation, which describes the change in the state variable with respect to time),

$K(0) = K_0$ (initial condition),

$K(t) \geq 0$ (non-negativity constraint on the state variable),

$K(\infty)$ free (terminal condition).

For this case, the current-value Hamiltonian can be expressed as given below:

$$\widetilde{H} = \alpha p(t) F\left(K\left(t\right), L\left(t\right)\right) - w(t) L\left(t\right) - \Re(t) I(t) + \mu(t)\left[I(t) - \delta K(t)\right]. \tag{42}$$

Maximizing conditions can be written as follows:

(i) $L^*(t)$ and $I^*(t)$ maximize $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial L} = 0$ and $\frac{\partial \widetilde{H}}{\partial I} = 0$,

(ii) $\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial K}$,

(iii) $\dot{K}^* = \frac{\partial \widetilde{H}}{\partial \mu}$ (this just gives back the state equation),

(iv) $\lim\limits_{t\to\infty} \mu(t) K(t) e^{-rt} = 0$ (the transversality condition).

The first two maximizing conditions can be expressed as given below:

$$\frac{\partial \widetilde{H}}{\partial L} = \alpha p(t) F_2'\left(K\left(t\right), L\left(t\right)\right) - w(t) = 0, \tag{43}$$

$$\frac{\partial \widetilde{H}}{\partial I} = -\Re(t) + \mu(t) = 0, \tag{44}$$

and

$$\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial K} = -\left[\alpha p(t) F_1'\left(K\left(t\right), L\left(t\right)\right) - \delta\mu(t)\right]. \tag{45}$$

Substituting $\dot{\mu}$ and μ from eq. (44) into (45) yields

$$\alpha p(t) F_1'\left(K\left(t\right), L\left(t\right)\right) - (r+\delta)\Re(t) + \dot{\Re}(t) = 0.$$

If $p(t)$ goes up (at the previous level of investment and labor), the producer faces the following expressions:

$$\alpha p(t) F_2'\left(K\left(t\right), L\left(t\right)\right) - w(t) > 0,$$

$$\alpha p(t) F_1'\left(K\left(t\right), L\left(t\right)\right) - (r+\delta)\Re(t) + \Re(t) > 0.$$

After the price increase, the producer needs to increase production to satisfy the new dynamic optimization condition (see appendix). Let $p =$ be the market price, $c =$ a reference price (e.g., retail price including production cost, and

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profits of the producer and middleman), i.e., a reference point parameter with respect to which variation in p is the basis of the producer's decision-making regarding production.

$$W_m = \textit{Change in production due to change in price},$$

A higher value of $(p - c)$ provides the producer an incentive to produce more. Therefore,

$$W_m \propto \alpha(p - c), \text{ or}$$

$$W_m = K_s(p - c). \tag{46}$$

During market equilibrium, $W_m = 0$, which implies that

$$0 = K_s(p_s - c_s). \tag{47}$$

K_s is the constant of proportionality; p_s and c_s are values in steady state equilibrium. Subtracting eq. (47) from (46) gives the following expression:

$$W_m = K_s\left[(p - p_s) - (c - c_s)\right] = -K_s\left(C - P\right) = -K_s\varepsilon, \tag{48}$$

where W_m, C and P reflect deviation from initial equilibrium values, and hence have initial values equal to zero.

6.3. Solution of the Model with a Consumption Tax

Eqs. (11a), (19) and (24) are reproduced as follows:

$$\frac{dP(t)}{dt} = -K_m W(t),$$

$$W_m(t) = -K_s\varepsilon(t),$$

$$\varepsilon(t) = C(t) - P(t),$$

$$W_d(t) = -K_d P(t),$$

and

$$W(t) = W_m(t) - W_d(t),$$

in the absence of an exogenous supply/demand shock. The above equations can be combined as given below:

$$\begin{aligned}\frac{dP(t)}{dt} &= -K_m\left[W_m(t) - W_d(t)\right] \\ &= -K_m\left[-K_s\varepsilon(t) + K_d P(t)\right] \\ &= -K_m\left[-K_s C(t) + (K_s + K_d)P(t)\right].\end{aligned}$$

After rearranging the above expression, we get:

$$\frac{dP(t)}{dt} + K_m(K_s + K_d)P(t) = K_m K_s C(t). \tag{49}$$

If an ad valorem consumption tax T is imposed on the buyer, the market price the buyer will be paying will be inclusive of the consumption tax; however, the price consideration for the producer's decision-making regarding how much to produce will be the one before tax, i.e.,

$$\varepsilon(t) = Tp(t) - P(t). \tag{50}$$

This implies that

$$\frac{dP(t)}{dt} = -K_m \left[K_s \left\{P(t) - Tp(t)\right\} + K_d P(t)\right],$$
$$\frac{dP(t)}{dt} = -K_m \left[K_s \left\{P(t) - TP(t) - Tp_s\right\} + K_d P(t)\right].$$

After rearranging, the following expression is obtained:

$$\frac{dP(t)}{dt} + K_m \left\{K_s(1-T) + K_d\right\} P(t) = K_m K_s T p_s. \tag{51}$$

The characteristic function of the above differential equation is as given below:

$$x + K_m \left\{K_s(1-T) + K_d\right\} = 0.$$

The single root of the characteristic function is given by:

$$x = -K_m \left\{K_s(1-T) + K_d\right\},$$

with the following complementary solution:

$$P_c(t) = C_2 e^{-[K_m\{K_s(1-T)+K_d\}]t}.$$

The particular solution can be expressed as given below:

$$P_p(t) = C_1.$$

Therefore, the solution can be written in the following form:

$$P(t) = C_1 + C_2 e^{-[K_m\{K_s(1-T)+K_d\}]t}. \tag{52}$$

After substitution of the above expression in the differential equation, the following equation is obtained:

$$-K_m \left\{K_s(1-T) + K_d\right\} C_2 e^{-[K_m\{K_s(1-T)+K_d\}]t} + K_m \left\{K_s(1-T) + K_d\right\} C_1 +$$

$$K_m \left\{K_s(1-T) + K_d\right\} C_2 e^{-[K_m\{K_s(1-T)+K_d\}]t} = K_m K_s T p_s,$$

$$C_1 = \frac{K_s T p_s}{\left\{K_s(1-T) + K_d\right\}}.$$

Putting initial conditions, the value of C_2 can be determined as given below:

$$\begin{aligned}
P(0) &= \frac{K_s T p_s}{\left\{K_s(1-T) + K_d\right\}} + C_2 = Tp_s, \\
C_2 &= Tp_s - \frac{K_s T p_s}{\left\{K_s(1-T) + K_d\right\}} \\
&= \frac{K_s T p_s - K_s T^2 p_s + K_d T p_s - K_s T p_s}{\left\{K_s(1-T) + K_d\right\}}, \\
&= \frac{(K_d - K_s T) T p_s}{\left\{K_s(1-T) + K_d\right\}}.
\end{aligned}$$

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After putting values of C_1 and C_2 in eq. (52), we obtain the following expression:

$$P(t) = \frac{K_s T p_s}{\{K_s(1-T)+K_d\}} + \frac{(K_d - K_s T)Tp_s}{\{K_s(1-T)+K_d\}} e^{-[K_m\{K_s(1-T)+K_d\}]t}. \tag{53}$$

Initial conditions, i.e., $t=0$, $P(0)=Tp_s$ are being satisfied. When $t=\infty$, $P(\infty)=\frac{K_s T p_s}{\{K_s(1-T)+K_d\}}$, which is the final steady-state equilibrium value. In final equilibrium, quantity demanded must equal quantity supplied. This is verified as follows: From eq. (??), change in demand due to a change in price after imposition of tax is as given below:

$$W_d(t) = -K_d P(t),$$

$$\text{or } w_{nd}(t) - w_{id}(0) = -K_d P(t),$$

where $w_{id}(0)$ is initial demand and $w_{nd}(t)$ is new demand after tax, because $W_d(t)$ is a deviation variable, i.e., deviation from the initial equilibrium value. Similarly, from eq. (48), for supply we have,

$$W_m(t) = -K_s \varepsilon(t),$$

$$w_{nm}(t) - w_{im}(0) = -K_s\left[Tp(t) - P(t)\right], \text{ or}$$

$$w_{nm}(t) - w_{im}(0) = -K_s\left[TP(t) + Tp_s - P(t)\right].$$

In final equilibrium

$$w_{nm}(\infty) = w_{nd}(\infty), \text{ or}$$

$$w_{im}(0) - K_s\left[\{T-1\}P(\infty) + Tp_s\right] = w_{id}(0) - K_d P(\infty),$$

which holds as in initial equilibrium, quantity demanded equals quantity supplied, i.e.

$$w_{im}(0) = w_{id}(0).$$

6.4. An Optimal Ad Valorem Tax

Deadweight loss due to the imposition of a tax in post-tax equilibrium is the only efficiency loss taken into consideration in the existing literature for the mathematical derivation of an optimal tax; however, there is also a loss of efficiency during the adjustment of the market to final equilibrium after the imposition of tax. When a tax is imposed on goods, the price jumps to the initial price plus the amount of tax, and gradually adjusts to bring the final post-tax equilibrium in which the price is higher than the initial price and less than the one existing when the tax was imposed, depending on the elasticities of demand and supply schedules. If inventory grows in size, it indicates that supply is higher than demand, and vice versa. When supply becomes equal to demand after adjustment of the market, the final equilibrium has arrived. During adjustment, due to supply and demand not being equal, there is a loss of output/consumption. Also, there is a lower level of production in post-tax equilibrium, which implies that not all resources are fully employed in the final equilibrium and hence some efficiency loss. By summing up the total consumption or production lost, total efficiency loss can be computed and is as follows:

$$EL = -\int_{0}^{\infty} W_d(t)dt = M(t) - \int_{0}^{\infty} W_m(t)dt. \tag{54}$$

For each unit of consumption/production in the economy, efficiency gain is as follows:

$$\begin{aligned}
\textit{Efficiency gain} &= (\textit{willingness to pay} - \textit{consumer price}) + \textit{tax} + \\
&\quad (\textit{producer price} - \textit{factors of production cost}) + (\textit{factors of production cost} - \textit{natural resources}). \\
&= \textit{willingness to pay} - \textit{natural resources.} \\
&\simeq \textit{willingness to pay.}
\end{aligned}$$

Using the above concept of efficiency, the loss in efficiency in value terms due to the imposition of tax is as follows:

$$\begin{aligned}
EL &= -\int_{0}^{\infty} \left[\frac{1}{2}\{\text{new price}(t) - \text{old equilibrium price}(t)\} + \{\text{old equilibrium price}(t)\} \right] W_d(t)dt \\
&= -\int_{0}^{\infty} \left[\frac{1}{2} P(t) + p_s \right] W_d(t)dt.
\end{aligned}$$

The above expression can be written as given below:

$$EL = \int_{0}^{\infty} K_d P(t) \left[\frac{1}{2} P(t) + p_s \right] dt, \tag{62a}$$

p_s is the price in the initial equilibrium. Efficiency loss in value terms gets minimized when it is minimized in terms of quantity. The initial value of $-W_d(t)$ is as follows: $-W_d(0) = K_d T p_s$ (i.e., decrease in demand due to tax at $t = 0$ from eq. (24)). Figure 3 illustrates that the consumption change jumps to $K_d T p_s$, i.e., there is a decrease in demand due to the imposition of tax at $t = 0$. Demand is not equal to supply any longer, and market forces come into play. Price along with demand adjusts over time until the final equilibrium arrives, i.e., $W_d(\infty)$. The shaded area in the figure is the efficiency loss, i.e., consumption lost during the adjustment of the market to the final equilibrium. The area between lines $-W_d(t) = 0$, and $-W_d(t) = -W_d(\infty)$ is the efficiency loss due to a shift in equilibrium caused by tax. From eq. (??), the change in demand due to the change in price after the imposition of tax is as given below:

$$\begin{aligned}
W_d(t) &= -K_d P(t), \\
\text{or } w_{nd}(t) - w_{id}(0) &= -K_d P(t),
\end{aligned}$$

$w_{id}(0)$ is the initial value of demand, and $w_{nd}(t)$ is the value after tax as $W_d(t)$ is a deviation variable, i.e., deviation from the initial value. The tax revenue expression is as given below:

$$TR = Tp(t) \left[w_{id}(0) - K_d P(t) \right]. \tag{55}$$

The problem of minimizing efficiency loss subject to the revenue constraint, i.e., tax revenue generated is greater than or equal to G in a given time period, is as given below:

$$\min_{T} EL \quad \text{s.t.} \quad TR \geq G.$$

The tax rate is the control variable with the constraint being binding. No closed-form solution exists; hence, a numerical solution is found as given below:

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Suppose the tax revenue target is \$1000. With $K_m = K_d = K_s = 1, w_{id}(0) = 100$, and $p_s = 10$, the tax revenue expression can be written as given below:

$$T\left[P(t) + 10\right]\left[100 - P(t)\right] = 1000.$$

For $t = 0$, $P(0) = Tp_s = 10T$, we have the above expression as follows:

$$T\left[T + 1\right]\left[10 - T\right] = 10,$$

$$T(9T - T^2 + 10) - 10 = 0,$$

$$T^3 - 9T^2 - 10T + 10 = 0.$$

This implies that

$$T = -1.556098,\ 0.648635, 9.907463.$$

As the price has to be positive, $T = -1.556098$ gets overruled. Among the other two values, expression $(62a)$ gets minimized for $T = 0.648635$. For $t = \infty$, the expression for tax revenue is as given below:

$$T\left[P(\infty) + 10\right]\left[100 - P(\infty)\right] = 1000,$$

$$T\left[\frac{T}{(2-T)} + 1\right]\left[10 - \frac{T}{(2-T)}\right] = 10,$$

$$T(20 - 11T) = 5(2 - T),$$

$$20T - 11T^2 - 10 + 5T = 0,$$

$$25T - 11T^2 - 10 = 0.$$

This implies that

$$T = 0.518115,\ 1.754612.$$

Expression $(62a)$ for efficiency loss is minimized for $T = 0.518115$. The optimal consumption tax is that the government initially imposes a tax $T = 0.648635$, and then gradually decreases it to a final value of $T = 0.518115$.

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