

# A Dynamic Model for an Optimal Consumption Tax Rate

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#### Abstract

Following Ramsey, the existing literature on optimal taxation only compares the pre and the post-tax market equilibriums in order to account for the efficiency losses. However, when the government imposes an ad valorem tax on the consumer, the buyer's price jumps to the pre-tax equilibrium price plus the amount of the tax, and the supply and the demand of the taxed commodity then adjust over time to bring the new post-tax market equilibrium. The existing literature does not take into account the efficiency losses during the adjustment process while computing the optimal ad valorem taxes. This paper shows how adjustment process influences the optimal ad valorem taxes, minimizing the efficiency losses (output and/ or consumption lost) during the dynamic adjustment process as well as the post-tax market equilibrium. (JEL H20, H21, H22)

Keywords: Consumption tax, Optimal rate, Dynamic efficiency, Adjustment path, Equilibrium

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#### 1 Introduction

A tax on consumption is a consumer tax levied through imposing a tax on purchase of goods and services. It takes the form of a direct or an indirect tax, such as a value-added tax (VAT), sales tax, excise tax on consumption, tariffs at the import stage, or an income tax where savings are tax free. The tax base is the price of consumption goods before tax.

A value-added tax is a consumption tax which is levied and collected at each stage in the supply chain on the difference of sales and purchases of all agents, such as importer, manufacturer, wholesaler, distributer, and retailer, etc. A simple VAT is proportional to consumption, and is regressive in nature, as with an increase in income, the proportion of consumption in total income falls. Investment and savings do not get taxed under the ambit of consumption tax, however, as soon as they get converted to consumption, they are taxed. In some countries, such as European Union, it is common to exclude certain goods from VAT to make it less regressive. Consumption tax can have different nomenclature in different countries, e.g., it is called as a "Goods and Services Tax", in New Zealand, Australia, Singapore, India, and Canada (in Canada, also known as Harmonized Sales Tax, when combined with a provincial sales tax). It is also known as a sales tax in some countries, as it is applied at final point of sale in a supply chain, and is applicable on sale of goods, and/or services. It is an ad valorem tax, i.e., a percentage of final price of goods and services. It is also called as a use tax, when consumers are liable to deposit the tax directly into government treasury. Laws may exempt certain items from consumption tax. An excise tax is also a sales tax, and is applied at times to reduce consumption of certain goods, such as tobacco, alcohol, etc. However, it is also applied for revenue generation, e.g., gasoline (petrol), tourism, etc.

Consumption tax can also be in the form of a direct tax as an expenditure tax, which is an income tax after deduction of savings and investment, such as the Hall–Rabushka flat tax. In the form of a direct tax, it is generally called as an expenditure tax, a cash-flow tax, or a consumed-income tax and can be either a flat or progressive. In the past, some countries had implemented a direct consumption tax, such as India and Sri Lanka. The base of this kind of tax is income minus savings. If the direct consumption tax rate is flat, it is regressive with respect to income, however, it can be made progressive by applying progressive tax rates, i.e., an increase in the tax rate with an increase in personal consumption.

An optimal taxation is the one which minimizes efficiency losses and distortion in the market as a result of deviation from pre-policy efficient market equilibrium, given the economic constraints when a tax is imposed. The first contribution to theory of optimal taxation was made by Ramsey (1927), through developing a theory for optimal commodity taxes and proposed a theoretical solution that consumption tax on each good should be "proportional to the sum of the reciprocals of its supply and demand elasticities". Suits and Musgrave (1953) find that ad valorem taxation yields a larger total

surplus than unit taxes provided they give the same yield. Diamond and Mirrlees (1971) consider commodity taxation along with the other kinds of taxes. Mirrless (1975) modified the standard problem by considering simultaneously excise taxes and a poll tax. Diamond (1975) examines the Ramsey rule for a many-person economy with excise taxes and a poll tax. Atkinson and Stiglitz (1976) show that with an optimal nonlinear income tax, discriminatory commodity taxes are only necessary to the extent that individual commodities are not weakly separable from leisure. In Deaton (1981), rules for optimal differential commodity taxes have been derived for the three different cases usually studied in the literature: the one consumer economy, the unidimensional continuum of consumers economy, and the finite number of discrete consumers economy. Lucas and Stokey (1983) derive a time consistent optimal fiscal policy in an economy without capital maximizing the consumer welfare subject to the condition that a competitive equilibrium holds in each time period.

In Judd (1985), the government taxes capital income net of depreciation at a proportional rate, which is assumed to be constant. Chamley (1986) analyzes the optimal tax on capital income in general equilibrium models of the second best. Deaton and Stern (1986) show that optimal commodity taxes for an economy with many households should be at a uniform proportional rate under certain conditions. Cremer and Gahvari (1993) incorporate tax evasion into Ramsey's optimal taxation problem. Skeath and Trandel (1994) show that ad valorem taxes Pareto dominate specific taxes. Cremer and Gahvari (1995) prove that optimal taxation requires a mix of differential commodity taxes and a uniform lump-sum tax. Naito (1999) shows that imposing a non-uniform commodity tax can Pareto-improve welfare even under nonlinear income taxation if the production side of an economy is taken into the consideration. Saez (2002b) shows that a small tax on a given commodity is desirable if high income earners have a relatively higher taste for this commodity or if consumption of this commodity increases with leisure.

Nordhaus (1993) proposes an optimal carbon tax (tax per ton of carbon). Chari, Christiano and Kehoe (1994) deal with the labor and capital income taxes instead of an advalorem tax as in our model. Ekins (1996) takes into account the secondary benefits of Carbon dioxide abatement for an optimal carbon tax. Coleman (2000) derives the optimal dynamic taxation of consumption, income from labor, and income from capital, and estimates the welfare gain that the US could attain by switching from its current income tax policy to an optimal dynamic tax policy. Pizer (2002) explores the possibility of a hybrid permit system and a dynamic optimal policy path in order to accommodate growth and not because of the adjustment over time to equalize the marginal benefit and cost. It is implicitly assumed that the marginal cost equals the marginal benefit in each time period. Jensen and Schjelderup (2011) study how a change in specific and ad valorem taxes under nonlinear pricing affects tax incidence. Aiura and Ogawa (2013) examine the choice of tax

method between ad valorem tax and specific tax. Blackorby and Murty (2013) employ a general equilibrium framework with a monopoly, hundred percent profit taxation and uniform lump-sum transfers. Nawaz (2017) discusses the optimal quantity taxes in a dynamic setting.

In existing literature, while deriving an optimal tax, efficiency loss in post-tax policy equilibrium/dead weight loss is minimized, however, there are additional efficiency losses during market adjustment toward final post-tax policy equilibrium. Without minimizing total efficiency loss, i.e., the one during market adjustment as well as the dead weight loss in final equilibrium, the derived tax policy cannot be optimal in true sense, and can be improved upon. When a consumption tax is imposed, the buyer's price jumps to the initial price plus the amount of tax. The price adjusts over time to bring final post-tax market equilibrium with some dead weight loss. Supply and demand also adjusts along with the price including tax until the new equilibrium arrives. Existing literature ignores efficiency losses on the adjustment path of the market to final equilibrium after imposition of a tax to derive an optimal consumption tax. The quantum of efficiency loss during market adjustment is contingent upon market parameters as shown in later part (section 4) of this article, however, theoretically ideally total efficiency loss, i.e., during adjustment of market as well as that in final post-tax policy equilibrium must be minimized to derive an optimal tax. This paper considers total efficiency loss (output and/or consumption lost), i.e., during market adjustment as well as the dead weight loss in post-tax policy equilibrium as an objective function to be minimized subject to tax revenue constraint to derive an optimal consumption tax.

The remainder of this paper is organized as follows: Section 2 explains how individual components of market system are joined together to form a dynamic market model. Section 3 provides the solution of the model with a consumption tax. Section 4 derives an optimal consumption tax minimizing efficiency losses subject to a tax revenue constraint. Section 5 summarizes findings and concludes. Section 6 constitutes appendix.

#### 2 The Model

Suppose there is a perfectly competitive market of a homogeneous good, and the market is in equilibrium, i.e., initial conditions of a market equilibrium apply. Four types of infinitely-lived market agents are there, i.e., a representative -or a unit mass of- producer, who demands capital and labor to produce goods; a middleman who purchases goods from producer, holds an inventory of goods, i.e., store them to be subsequently sold, and sells those to consumer; a representative –or a unit mass of– consumer who supplies labor inelastically, accumulates capital through investment and buys goods from the middleman; and government. Middleman's role is instrumental to capture adjustment of market to final equilibrium as producer is a price taker and cannot change price. Middleman is shown to have an incentive to change price only during market adjustment and is not better off deviating from market price once equilibrium is achieved. Middleman sells goods

to the consumer at market price p, which is chosen by maximizing the difference between revenue generated by selling goods to consumer and the cost of holding/storing goods after buying from producer, i.e., cost of holding inventory of goods. Middleman pays a fixed fraction of initial market price to producer, i.e.,  $\alpha p$  with  $\alpha < 1$ , and with fixed  $\alpha$  and p, producer is a price taker.

Price adjustment mechanism has the basis of lack of coordination among buyers and suppliers at current prices when a shock puts the market out of equilibrium, and is illustrated as given below: Suppose the market is in an initial equilibrium, and middleman holds an equilibrium level of inventory due to supply and demand rates being the same. Inventory is the stock variable and reflects the difference between supply and demand rates accumulated over time. A change in inventory happens when either supply or demand or both rates change by different magnitudes. Supply and demand rates are flow variables, i.e., the quantity supplied/demanded per unit time. If an exogenous shock leading to a demand contraction happens to the market, the stock of inventory will pile up at the existing price as the supply from producer continues to be the same as before. Middleman will reduce price which will increase demand along demand schedule, and producer will find optimal to produce a lower quantity than before. A new equilibrium with both lower price and output will be reached. The equilibrium is defined as follows:

(i) The middleman and the producer maximize their profits and the consumer maximizes utility subject to their respective constraints (see Section 2).

(ii) The quantity consumed by the consumer equals the quantity supplied by the producer (the inventory remains the same, when the market is in equilibrium).

The equilibrium conditions, i.e., Routh–Hurwitz stability criterion, which provides a necessary and sufficient condition for the stability of a linear dynamical system, are mentioned in Section 3.

When the market is in equilibrium, the middleman is a price taker and sells the goods to the consumer at the given market price on account of the fact that the set up is for a perfectly competitive market. The middleman can change the price along the dynamic adjustment path when the market is out of equilibrium, until the middleman again becomes a price taker when the new equilibrium arrives. An ad valorem tax is announced and implemented at the same time by the government (the agents' expectations will be taken into account in a future research project when the announcement and implementation dates of the tax could be different). The market does not suddenly jump to the post-tax market equilibrium after the imposition of an ad valorem tax, rather the adjustment of price takes place over time to bring the new equilibrium. The adjustment of price involves endogenous decision making (on the basis of self-interest) by all the agents in the market, i.e., producer, consumer, and the middleman. Suppose a producer produces a perishable good and sells it to a middleman who subsequently sells it to a consumer living in a community. The middleman and the producer sell a quantity equal to the quantity produced by the producer in each time period, and the market stays in equilibrium. Suppose that the government announces and imposes an ad valorem tax on the consumer, which decreases the demand of the good, part of the production sold to the middleman by the producer will remain unsold to the consumer by the end of the time period in which the tax was imposed, and be wasted. If we assume that the middleman and the producer can change the price and the production respectively, immediately, had they known the pattern of new demand, they would immediately pick the price (by the middleman) and quantity (by the producer) to maximize their profits and clear the market without any waste of production. However, this information is lacking, so the middleman could decrease the price based on the best guess he could have about the new demand (based on the amount of unsold production), which would drive the market close to the new equilibrium. The producer produces a lower quantity at the lower price. If the producer's production is fully sold out to the consumer by the middleman in the following time period, he will not be changing the production anymore knowing that the new equilibrium has arrived, however, if some of his production still remains unsold, the middleman will choose to reduce the price further (and the producer, the production accordingly) to bring the market closer to the new equilibrium. The market eventually settles at a new equilibrium after some efficiency losses. The resources which went into the unsold production in each time period by the imposition of the tax are wasted. A new equilibrium will be finally arrived at, with a deadweight loss due to ad valorem tax. As a result of an ad valorem tax, the efficiency loss is the waste of resources during the adjustment period plus the loss in the final equilibrium.

In mathematical terms, the objective function of all the market agents is maximized through the first order conditions and the equations representing their individual actions are solved simultaneously, to capture the collective result of their individual actions. We assume for simplification, that the new equilibrium is not too off from the initial equilibrium after the imposition of the ad valorem tax. This makes the linearization of demand and supply curves a reasonable approach. In figure 1, linearization seems to be a good approximation when moving from point a to b, whereas it does not seem to be a good approximation in the movement from point a to c. For the movement of the market equilibrium from point a to c, we need to model a non-linear dynamical system (not covered under the scope of this article).

#### 2.1 Middleman

The middleman buys goods from the producer and sells those to the consumer for profit. The middleman does not purchase and sell the same quantity at all points in time, and hence holds an inventory of the goods purchased to be sold subsequently. Inventory is an intermediary stage between demand and supply, which reflects the quantum of difference between demand and supply of the goods in the market. If there is no change in inventory, it implies that supply and demand rates are the same. A decrease or increase in inventory implies a change in demand, supply, or both at different rates.

Figure 2 illustrates the link between supply, demand, inventory, and prices. When the supply shifts to the right (while demand does not change), the inventory increases at the initial price, and the new equilibrium brings the price down. Similarly, when the demand shifts to the right (while supply remains the same), the inventory depletes from the market at the initial price and the new equilibrium brings the price up. This implies that there is an inverse relationship between an inventory change and a price change (ceteris peribus). If both the demand and supply curves shift by the same magnitude such that the inventory remains constant, then the price will also not change. Inventory unifies the demand and supply shocks in the sense that they are both affecting the same factor, i.e., inventory. Therefore, each kind of shock is just an inventory shock. According to the above discussion, there is an inverse relationship between an inventory change and a price change; let us discuss the mechanism which brings about such a change. Consider a market of homogeneous goods where the middlemen, such as whole salers, retailers, etc., hold inventories of goods, incur some cost for holding those, and sell the goods to the consumers to make profits. The cost of holding an inventory is a positive function of the size of the inventory, i.e., a larger inventory is more costly to hold as compared to a smaller inventory. In the absence of an exogenous shock, if the demand and supply rates are equal, then the market is in equilibrium and the price does not change with time. Suppose that a technological advancement decreases the marginal cost of production and increases the supply rate, whereas the demand remains the same. As the supply and demand rates are no longer equal, the difference will appear somewhere in the economy in the form of piled up inventories. As the production flows from producers to the consumers through the middlemen, it is reasonable to assume that the middlemen will be holding the net difference (Note: The piled up inventories can also appear as producers' finished goods, however, the key point is that a difference of demand and supply rates directly affect the inventories in the economy). The economy cannot sustain this situation for an indefinite period of time, and the middlemen have to think of some means of getting rid of piled up inventories. The only resort they have is to decrease the price to bring the demand up along the demand curve.

The price, in a perfectly competitive market, will eventually come down to equalize the new marginal cost, however the adjustment path depends on the actions of the middlemen, i.e., how they react to the change in their inventories. Notice that although the marginal cost of production for the producer has decreased, however, the marginal cost of holding an extra unit of inventory for the middleman has increased. This intuitive explanation is theoretically consistent with the supply, demand, profit, and utility maximization by a producer and a consumer respectively. In the real world, the examples of this kind of behavior of middlemen are as follows: we enjoy the *end of year sales* as consumers, there are offers such as *buy one get one free, gift offers* if consumers buy above a certain quantity threshold, etc. For a mathematical picture, let us consider the profit maximization problem of the *middleman* as follows:

#### 2.1.1 Short-run Problem

Let us first consider the short-run problem (Note: The middleman has a myopic objective, rather than doing dynamic optimization. This is a one period analysis with discrete analog, and is presented for an intuitive purpose for clarity of the more complicated dynamic problem in section 2.1.2) of the middleman as follows:

$$\Pi = pq(p) - \varsigma(m(p, e)), \tag{1}$$

where

 $\Pi = \text{profit},$ 

p = market price,

q(p) = quantity sold at price p,

m =inventory (total number of goods held by the *middleman*),

e = other factors which influence inventory other than the market price including the middleman's purchase price from the producer,

 $\varsigma(m(p,e)) = \text{cost}$  as a function of inventory (increasing in inventory).

The first order condition (with respect to price) is as follows:

$$pq'(p) + q(p) - \varsigma'(m(p,e))m'_1(p,e) = 0,$$
(2)

The middleman's incentive to change the price is only during the adjustment process, and will incur losses by deviating from the price equal to the marginal cost, when the market is in equilibrium. The supply does not equal the demand during the adjustment process, and the market drifts toward the new equilibrium. However, the price cannot change automatically and has to be changed by some economic agent in his/ her own benefit, therefore a change in price by the middleman in the direction of bringing the new equilibrium is not against the market forces, so he/ she does not lose business by changing the price on the dynamic adjustment path to the new equilibrium, unlike when the market is already in equilibrium and the middleman faces an infinitely elastic demand as given below:

$$\begin{split} pq'(p) + q(p) &= \varsigma'(m(p,e))m_1'(p,e), \\ p\left[1 + \frac{1}{demand\ elasticity}\right] &= \varsigma'(m(p,e))\frac{m_1'(p,e)}{q'(p)}. \end{split}$$

The expression on the right hand side is the marginal cost, which equals price when middleman faces an infinitely elastic demand. Suppose a supply shock shifts supply curve downward due to a reduced marginal cost of production, e.g., due to a technological innovation. The competitive market is no longer in equilibrium as supply does not equal demand after supply shock at initial equilibrium price. As supply has expanded, price will come down in final equilibrium, however, there cannot be a sudden jump in price from one equilibrium to the other, and rather middleman will continue charging the same price as before, i.e., higher than the new lower marginal cost until his inventory piles up enough and market forces make him realize that his profit maximization condition has changed due to an expansion in market supply and he needs to reduce price to meet his new profit maximizing condition after supply shock. The same reasoning goes for a reverse supply shock, i.e., in case of shrinkage in market supply, price will increase in final equilibrium, and middleman will not change price and continue charging a price lower than the new higher marginal cost until inventory level goes down substantially to make middleman choose a higher selling price than before. In this scenario, consumer will be the short term beneficiary for paying a price lower than marginal cost. In the first scenario, middleman was a short term beneficiary as he charged a price higher than the marginal cost during adjustment period of the market. Final equilibrium price is equal to marginal cost of producer plus marginal cost of middleman for storing goods, i.e., total marginal cost, in absence of a tax/subsidy, so neither does the middleman nor does the consumer get any economic rent or economic benefit respectively by charging and paying a price respectively different from marginal cost when market is in equilibrium.

To put it in mathemaical terms, suppose due to a positive supply shock while demand stays the same, such as a technological advancement which brings down the marginal cost of production and shifts supply downward, for middleman to have another unit of inventory, the marginal cost, i.e.,  $\varsigma'(m(p,e))\frac{m'_1(p,e)}{q'(p)}$  is higher at existing price due to the term  $\varsigma'(m(p,e))$  being higher at current price. This could be due to higher storage charges as a results of increased demand of storage places after positive supply shock. The second term, i.e.,  $\frac{m'_1(p,e)}{q'(p)}$ , being a function of price is the same as before until the price changes. Middleman's purchase price is the same as before due to producer being a price taker during adjustment of the market too, and charging a fixed fraction of market price to the middleman. To understand the concept intuitively, in a discrete analog of above scenario, middleman maximizes profits in each time period taking the purchase price from producer as given and choosing the selling market price without considering future time periods. Middleman faces following inequality at existing price:

$$\frac{\partial \Pi}{\partial p} = pq'(p) + q(p) - \varsigma'(m(p,e))m'_1(p,e) < 0, \tag{3}$$

implying that middleman decreases price after supply shock to have another unit of inventory to maximize profits after the shock. In above scenario, producer is the short-term beneficiary due to a reduced marginal cost, but receiving the same price as before from middleman. If profit maximizing combinations of inventories and respective prices for middleman are plotted together with price on y-axis and inventory on x-axis, a downward sloping *inventory curve* results, which is analogous

to traditional *supply* and *demand curves* for profit maximizing producers and utility maximizing consumers respectively.

#### 2.1.2 Dynamic Problem

In this section, dynamic problem of middleman is discussed. Middleman maximizes present discounted value of series of future profits with zero time value as given below:

$$V(0) = \int_{0}^{\infty} [pq(p) - \varsigma(m(p, e))] e^{-rt} dt,$$
(4)

with following description of variables in the above expression: r as discount rate, p(t) as *control variable* and m(t) as *state variable*. Middleman's maximization problem is as given below:

$$\underset{\{p(t)\}}{MaxV(0)} = \int_{0}^{\infty} [pq(p) - \varsigma(m(p,e))] e^{-rt} dt,$$

subject to the following constraints:

 $\dot{m}(t) = m'_1(p(t), e(p(t), z))\dot{p}(t) + m'_2(p(t), e(p(t), z))e'_1(p(t), z)\dot{p}(t)$  (state equation, which describes change in state variable with respect to time; and z being exogenous inputs in the model),  $m(0) = m_s$  (initial condition),

 $m(t) \ge 0$  (non-negativity constraint on state variable),

 $m(\infty)$  free (terminal condition).

For this case, current-value Hamiltonian can be expressed as given below:

$$\widetilde{H} = p(t)q(p(t)) - \varsigma(m(p(t), e(p(t), z))) + \mu(t)\dot{p}(t) \begin{bmatrix} m'_1(p(t), e(p(t), z)) + m'_2(p(t), e(p(t), z)) * \\ e'_1(p(t), z) \end{bmatrix}.$$
(5)

Maximizing conditions can be written as follows:

(i)  $p^*(t)$  maximizes  $\tilde{H}$  for all t:  $\frac{\partial \tilde{H}}{\partial p} = 0$ , (ii)  $\dot{\mu} - r\mu = -\frac{\partial \tilde{H}}{\partial m}$ , (iii)  $\dot{m}^* = \frac{\partial \tilde{H}}{\partial \mu}$  (this just gives back the state equation), (iv)  $\lim_{t \to \infty} \mu(t)m(t)e^{-rt} = 0$  (the transversality condition). The first two maximizing conditions can be expressed as given below:

$$\frac{\partial \widetilde{H}}{\partial p} = 0,\tag{6}$$

and

$$\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial m} = \varsigma'(m(p(t), e(p(t), z))).$$
(7)

In equilibrium,  $\dot{p}(t) = 0$ , and  $\frac{\partial \tilde{H}}{\partial p}$  reduces to the following expression (see appendix):

$$p(t)\left[1 + \frac{1}{demand\ elasticity}\right] = \varsigma'(m(p(t), e(p(t), z))) \left\{\frac{m'_1(p(t), e(p(t), z))}{q'(p(t))} + \frac{m'_2(p(t), e(p(t), z))e'_1(p(t), z)}{q'(p(t))}\right\}$$

In a dynamic setting, the right side of above expression is the marginal cost, which is quite different from that in the short-term/myopic problem, due to the fact that for dynamic problem, middleman also considers the impact of price he/she chooses on future purchase price from producer. Price equals marginal cost for an infinitely elastic demand. Suppose a positive supply shock hits the market, and middleman wants to increase the size of inventory. To have an extra unit in inventory, middleman's marginal cost is higher at existing price due to the term  $\varsigma'(m(p(t), e(p(t), z)))$ , which is higher at previous price at that time. The term in parantheses in above expression, i.e.,  $\frac{m'_1(p(t), e(p(t), z))}{q'(p(t))} + \frac{m'_2(p(t), e(p(t), z))e'_1(p(t), z)}{q'(p(t))}$  being a function of price is the same as before until price gets changed by middleman. Therefore, on existing price, middleman's profit maximizing expression changes to the following:

$$\frac{\partial \widetilde{H}}{\partial p} < 0.$$

This implies at previous price after supply shock, middleman's profit maximizing condition is not being satisfied if he wants to have an extra unit of inventory, therefore, after supply shock, middleman must decrease price to have another unit and to maximize profits. To increase inventory, price must be decreased, therefore, there is a negative relationhip between price and inventory change. Inventory is the state between supply and demand, and hence unifies both kinds of shocks, i.e., each kind of shock influences the inventory size, therefore, each kind of shock is just an inventory shock. If supply equals demand, market is in equilibrium, however, if any kind of shock happens and either supply or demand or both rates get changed, and the economic agents do not respond to the shock, price will be changing continuously until the system saturates, e.g., if a positive exogenous supply shock happens, and the producer and consumer do not modify their responses with a change in price, the market will get flooded with supply till the point of saturation. This response can be depicted by the following mathematical expression: Price change  $\propto$  change in market inventory.

$$\begin{split} P &= price \ change. \\ M &= m - m_s = change \ in \ inventory \ in \ the \ market, \\ m &= inventory \ at \ time \ t, \\ m_s &= inventory \ in \ steady \ state \ equilibrium. \\ Input - output &= \frac{dm}{dt} = \frac{d(m - m_s)}{dt} = \frac{dM}{dt}, \\ \text{or } M &= \int (input - output) \ dt. \\ Price \ change \propto \int (supply \ rate - \ demand \ rate) \ dt, \ \text{or} \\ P &= -K_m \int (supply \ rate - \ demand \ rate) \ dt, \end{split}$$

where  $K_m$  is the costant of proportionality. Supply and demand rates are flow variables and reflect the flow of supply and demand respectively per unit time in the market. When (supply rate – demand rate) is positive, P is negative, i.e., excessive supply than demand leads to a decrease in market price and vice versa. Above equation can also be written as follows:

$$\int (supply \ rate - demand \ rate) dt = -\frac{P}{K_m}, \ \text{or}$$

$$\int (w_i - w_0) dt = -\frac{P}{K_m}, \tag{8}$$

 $w_i = supply \ rate,$  $w_0 = demand \ rate,$  $K_m = dimensional \ constant.$ 

At t = 0, market is in a steady state equilibrium, and *supply rate* equals *demand rate*. Putting initial conditions in eq. (8), it can be expressed as given below:

$$\int (w_{is} - w_{0s}) \, dt = 0. \tag{9}$$

Subscript s stands for steady state equilibrium, the state which reflects initial values of the market, and P = 0, when market is in a steady state equilibrium. Subtracting eq. (9) from (8), results in the following expression:

$$\int (w_i - w_{is}) dt - \int (w_0 - w_{0s}) dt = -\frac{P}{K_m}, \text{ or}$$

$$\int (W_i - W_0) \, dt = -\frac{P}{K_m},\tag{10}$$

where 
$$w_i - w_{is} = W_i$$
 = change in supply rate,  
 $w_0 - w_{0s} = W_0$  = change in demand rate.

 $P, W_i$  and  $W_0$  reflect deviation from initial equilibrium values, and hence have initial values equal to zero. Eq. (10) can also be expressed as given below:

$$P = -K_m \int W dt = -K_m M,\tag{11}$$

where  $W = W_i - W_0$ .

If price gets a jump due to an input other than change in inventory, the output is just a sum of impact of various inputs for a linear dynamical system, therefore P in eq. (11) can be expressed as given below:

$$P = -K_m \int W dt + J = -K_m M + J. \tag{11a}$$

Inventory can also get an exogenous shock, such as fire, etc.

#### 2.2 Producer

Producer maximizes present discounted value of series of future profits with zero time value as given below:

$$V(0) = \int_{0}^{\infty} \left[ \alpha p(t) F(K(t), L(t)) - w(t) L(t) - \Re(t) I(t) \right] e^{-rt} dt,$$
(12)

with following description of variables in the above expression:  $\alpha$  as the fraction of the market price charged by producer to middleman, L(t) (labor) and I(t) (level of investment) as control variables and K(t) as state variable. Producer's maximization problem is as given below:

$$\underset{\left\{L(t),I(t)\right\}}{Max}V(0) = \int_{0}^{\infty} [\alpha p(t)F\left(K\left(t\right),L\left(t\right)\right) - w(t)L\left(t\right) - \Re(t)I(t)] e^{-rt} dt,$$

subject to the following constraints:

 $K(t) = I(t) - \delta K(t)$  (state equation, which describes change in state variable with respect to time),  $K(0) = K_0$  (initial condition),

 $K(t) \ge 0$  (non-negativity constraint on state variable),

 $K(\infty)$  free (terminal condition).

For this case, current-value Hamiltonian can be expressed as given below:

$$\widetilde{H} = \alpha p(t) F(K(t), L(t)) - w(t) L(t) - \Re(t) I(t) + \mu(t) [I(t) - \delta K(t)].$$
(13)

Maximizing conditions can be written as follows:

(i)  $L^*(t)$  and  $I^*(t)$  maximize  $\tilde{H}$  for all t:  $\frac{\partial \tilde{H}}{\partial L} = 0$  and  $\frac{\partial \tilde{H}}{\partial I} = 0$ , (ii)  $\dot{\mu} - r\mu = -\frac{\partial \tilde{H}}{\partial K}$ , (iii)  $\dot{K}^* = \frac{\partial \tilde{H}}{\partial \mu}$  (this just gives back the state equation), (iv)  $\lim_{t \to \infty} \mu(t)K(t)e^{-rt} = 0$  (the transversality condition). The first two maximizing conditions can be expressed as given below:

$$\frac{\partial \widetilde{H}}{\partial L} = 0, \tag{14}$$

$$\frac{\partial \widetilde{H}}{\partial I} = 0, \tag{15}$$

and

$$\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial K}.$$
(16)

After price increase, producer needs to increase production to satisfy new dynamic optimization condition (see appendix). Let p = market price, c = a reference price (e.g., retail price including production cost, and profits of producer and middleman), i.e., a reference point parameter with respect to which variation in p is the basis of producer's decision making regarding production.

## $W_m = Change in production due to change in price,$

A higher value of (p-c) provides producer an incentive to produce more. Therefore,

$$W_m \propto \alpha(p-c), \text{ or}$$
  
 $W_m = K_s(p-c).$  (17)

During market equilibrium,  $W_m = 0$ , which implies that

$$0 = K_s(p_s - c_s). (18)$$

 $K_s$  is the constant of proportionality;  $p_s$  and  $c_s$  are values in steady state equilibrium. Subtracting eq. (18) from (17) gives the following expression:

$$W_m = K_s \left[ (p - p_s) - (c - c_s) \right] = -K_s \left( C - P \right) = -K_s \varepsilon,$$
(19)

where  $W_m, C$  and P reflect deviation from initial equilibrium values, and hence have initial values equal to zero.

#### 2.3 Consumer

Consumer maximizes present discounted value of series of future utilities with zero time value as given below:

$$V(0) = \int_{0}^{\infty} U(x(t))e^{-\rho t}dt,$$
(20)

with following description of variables in the above expression:  $\rho$  as discount rate and x(t) as *control variable*. The maximization problem can be written as

$$\underset{\{x(t)\}}{MaxV(0)} = \int_{0}^{\infty} U(x(t))e^{-\rho t}dt,$$

subject to the following constraints:

 $\dot{a}(t) = R(t)a(t) + w(t) - p(t)x(t)$  (state equation, which describes change in state variable with respect to time). a(t) as asset holdings is a *state variable*, and w(t) and R(t) are exogenous time path of wages and return on assets.

 $a(0) = a_s$  (initial condition),

 $a(t) \ge 0$  (non-negativity constraint on state variable),

 $a(\infty)$  free (terminal condition).

For this case, current-value Hamiltonian can be expressed as given below:

$$\widetilde{H} = U(x(t)) + \mu(t) \left[ R(t) a(t) + w(t) - p(t) x(t) \right].$$
(21)

Maximizing conditions can be written as follows:

(i) 
$$x^*(t)$$
 maximizes  $\tilde{H}$  for all  $t$ :  $\frac{\partial H}{\partial x} = 0$ ,  
(ii)  $\dot{\mu} - \rho\mu = -\frac{\partial \tilde{H}}{\partial a}$ ,  
(iii)  $\dot{a}^* = \frac{\partial \tilde{H}}{\partial \mu}$  (this just gives back the state equation),  
(iv)  $\lim_{t \to \infty} \mu(t)a(t)e^{-\rho t} = 0$  (the transversality condition).  
The first two maximizing conditions can be expressed as given below:

$$\frac{\partial H}{\partial x} = U'(x(t)) - \mu(t)p(t) = 0, \qquad (22)$$

and

$$\dot{\mu} - \rho \mu = -\frac{\partial \widetilde{H}}{\partial a} = -\mu(t)R(t).$$
(23)

For an increase in price of the consumption good, consumer faces the following expression at existing level of consumption:

$$\frac{\partial \widetilde{H}}{\partial x} = U'(x(t)) - \mu(t)p(t) < 0.$$

To satisfy modified dynamic optimization condition after a price increase, consumer must reduce consumption. If change in consumption is proportional to change in price, the following formulation results:

#### Change in demand $\propto P$ , or

$$W_d = -K_d P. \tag{24}$$

 $W_d$  is change in demand due to change in price, i.e., when P is positive  $W_d$  is negative.

## 3 Solution of the Model with a Consumption Tax

The model has been solved (see appendix) resulting in the following expression:

$$\frac{dP(t)}{dt} = -K_m \left[ -K_s \varepsilon(t) + K_d P(t) \right].$$
(25)

If an ad valorem consumption tax T is imposed on buyer, the market price the buyer will be paying will be inclusive of the consumption tax, however, price consideration for producer's decision making regarding how much to produce will be the one before tax, i.e.,

$$\varepsilon(t) = Tp(t) - P(t). \tag{26}$$

This implies that,

$$\frac{dP(t)}{dt} + K_m \{K_s(1-T) + K_d\} P(t) = K_m K_s T p_s.$$
(27)

For above differential equation, the stability criterion, i.e., Routh–Hurwitz is as follows:  $K_m \{K_s(1-T) + K_d\} > 0$ . If this criterion is met, market will always adjust on its own to another equilibrium after a shock. Solving the differential equation with initial conditions, t = 0,  $P(0) = Tp_s$ , we obtain the following expression:

$$P(t) = C_1 + C_2 e^{-[K_m \{K_s(1-T) + K_d\}]t}.$$
(28)

After putting values of  $C_1$  and  $C_2$  in eq. (28), we obtain the following expression:

$$P(t) = \frac{K_s T p_s}{\{K_s(1-T) + K_d\}} + \frac{(K_d - K_s T) T p_s}{\{K_s(1-T) + K_d\}} e^{-[K_m \{K_s(1-T) + K_d\}]t}.$$
(29)

Initial conditions, i.e., t = 0,  $P(0) = Tp_s$  are being satisfied. When  $t = \infty$ ,  $P(\infty) = \frac{K_s Tp_s}{\{K_s(1-T)+K_d\}}$ , which is the final steady state equilibrium value. In final equilibrium, quantity demanded equals quantity supplied (see appendix).

#### 4 An Optimal Ad Valorem Tax

Deadweight loss due to imposition of a tax in post-tax equilibrium is the only efficiency loss taken into consideration in the existing literature for mathematical derivation of an optimal tax, however, there is also loss of efficiency during adjustment of market to final equilibrium after imposition of tax. When a tax is imposed on goods, price jumps to initial price plus the amount of tax, and gradually adjusts to bring final post-tax equilibrium in which price is higher than initial price and less the one existing when tax was imposed depending on demand and supply schedules' elasticities. If inventory grows in size, it indicates that supply is higher than demand and vice versa. When supply becomes equal to demand after adjustment of market, final equilibrium has arrived. During adjustment due to supply and demand not being equal there is a loss of output/consumption. Also, there is a lower level of production in post-tax equilibrium which implies that all resources are not fully employed in final equilibrium and hence some efficiency loss. By summing up total consumption or production lost, total efficiency loss can be computed and is as follows:

$$EL = -\int_{0}^{\infty} W_{d}(t)dt = M(t) - \int_{0}^{\infty} W_{m}(t)dt.$$
 (30)

For each unit of consumption/production in economy, efficiency gain is as follows:

Efficiency gain = (willingness to pay - consumer price) + 
$$(tax)$$
 +  
(producer price - factors of production cost) + (factors of production cost - natural resources).  
= willingness to pay - natural resources.  
 $\simeq$  willingness to pay.

Using above concept of efficiency, loss in efficiency in value terms due to imposition of tax is as follows:

$$\begin{split} EL &= -\int_{0}^{\infty} \left[ \frac{1}{2} \{ \text{new price}(t) - \text{old equilibrium price}(t) \} + \{ \text{old equilibrium price}(t) \} \right] W_d(t) dt \\ &= -\int_{0}^{\infty} \left[ \frac{1}{2} P(t) + p_s \right] W_d(t) dt. \end{split}$$

Above expression can be written as given below:

$$EL = \int_{0}^{\infty} K_d P(t) \left[ \frac{1}{2} P(t) + p_s \right] dt, \qquad (31)$$

 $p_s$  is the price in initial equilibrium. Efficiency loss in value terms gets minimized when it is minimized in terms of quantity. Initial value of  $-W_d(t)$  is as follows:  $-W_d(0) = K_d T p_s$  (i.e., decrease in demand due to tax at t = 0 from eq. (24)). Figure 3 illustrates that consumption change jumps to  $K_d T p_s$ , i.e., there is a decrease in demand due to imposition of tax at t = 0. Demand is not equal to supply any longer, and market forces come into play. Price along with demand adjusts over time until final equilibrium arrives, i.e.,  $W_d(\infty)$ . Shaded area in the figure is the efficiency loss, i.e., consumption lost during adjustment of market to final equilibrium. Area between lines  $-W_d(t) = 0$ , and  $-W_d(t) = -W_d(\infty)$  is efficiency loss due to a shift in equilibrium due to tax. Tax revenue expression is as given below:

$$TR = Tp(t) \left[ w_{id}(0) - K_d P(t) \right].$$
(32)

The problem of minimizing efficiency loss subject to revenue constraint, i.e., tax revenue generated is greater than or equal to G in a given time period, is as given below:

$$\min_{\mathcal{T}} EL \quad \text{s.t.} \quad TR \ge G.$$

Tax rate is control variable with constraint being binding. No closed form solution exists, hence numerical solution is found as given below:

Suppose the tax revenue target is \$1000. With  $K_m = K_d = K_s = 1$ ,  $w_{id}(0) = 100$ , and  $p_s = 10$ , tax revenue expression can be written as given below:

$$T[P(t) + 10][100 - P(t)] = 1000.$$

For t = 0,  $P(0) = Tp_s = 10T$ , we have the above expression is follows:

$$T^3 - 9T^2 - 10T + 10 = 0.$$

This implies that

$$T = -1.556098, 0.648635, 9.907463.$$

As price has to be positive, T = -1.556098 gets overruled. Among other two values, expression (31) gets minimized for T = 0.648635. For  $t = \infty$ , expression for tax revenue is as given below:

$$11T^2 - 25T + 10 = 0.$$

This implies that

 $T = 0.518115, \ 1.754612.$ 

Expression (31) for efficiency loss is minimized for T = 0.518115. The optimal consumption tax is that government initially imposes a tax T = 0.648635, and then gradually decreases it to a final value of T = 0.518115.

#### 5 Conclusion

When an advalorem consumption tax is imposed on buyer, price of goods jumps to initial price plus tax. Market forces come into play making supply and demand adjust to result in final equilibrium. Traditionally, it is just the efficiency loss in final equilibrium which is considered for derivation of an optimal tax, however, when market adjusts to arrive at a new equilibrium, there are extra efficiency losses as supply and demand are not equal during market adjustment. If efficiency losses during market adjustment are ignored, the tax schedule derived is not optimal. Last section dealt with an optimal tax schedule derivation generating target amount of tax revenue considering adjustment of market supply and demand. For estimation of an optimal consumption tax schedule, data to estimate slopes of supply, demand and inventory curve; price, and quantity in initial equilibrium is required.

## 6 Appendix:

#### 6.1 Dynamic Problem of the Middleman

In this section, dynamic problem of middleman is discussed. Middleman maximizes present discounted value of series of future profits with zero time value as given below:

$$V(0) = \int_{0}^{\infty} [pq(p) - \varsigma(m(p, e))] e^{-rt} dt,$$
(33)

with following description of variables in the above expression: r as discount rate, p(t) as *control* variable and m(t) as state variable. Middleman's maximization problem is as given below:

$$\underset{\{p(t)\}}{Max}V(0) = \int_{0}^{\infty} [pq(p) - \varsigma(m(p,e))] e^{-rt} dt,$$

subject to the following constraints:

 $\dot{m}(t) = m'_1(p(t), e(p(t), z))\dot{p}(t) + m'_2(p(t), e(p(t), z))e'_1(p(t), z)\dot{p}(t)$  (state equation, which describes change in state variable with respect to time; and z being exogenous inputs in the model),  $m(0) = m_s$  (initial condition),

 $m(t) \ge 0$  (non-negativity constraint on state variable),

 $m(\infty)$  free (terminal condition).

For this case, current-value Hamiltonian can be expressed as given below:

$$\widetilde{H} = p(t)q(p(t)) - \varsigma(m(p(t), e(p(t), z))) + \mu(t)\dot{p}(t) \begin{bmatrix} m'_1(p(t), e(p(t), z)) + m'_2(p(t), e(p(t), z)) * \\ e'_1(p(t), z) \end{bmatrix}.$$
(34)

Maximizing conditions can be written as follows:

(i)  $p^*(t)$  maximizes  $\tilde{H}$  for all  $t: \frac{\partial \tilde{H}}{\partial p} = 0$ , (ii)  $\dot{\mu} - r\mu = -\frac{\partial \tilde{H}}{\partial m}$ , (iii)  $\dot{m}^* = \frac{\partial \tilde{H}}{\partial \mu}$  (this just gives back the state equation), (iv)  $\lim_{t \to \infty} \mu(t)m(t)e^{-rt} = 0$  (the transversality condition). The first two maximizing conditions can be supressed as given below

The first two maximizing conditions can be expressed as given below:

$$\frac{\partial \widetilde{H}}{\partial p} = q(p(t)) + p(t)q'(p(t)) - \varsigma'(m(p(t), e(p(t), z))) \left\{ \begin{array}{c} m_1'(p(t), e(p(t), z)) + m_2'(p(t), e(p(t), z)) * \\ e_1'(p(t), z) \end{array} \right\} \\
+ \mu(t)\dot{p}(t) * \left[ \begin{array}{c} m_{11}''(p(t), e(p(t), z)) + m_{12}''(p(t), e(p(t), z))e_1'(p(t), z) + \\ m_{21}''(p(t), e(p(t), z))e_1'(p(t), z) + m_{22}''(p(t), e(p(t), z))e_1'^2(p(t), z) + \\ m_2'(p(t), e(p(t), z))e_{11}''(p(t), z) \end{array} \right] \\
= 0,$$
(35)

and

$$\dot{\mu} - r\mu = -\frac{\partial \dot{H}}{\partial m} = \varsigma'(m(p(t), e(p(t), z))).$$
(36)

In equilibrium,  $\dot{p}(t) = 0$ , and  $\frac{\partial \tilde{H}}{\partial p}$  reduces to the following expression:

$$q(p(t)) + p(t)q'(p(t)) - \varsigma'(m(p(t), e(p(t), z))) \left\{ \begin{array}{c} m'_1(p(t), e(p(t), z)) + m'_2(p(t), e(p(t), z)) * \\ e'_1(p(t), z) \end{array} \right\}$$
  
= 0,

$$p(t)q'(p(t)) + q(p(t)) = \varsigma'(m(p(t), e(p(t), z))) \left\{ \begin{array}{c} m_1'(p(t), e(p(t), z)) + m_2'(p(t), e(p(t), z)) * \\ e_1'(p(t), z) \end{array} \right\},$$

$$p(t) \left[ 1 + \frac{1}{demand \ elasticity} \right] = \varsigma'(m(p(t), e(p(t), z))) \left\{ \frac{m_1'(p(t), e(p(t), z))}{q'(p(t))} + \frac{m_2'(p(t), e(p(t), z))e_1'(p(t), z)}{q'(p(t))} \right\}$$

In a dynamic setting, the right side of above expression is the marginal cost, which is quite different from that in the short-term/myopic problem, due to the fact that for dynamic problem, middleman also considers the impact of price he/she chooses on future purchase price from producer. Price equals marginal cost for an infinitely elastic demand. Suppose a positive supply shock hits the market, and middleman wants to increase the size of inventory. To have an extra unit in inventory, middleman's marginal cost is higher at existing price due to the term  $\varsigma'(m(p(t), e(p(t), z)))$ , which is higher at previous price at that time. The term in parantheses in above expression, i.e.,  $\frac{m'_1(p(t), e(p(t), z))}{q'(p(t))} + \frac{m'_2(p(t), e(p(t), z))e'_1(p(t), z)}{q'(p(t))}$  being a function of price is the same as before until price gets changed by middleman. Therefore, on existing price, middleman's profit maximizing expression changes to the following:

$$\begin{split} \frac{\partial \widetilde{H}}{\partial p} &= q(p(t)) + p(t)q'(p(t)) - \varsigma'(m(p(t), e(p(t), z))) \left\{ \begin{array}{c} m_1'(p(t), e(p(t), z)) + m_2'(p(t), e(p(t), z)) * \\ e_1'(p(t), z) \end{array} \right\} \\ &+ \mu(t)\dot{p}(t) * \left[ \begin{array}{c} m_{11}''(p(t), e(p(t), z)) + m_{12}''(p(t), e(p(t), z))e_1'(p(t), z) + \\ m_{21}''(p(t), e(p(t), z))e_1'(p(t), z) + m_{22}''(p(t), e(p(t), z))e_1'^2(p(t), z) + \\ m_2'(p(t), e(p(t), z))e_{11}''(p(t), z) \end{array} \right] \\ &\leq 0. \end{split}$$

This implies at previous price after supply shock, middleman's profit maximizing condition is not being satisfied if he wants to have an extra unit of inventory, therefore, after supply shock, middleman must decrease price to have another unit and to maximize profits. To increase inventory, price must be decreased, therefore, there is a negative relationhip between price and inventory change. Inventory is the state between supply and demand, and hence unifies both kinds of shocks, i.e., each kind of shock influences the inventory size, therefore, each kind of shock is just an inventory shock. If supply equals demand, market is in equilibrium, however, if any kind of shock happens and either supply or demand or both rates get changed, and the economic agents do not respond to the shock, price will be changing continuously until the system saturates, e.g., if a positive exogenous supply shock happens, and the producer and consumer do not modify their responses with a change in price, the market will get flooded with supply till the point of saturation. This response can be depicted by the following mathematical expression:

Price change  $\propto$  change in market inventory.

$$\begin{split} P &= price \ change. \\ M &= m - m_s = change \ in \ inventory \ in \ the \ market, \\ m &= \ inventory \ at \ time \ t, \\ m_s &= \ inventory \ in \ steady \ state \ equilibrium. \\ Input - \ output &= \ \frac{dm}{dt} = \ \frac{d(m - m_s)}{dt} = \ \frac{dM}{dt}, \\ \text{or} \ M &= \ \int \ (input - \ output) \ dt. \\ Price \ change \propto \ \int \ (supply \ rate - \ demand \ rate) \ dt, \ \text{or} \\ P &= -K_m \ \int \ (supply \ rate - \ demand \ rate) \ dt, \end{split}$$

where  $K_m$  is the costant of proportionality. Supply and demand rates are flow variables and reflect the flow of supply and demand respectively per unit time in the market. When (supply rate – demand rate) is positive, P is negative, i.e., excessive supply than demand leads to a decrease in market price and vice versa. Above equation can also be written as follows:

$$\int (supply \ rate - demand \ rate) dt = -\frac{P}{K_m}, \ \text{or}$$

$$\int (w_i - w_0) dt = -\frac{P}{K_m},$$
(37)

 $w_i = supply \ rate,$  $w_0 = demand \ rate,$  $K_m = dimensional \ constant.$ 

At t = 0, market is in a steady state equilibrium, and supply rate equals demand rate. Putting initial conditions in eq. (37), it can be expressed as given below:

$$\int (w_{is} - w_{0s}) dt = 0. \tag{38}$$

Subscript s stands for steady state equilibrium, the state which reflects initial values of the market, and P = 0, when market is in a steady state equilibrium. Subtracting eq. (38) from (37), results in the following expression:

$$\int (w_i - w_{is}) dt - \int (w_0 - w_{0s}) dt = -\frac{P}{K_m}, \text{ or}$$

$$\int (W_i - W_0) dt = -\frac{P}{K_m},$$
(39)

where 
$$w_i - w_{is} = W_i$$
 = change in supply rate,  
 $w_0 - w_{0s} = W_0$  = change in demand rate.

 $P, W_i$  and  $W_0$  reflect deviation from initial equilibrium values, and hence have initial values equal to zero. Eq. (39) can also be expressed as given below:

$$P = -K_m \int W dt = -K_m M, \tag{40}$$

where 
$$W = W_i - W_0$$
.

If price gets a jump due to an input other than change in inventory, the output is just a sum of impact of various inputs for a linear dynamical system, therefore P in eq. (40) can be expressed as given below:

$$P = -K_m \int W dt + J = -K_m M + J. \tag{43a}$$

#### 6.2 Producer

Producer maximizes present discounted value of series of future profits with zero time value as given below:

$$V(0) = \int_{0}^{\infty} [\alpha p(t) F(K(t), L(t)) - w(t) L(t) - \Re(t) I(t)] e^{-rt} dt, \qquad (41)$$

with following description of variables in the above expression:  $\alpha$  as the fraction of the market price charged by producer to middleman, L(t) (labor) and I(t) (level of investment) as *control variables* and K(t) as *state variable*. Producer's maximization problem is as given below:

$$\underset{\{L(t),I(t)\}}{Max}V(0) = \int_{0}^{\infty} \left[ \alpha p(t)F(K(t),L(t)) - w(t)L(t) - \Re(t)I(t) \right] e^{-rt} dt,$$

subject to the following constraints:

 $K(t) = I(t) - \delta K(t)$  (state equation, which describes change in state variable with respect to time),  $K(0) = K_0$  (initial condition),

 $K(t) \ge 0$  (non-negativity constraint on state variable),

 $K(\infty)$  free (terminal condition).

For this case, current-value Hamiltonian can be expressed as given below:

$$\widetilde{H} = \alpha p(t) F(K(t), L(t)) - w(t) L(t) - \Re(t) I(t) + \mu(t) [I(t) - \delta K(t)].$$
(42)

Maximizing conditions can be written as follows:

(i)  $L^*(t)$  and  $I^*(t)$  maximize  $\tilde{H}$  for all  $t: \frac{\partial \tilde{H}}{\partial L} = 0$  and  $\frac{\partial \tilde{H}}{\partial I} = 0$ , (ii)  $\dot{\mu} - r\mu = -\frac{\partial \tilde{H}}{\partial K}$ , (iii)  $\dot{K}^* = \frac{\partial \tilde{H}}{\partial \mu}$  (this just gives back the state equation), (iv)  $\lim_{t \to \infty} \mu(t) K(t) e^{-rt} = 0$  (the transversality condition).

The first two maximizing conditions can be expressed as given below:

$$\frac{\partial \widetilde{H}}{\partial L} = \alpha p(t) F_2'(K(t), L(t)) - w(t) = 0, \qquad (43)$$

$$\frac{\partial \tilde{H}}{\partial I} = -\Re(t) + \mu(t) = 0, \qquad (44)$$

and

$$\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial K} = -\left[\alpha p(t)F_1'(K(t), L(t)) - \delta\mu(t)\right].$$
(45)

Substituting  $\dot{\mu}$  and  $\mu$  from eq. (44) in (45) yields

 $\alpha p(t)F_{1}^{\prime}\left(K\left(t\right),L\left(t\right)\right)-(r+\delta)\Re(t)+\dot{\Re}(t)=0.$ 

If p(t) goes up (at previous level of investment and labor), producer faces the following expressions:

$$\alpha p(t) F_{2}'(K(t), L(t)) - w(t) > 0,$$
  
$$\alpha p(t) F_{1}'(K(t), L(t)) - (r+\delta) \Re(t) + \dot{\Re}(t) > 0.$$

After price increase, producer needs to increase production to satisfy new dynamic optimization condition (see appendix). Let p = market price, c = a reference price (e.g., retail price including production cost, and profits of producer and middleman), i.e., a reference point parameter with respect to which variation in p is the basis of producer's decision making regarding production.

#### $W_m = Change in production due to change in price,$

A higher value of (p-c) provides producer an incentive to produce more. Therefore,

$$W_m \propto \alpha(p-c), \text{ or}$$
  
 $W_m = K_s(p-c).$  (46)

During market equilibrium,  $W_m = 0$ , which implies that

$$0 = K_s(p_s - c_s). (47)$$

 $K_s$  is the constant of proportionality;  $p_s$  and  $c_s$  are values in steady state equilibrium. Subtracting eq. (47) from (46) gives the following expression:

$$W_m = K_s [(p - p_s) - (c - c_s)] = -K_s (C - P) = -K_s \varepsilon,$$
(48)

where  $W_m, C$  and P reflect deviation from initial equilibrium values, and hence have initial values equal to zero.

#### 6.3 Solution of the Model with a Consumption Tax

Eqs. (11a), (19) and (24) are reproduced as follows:

$$\frac{dP(t)}{dt} = -K_m W(t),$$
  

$$W_m(t) = -K_s \varepsilon(t),$$
  

$$\varepsilon(t) = C(t) - P(t),$$
  

$$W_d(t) = -K_d P(t),$$

and

$$W(t) = W_m(t) - W_d(t),$$

in absence of an exogenous supply/demand shock. Above equations can be combined as given below:

$$\begin{aligned} \frac{dP(t)}{dt} &= -K_m \left[ W_m(t) - W_d(t) \right] \\ &= -K_m \left[ -K_s \varepsilon(t) + K_d P(t) \right] \\ &= -K_m \left[ -K_s C(t) + (K_s + K_d) P(t) \right]. \end{aligned}$$

After rearranging above expression, we get:

$$\frac{dP(t)}{dt} + K_m(K_s + K_d)P(t) = K_m K_s C(t).$$
(49)

If an ad valorem consumption tax T is imposed on buyer, the market price the buyer will be paying will be inclusive of the consumption tax, however, price consideration for producer's decision making regarding how much to produce will be the one before tax, i.e.,

$$\varepsilon(t) = Tp(t) - P(t). \tag{50}$$

This implies that

$$\frac{dP(t)}{dt} = -K_m \left[ K_s \left\{ P(t) - Tp(t) \right\} + K_d P(t) \right],\\ \frac{dP(t)}{dt} = -K_m \left[ K_s \left\{ P(t) - TP(t) - Tp_s \right\} + K_d P(t) \right]$$

After rearranging, following expression is obtained:

$$\frac{dP(t)}{dt} + K_m \{K_s(1-T) + K_d\} P(t) = K_m K_s T p_s.$$
(51)

Characteristic function of above differential equation is as given below:

 $x + K_m \{K_s(1 - T) + K_d\} = 0.$ 

Single root of characteristic function is given by:

$$x = -K_m \{K_s(1-T) + K_d\}$$

with the following complementary solution:

$$P_c(t) = C_2 e^{-[K_m \{K_s(1-T) + K_d\}]t}.$$

Particular solution can be expressed as given below:

$$P_p(t) = C_1.$$

Therefore, solution can be written in the following form:

$$P(t) = C_1 + C_2 e^{-[K_m \{K_s(1-T) + K_d\}]t}.$$
(52)

After substitution of above expression in the differential equation, following equation is obtained:

$$-K_m \{K_s(1-T) + K_d\} C_2 e^{-[K_m \{K_s(1-T) + K_d\}]t} + K_m \{K_s(1-T) + K_d\} C_1 + K_m \{K_s(1-T) + K_d\} C_2 e^{-[K_m \{K_s(1-T) + K_d\}]t} = K_m K_s T p_s,$$
$$C_1 = \frac{K_s T p_s}{\{K_s(1-T) + K_d\}}.$$

Putting initial conditions, the value of  $C_2$  can be determined as given below:

$$P(0) = \frac{K_s T p_s}{\{K_s(1-T) + K_d\}} + C_2 = T p_s,$$

$$C_2 = T p_s - \frac{K_s T p_s}{\{K_s(1-T) + K_d\}}$$

$$= \frac{K_s T p_s - K_s T^2 p_s + K_d T p_s - K_s T p_s}{\{K_s(1-T) + K_d\}}$$

$$= \frac{(K_d - K_s T) T p_s}{\{K_s(1-T) + K_d\}}.$$

After putting values of  $C_1$  and  $C_2$  in eq. (52), we obtain the following expression:

$$P(t) = \frac{K_s T p_s}{\{K_s(1-T) + K_d\}} + \frac{(K_d - K_s T) T p_s}{\{K_s(1-T) + K_d\}} e^{-[K_m \{K_s(1-T) + K_d\}]t}.$$
(53)

Initial conditions, i.e., t = 0,  $P(0) = Tp_s$  are being satisfied. When  $t = \infty$ ,  $P(\infty) = \frac{K_s Tp_s}{\{K_s(1-T)+K_d\}}$ , which is the final steady state equilibrium value. In final equilibrium, quantity demanded must equal quantity supplied. This is verified as follows: From eq. (??), change in demand due to a change in price after imposition of tax is as given below:

$$W_d(t) = -K_d P(t),$$
  
or  $w_{nd}(t) - w_{id}(0) = -K_d P(t),$ 

where  $w_{id}(0)$  is initial demand and  $w_{nd}(t)$  is new demand after tax, because  $W_d(t)$  is a deviation variable, i.e., deviation from initial equilibrium value. Similarly from eq. (48), for supply we have,

$$W_m(t) = -K_s \varepsilon(t),$$
  

$$w_{nm}(t) - w_{im}(0) = -K_s \left[ Tp(t) - P(t) \right], \text{ or}$$
  

$$w_{nm}(t) - w_{im}(0) = -K_s \left[ TP(t) + Tp_s - P(t) \right].$$

In final equilibrium

$$w_{nm}(\infty) = w_{nd}(\infty), \text{ or}$$
  
 $w_{im}(0) - K_s [\{T - 1\} P(\infty) + Tp_s] = w_{id}(0) - K_d P(\infty),$ 

which holds as in initial equilibrium, quantity demanded equals quantity supplied, i.e.

$$w_{im}(0) = w_{id}(0).$$

#### 6.4 An Optimal Ad Valorem Tax

Deadweight loss due to imposition of a tax in post-tax equilibrium is the only efficiency loss taken into consideration in the existing literature for mathematical derivation of an optimal tax, however, there is also loss of efficiency during adjustment of market to final equilibrium after imposition of tax. When a tax is imposed on goods, price jumps to initial price plus the amount of tax, and gradually adjusts to bring final post-tax equilibrium in which price is higher than initial price and less the one existing when tax was imposed depending on demand and supply schedules' elasticities. If inventory grows in size, it indicates that supply is higher than demand and vice versa. When supply becomes equal to demand after adjustment of market, final equilibrium has arrived. During adjustment due to supply and demand not being equal there is a loss of output/consumption. Also, there is a lower level of production in post-tax equilibrium which implies that all resources are not fully employed in final equilibrium and hence some efficiency loss. By summing up total consumption or production lost, total efficiency loss can be computed and is as follows:

$$EL = -\int_{0}^{\infty} W_d(t)dt = M(t) - \int_{0}^{\infty} W_m(t)dt.$$
 (54)

For each unit of consumption/production in economy, efficiency gain is as follows:

Efficiency gain = (willingness to pay - consumer price) + (tax) + (producer price - factors of production cost) + (factors of production cost - natural resources). = willingness to pay - natural resources. ~ willingness to pay.

Using above concept of efficiency, loss in efficiency in value terms due to imposition of tax is as follows:

$$EL = -\int_{0}^{\infty} \left[ \frac{1}{2} \{ \text{new price}(t) - \text{old equilibrium price}(t) \} + \{ \text{old equilibrium price}(t) \} \right] W_d(t) dt$$
$$= -\int_{0}^{\infty} \left[ \frac{1}{2} P(t) + p_s \right] W_d(t) dt.$$

Above expression can be written as given below:

$$EL = \int_{0}^{\infty} K_d P(t) \left[ \frac{1}{2} P(t) + p_s \right] dt, \qquad (62a)$$

 $p_s$  is the price in initial equilibrium. Efficiency loss in value terms gets minimized when it is minimized in terms of quantity. Initial value of  $-W_d(t)$  is as follows:  $-W_d(0) = K_d T p_s$  (i.e., decrease in demand due to tax at t = 0 from eq. (24)). Figure 3 illustrates that consumption change jumps to  $K_d T p_s$ , i.e., there is a decrease in demand due to imposition of tax at t = 0. Demand is not equal to supply any longer, and market forces come into play. Price along with demand adjusts over time until final equilibrium arrives, i.e.,  $W_d(\infty)$ . Shaded area in the figure is the efficiency loss, i.e., consumption lost during adjustment of market to final equilibrium. Area between lines  $-W_d(t) = 0$ , and  $-W_d(t) = -W_d(\infty)$  is efficiency loss due to a shift in equilibrium due to tax. From eq. (??), change in demand due to change in price after imposition of tax is as given below:

$$W_d(t) = -K_d P(t),$$
 or  $w_{nd}(t) - w_{id}(0) = -K_d P(t),$ 

 $w_{id}(0)$  is initial value of demand and  $w_{nd}(t)$  is the value after tax as  $W_d(t)$  is a deviation variable, i.e. deviation from initial value. Tax revenue expression is as given below:

$$TR = Tp(t) \left[ w_{id}(0) - K_d P(t) \right].$$
(55)

The problem of minimizing efficiency loss subject to revenue constraint, i.e., tax revenue generated is greater than or equal to G in a given time period, is as given below:

$$\min_{T} EL \quad \text{s.t.} \quad TR \ge G$$

Tax rate is control variable with constraint being binding. No closed form solution exists, hence numerical solution is found as given below:

Suppose the tax revenue target is \$1000. With  $K_m = K_d = K_s = 1$ ,  $w_{id}(0) = 100$ , and  $p_s = 10$ , tax revenue expression can be written as given below:

$$T[P(t) + 10][100 - P(t)] = 1000.$$

For t = 0,  $P(0) = Tp_s = 10T$ , we have the above expression as follows:

$$T [T + 1] [10 - T] = 10,$$
  

$$T(9T - T^{2} + 10) - 10 = 0,$$
  

$$T^{3} - 9T^{2} - 10T + 10 = 0.$$

This implies that

T = -1.556098, 0.648635, 9.907463.

As price has to be positive, T = -1.556098 gets overruled. Among other two values, expression (62a) gets minimized for T = 0.648635. For  $t = \infty$ , expression for tax revenue is as given below:

$$T \left[ P(\infty) + 10 \right] \left[ 100 - P(\infty) \right] = 1000,$$
  

$$T \left[ \frac{T}{(2-T)} + 1 \right] \left[ 10 - \frac{T}{(2-T)} \right] = 10,$$
  

$$T(20 - 11T) = 5(2 - T),$$
  

$$20T - 11T^2 - 10 + 5T = 0,$$
  

$$25T - 11T^2 - 10 = 0.$$

This implies that

## $T = 0.518115, \ 1.754612.$

Expression (62*a*) for efficiency loss is minimized for T = 0.518115. The optimal consumption tax is that government initially imposes a tax T = 0.648635, and then gradually decreases it to a final value of T = 0.518115.

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Figure 1: When is Linearity a Reasonable Assumption?



Figure 2: Movement of Price with Inventory.



Figure 3: Dynamic Efficiency Loss because of an Ad Valorem Tax.