Commentary

Extropy's Limitations as a Complementary Dual of Entropy and the Role of Informity

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In their 2015 paper, "Extropy: Complementary Dual of Entropy," Lad et al. introduced the concept of extropy as a proposed complementary dual of entropy, with distinct definitions for discrete and continuous distributions. This commentary argues that extropy suffers from two primary issues: an ad hoc transition from discrete to continuous extropy, lacking theoretical rigor, and a contradictory interpretation as a measure of certainty, undermined by its equivalence to entropy in the binary case and negative continuous form. Consequently, extropy is not a fully coherent complementary dual of entropy. We demonstrate that informity, introduced by Huang[1], quantifies certainty consistently across discrete and continuous distributions, serving as the true complementary dual to entropy's uncertainty measure. We compare informity, entropy, and extropy with two examples.

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1. Introduction

Lad et al.[2] introduced a mathematical quantity called "extropy" as a proposed complementary dual of entropy in information theory. For a discrete random variable *X* with probability mass function (PFM) $P(X)$, where $P\left(x_i\right)$ is the probability of outcome x_i, they defined discrete extropy as

$$J(X)=-\sum_{i=1}^{N}\left[1-P\left(x_i\right)\right]\ \log\left[1-P\left(x_i\right)\right] \tag{1}$$

where x_i is an outcome of *X* and *N* is the number of possible outcomes of *X*.

For a continuous random variable *Y* with the probability density function (PDF) $p(Y)$, they defined continuous (or differential) extropy as

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$$J(Y) = -\frac{1}{2}\int p(y)^2 dy = -\frac{1}{2}E[p(Y)] \quad (2)$$

where $E[\bullet]$ denotes the mathematical expectation. Notably, the definition of continuous extropy is inconsistent with the discrete extropy.

Lad et al.[2] claimed that extropy is a complementary dual of entropy, and stated that "entropy and extropy identify what many people think of as yin and yang, and what artists commonly refer to as positive and negative space (p. 53)." This analogy suggests that extropy measures an opposite quality to entropy's measure of uncertainty, such as certainty. However, upon careful review of their paper, we argue that this claim is contradictory and misleading. Moreover, we identify two primary issues with extropy: the problematic transition from discrete to continuous extropy and the lack of clear interpretability as a measure of certainty. In this commentary, we address these issues and argue that extropy is not a fully coherent complementary dual of entropy. Instead, we propose that "informity," introduced by Huang[1], better fulfills this role by directly quantifying certainty in a distribution.

In the following sections, Section 2 examines the issue of transitioning from discrete to continuous extropy, highlighting the ad hoc nature of the derivation. Section 3 discusses the interpretability issue, focusing on the contradiction between extropy's uncertainty-based formulation and its role as a certainty measure. Section 4 explains how informity serves as a true complementary dual of entropy. Section 5 gives two examples to compare informity, entropy and extropy. Section 6 presents conclusion and recommendation.

2. The transition issue

In the discrete case, extropy is defined as $J(X) = -\sum_{i=1}^{N}[1 - P(x_i)]log[1 - P(x_i)]$. However, extending this definition to the continuous case by replacing $P(x_i)$ with the density value $p(y)$ is problematic, as the expression $[1 - p(y)]log[1 - p(y)$ becomes undefined when $p(y) > 1$, which is possible for probability density functions. Recognizing this issue, Lad et al.[2] proposed an alternative approach to define continuous extropy.

To address this challenge, they expanded $[1 - P(x_i)]log[1 - P(x_i)]$ using the first three terms of its Maclaurin series with remainder

$$[1 - P(x_i)]\log[1 - P(x_i)] = -P(x_i) + \frac{1}{2}P(x_i)^2 + \frac{1}{6(1-r)^2}P(x_i)^3, r \in (0, P(x_i)). \quad (3)$$

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When N is large and the maximum $P\left(x_i\right)$ is small, the higher-order terms become negligible, and the discrete extropy can be approximated as

$$J(X) \approx 1-\frac{1}{2} \sum_{i=1}^N P\left(x_i\right)^2, \tag{4}$$

where the constant 1 arises from summing the leading term $P\left(x_i\right)$ over all outcomes, since $\sum_{i=1}^N P\left(x_i\right)=1$.

A continuous distribution for the random variable Y with PDF $p(Y)$ on the support $\left[y_1, y_2\right]$ can be approximated by dividing the support into N intervals of size $\Delta y=\left(y_2-y_1\right) / N$, forming a discrete probability system: $\left\{Y_{\Delta} ; p(y) \Delta y\right\}$. Applying the discrete extropy approximation, the extropy of the discretized distribution is

$$J\left(Y_{\Delta}\right) \approx 1-\frac{1}{2} \sum_{i=1}^N\left[p\left(y_i\right) \Delta y\right]^2=1-\frac{\Delta y}{2} \sum_{i=1}^N p\left(y_i\right)^2 \Delta y. \tag{5}$$

As $\Delta y \rightarrow 0$, the sum $\sum_{i=1}^N p\left(y_i\right)^2 \Delta y$ approximates the integral $\int_{y_1}^{y_2} p(y)^2 d y$. However, directly taking the limit of $J\left(Y_{\Delta}\right)$ yields

$$\lim _{\Delta y \rightarrow 0} J\left(Y_{\Delta}\right) \approx 1-\frac{\Delta y}{2} \int p(y)^2 d y \rightarrow 1, \tag{6}$$

since the term involving Δy vanishes. However, this result is not informative because it converges to a constant that is independent of the probability density $p(y)$.

To derive the continuous extropy, Lad et al.[2] proposed

$$J(Y)=-\frac{1}{2} \int p(y)^2 d y=\lim _{\Delta y \rightarrow 0}\left[\frac{J\left(Y_{\Delta}\right)-1}{\Delta y}\right], \tag{7}$$

where the subtraction of 1 and division by Δy isolate the quadratic term

$$J\left(Y_{\Delta}\right)-1 \approx-\frac{\Delta y}{2} \sum_{i=1}^N p\left(y_i\right)^2 \Delta y. \tag{8}$$

As $\Delta y \rightarrow 0$, this yields the desired integral form. However, Lad et al.[2] provided no detailed rationale for this specific adjustment. The subtraction of the constant 1 removes the offset from the discrete approximation $J\left(Y_{\Delta}\right)$, and the division by Δy rescales the quadratic term to match the desired integral form $-\frac{1}{2} \int p(y)^2 d y$. This approach appears ad hoc, as their paper offers no theoretical justification for discarding the constant term or choosing this particular limiting process. In contrast, the derivation of differential entropy involves a well-established adjustment (e.g., subtracting $\log \Delta y$) to account for discretization, which is rigorously grounded in information theory. The lack of a clear theoretical connection between the discrete and continuous definitions, combined with the absence of an explanation for the limiting adjustment, significantly undermines the robustness of the continuous extropy formulation.

3. The interpretability issue

Lad et al.'s interpretation of extropy is both confusing and contradictory, undermining its role as a coherent information-theoretic measure. On one hand, they presented extropy as a ″complementary dual″ of entropy, likening the relationship to "yin and yang, and what artists commonly refer to as positive and negative space" (p. 53). This analogy implies that extropy measures a quality that is the opposite of the ″uncertainty″ measured by entropy, such as ″certainty.″ On the other hand, they stated that "as is entropy, extropy is interpreted as a measure of the amount of uncertainty represented by the distribution for X (p. 41)," which directly contradicts the implication of certainty. This contradiction is evident in the binary case (e.g., a coin toss), where they noted that extropy equals entropy ($J(X) = H(X)$). This equality indicates that extropy quantifies the same uncertainty as entropy, not a complementary quality like certainty. The "yin and yang" analogy is thus misleading, as it overstates the opposition between extropy and entropy, creating significant confusion about extropy's intended meaning.

The interpretability of extropy is further compromised by its continuous formulation. Defined as $J(Y) = -\frac{1}{2}\int p(y)^2 dy$, the continuous extropy is always negative, which is counterintuitive for a measure purportedly capturing a complementary aspect to entropy's uncertainty measure, such as certainty. A measure of certainty would be expected to be positive and increase with the concentration of the distribution, yet the negative sign inverts this intuition. As discussed in Section 2, the derivation of this form relies on an ad hoc limiting process lacking theoretical justification, which further obscures its interpretive connection to entropy. This lack of a coherent interpretive framework undermines the claim that extropy serves as a true "complementary dual" of entropy.

In summary, while Lad et al.'s[2] introduction of extropy as a novel measure is a commendable contribution to information theory, its presentation as a "complementary dual" of entropy is misleading. The contradictory claims that extropy measures uncertainty while implying certainty through the "yin and yang" analogy, combined with the counterintuitive negativity of continuous extropy, create significant conceptual confusion. These issues highlight the need for a more interpretable measure to fulfill the role of a complementary dual of entropy.

4. Informity as the true complementary dual of entropy

Huang[1] recently introduced the concept of informity within a novel information-theoretic framework, which offers a robust measure of certainty in probability distributions. For a discrete random variable *X* with PMF $P(X)$, discrete informity is defined as

$$\beta(X)=\sum_{i=1}^{N} P\left(x_i\right)^2=E[P(X)]. \tag{9}$$

For a continuous random variable *Y* with PDF $p(Y)$, continuous informity is defined as

$$\beta(Y)=\int p(y)^2 dy=E[p(Y)]. \tag{10}$$

Unlike extropy, informity maintains consistency between its discrete and continuous forms, mirroring the relationship between discrete and differential entropy. Both discrete and continuous informity are always non-negative, aligning with the intuitive expectation for a measure of information content or certainty.

Informity directly quantifies the concentration of a distribution, with higher values indicating greater certainty (e.g., a distribution peaked around specific outcomes or values) and lower values indicating greater spread. This makes informity a clear and intuitive measure of certainty (the "yang" aspect) in a probability distribution, in direct contrast to entropy, which quantifies uncertainty (the “yin” aspect). Indeed, the meaning of informity is the opposite of entropy[1]. A distribution with minimum informity, such as a uniform distribution, corresponds to maximum entropy, highlighting their opposing behaviors. In contrast, extropy does not exhibit this opposition, as its maximum also occurs for the uniform distribution, indicating it measures uncertainty rather than certainty.

Furthermore, informity’s quadratic form, $\beta(X)=\sum_{i=1}^{N} P\left(x_i\right)^2$ is known as the Simpson index or “repeat rate,” a well-established measure of concentration[3]. It is closely related to the second-order Rényi entropy, $H_2(X)=-\log \sum_{i=1}^{N} P\left(x_i\right)^2$, Gini impurity, $G(X)=1-\sum_{i=1}^{N} P\left(x_i\right)^2$, and logical entropy, $S_L(X)=1-\sum_{i=1}^{N} P\left(x_i\right)^2$ [4][5][6][7]. These connections provide informity with a strong theoretical foundation for applications in information theory and statistics. While the quadratic form is not new, Huang[1] introduced a novel perspective by interpreting informity as the expected information content of a probability distribution, distinct from related measures. Unlike extropy's negative and counterintuitive continuous form, informity's positivity and consistency across discrete and continuous cases make it a unified and intuitive framework for quantifying certainty, complementing entropy's role in measuring uncertainty.

5. Examples

Example 1: A coin toss

Consider a coin toss, which forms a binary distribution with probabilities $P(head) = p$ and $P(tail) = 1 - p$. Its discrete informity is

$$\beta(X) = p^2 + (1 - p)^2, \tag{11}$$

its discrete entropy is

$$H(X) = -[p \log_2 p + (1 - p) \log_2 (1 - p)], \tag{12}$$

and its discrete extropy is

$$J(X) = -\left[(1 - p) \log_2 (1 - p) + p \log_2 p\right] = H(X). \tag{13}$$

Figure 1 plots these measures as a function of $P(head) = p$, with entropy and extropy measured in bits.

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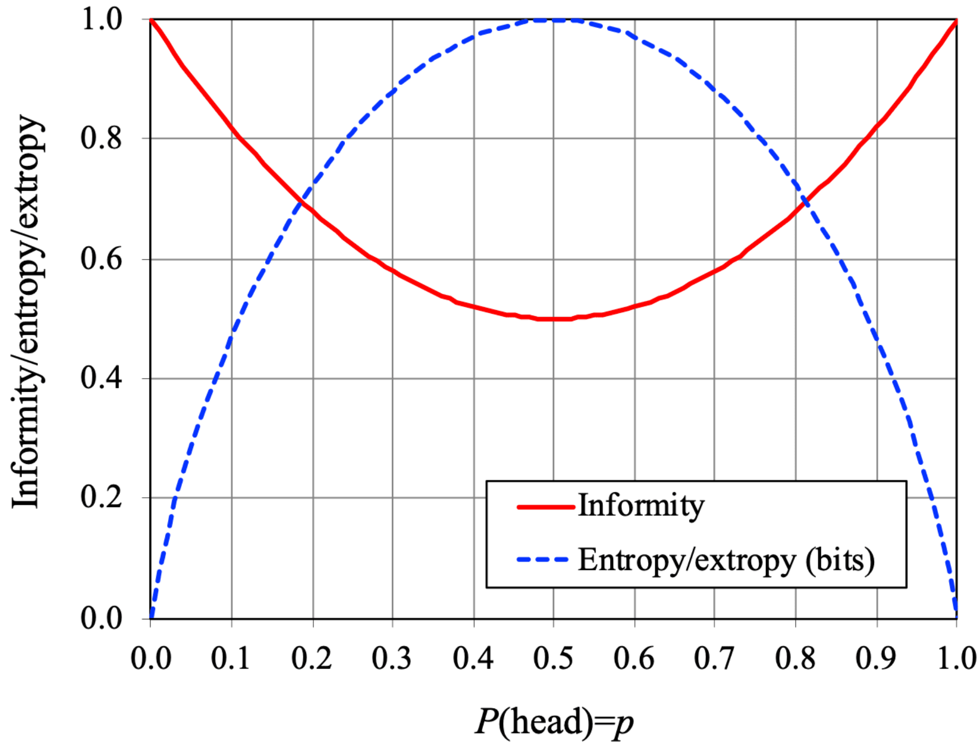

Figure 1. Plots of informity, entropy, and extropy of a binary distribution as a function of $P(head) = p$, with informity unitless and entropy/extropy in bits. Adapted from Huang[1], with extropy added for comparison

As shown in Figure 1, when the coin is fair ($P(head) = P(tail) = 0.5$), informity is minimized ($\beta(X) = 0.5$), while entropy and extropy are maximized ($H(X) = J(X) = 1\ bit$), indicating greatest uncertainty. For a biased coin (e.g., p→1 or p→0), informity increases to $\beta(X) = 1$, and entropy and extropy decrease to $H(X) = J(X) = 0$, reflecting maximum certainty. The equality $J(X) = H(X)$ demonstrates that extropy measures uncertainty, not certainty, contradicting its proposed role as a complementary dual to entropy (Section 3). This contrast highlights informity's role as a true measure of certainty, complementing entropy's uncertainty measure.

Example 2: A uniform distribution

Consider a uniform distribution over *N* outcomes: $P(x_i) = 1/N$. Its discrete informity is

$$\beta(X) = \sum_{i=1}^{N} \left(\frac{1}{N}\right)^2 = \frac{1}{N}, \tag{14}$$

its discrete entropy is

$$H(X) = -\sum_{i=1}^{N} \frac{1}{N} \log_2 \frac{1}{N} = \log_2 N, \tag{15}$$

and its discrete extropy is

$$J(X) = -\sum_{i=1}^{N} \left[1 - \frac{1}{N}\right] \log_2 \left[1 - \frac{1}{N}\right] = (N-1)\log_2 (\frac{N}{N-1}). \tag{16}$$

Figure 2 plots these measures as a function of *N*, with entropy and extropy in bits.

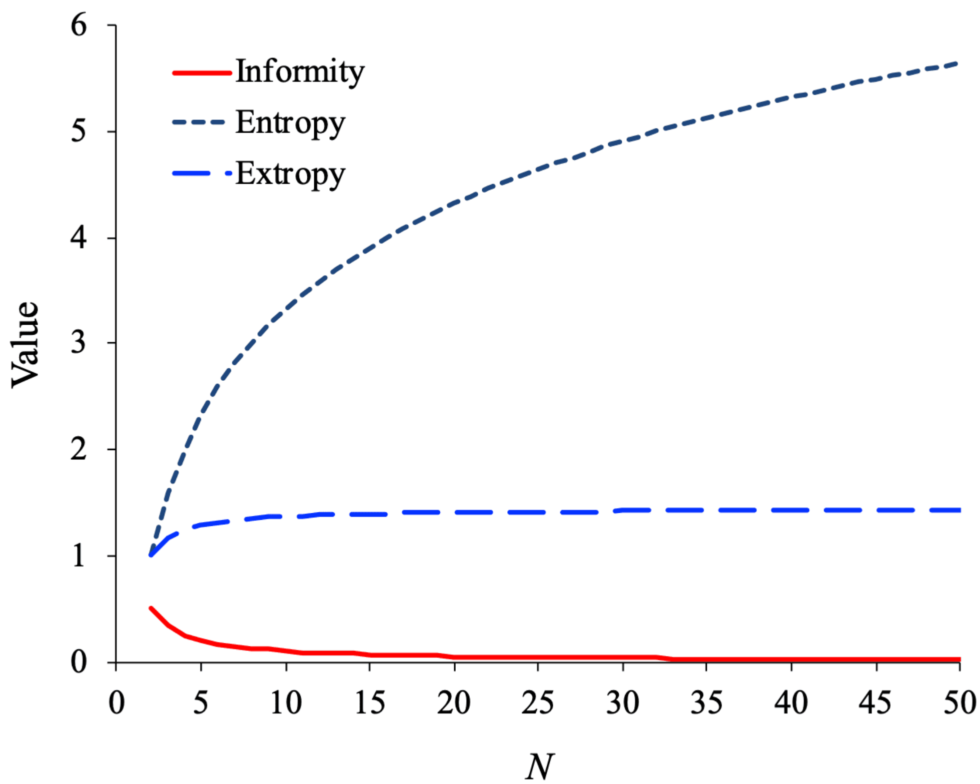

Figure 2. Plots of informity, entropy, and extropy of a uniform distribution as a function of the number of outcomes (*N*), with informity unitless and entropy/extropy in bits

As shown in Figure 2, both entropy and extropy increase with increasing *N*, reflecting greater uncertainty as the distribution becomes more spread. Extropy approaches 1.4427 bits as $N \to \infty$, confirming its role as an uncertainty measure, albeit on a different scale than entropy, which has no upper limit. In contrast, informity decreases as 1/*N*, indicating greater certainty for smaller *N*, where the distribution is more concentrated. This example underscores that extropy does not serve as a complementary dual of entropy, while informity's opposing behavior establishes it as the true dual.

6. Conclusion and recommendation

Lad et al.'s[2] introduction of extropy as a novel information-theoretic measure is a commendable contribution, but its presentation as a "complementary dual" of entropy is misleading and conceptually problematic. As discussed in Section 3, their contradictory claims—that extropy measures uncertainty while implying certainty through the "yin and yang" analogy—create significant confusion. The equality of extropy and entropy in the binary case ($J(X) = H(X)$) and the behavior of discrete extropy, which increases with the number of outcomes (approaching approximately 1.4427 bits for a uniform distribution as $N \to \infty$; see Section 5), confirm that extropy quantifies uncertainty, not certainty. Furthermore, the continuous extropy, defined as $J(Y) = -\frac{1}{2}\int p(y)^2 dy$), is negative and counterintuitive for a purported measure of certainty, and its ad hoc derivation (Section 2) lacks theoretical rigor. These issues—conceptual confusion and mathematical inconsistency—cannot be resolved within the extropy framework, highlighting the need for a more coherent measure to complement entropy.

Informity, introduced by Huang[1], addresses these shortcomings and serves as the true complementary dual of entropy. Unlike extropy, informity quantifies certainty, with higher values indicating greater concentration in a distribution (Section 4). Its consistent formulation across discrete ($\beta(X) = \sum_{i=1}^{N} P(x_i)^2$) and continuous ($\beta(Y) = \int p(y)^2 dy$) cases, along with its non-negativity, aligns with the intuitive expectation for a measure of information content. As shown in Figure 1 (Section 5), informity decreases for a uniform distribution as the number of outcomes increases, contrasting with the increasing behavior of entropy and extropy, thus fulfilling the "yang" role to entropy's "yin." By providing a unified and interpretable framework, informity overcomes the limitations of extropy and offers a robust alternative for quantifying certainty in probability distributions.

We recommend adopting informity in information-theoretic applications where certainty or concentration is of interest, such as diversity analysis, machine learning, and statistical modeling. For example, informity's quadratic form, equivalent to the Simpson index, can enhance measures of diversity in ecological or economic studies, while its connection to Rényi entropy supports its use in data compression and classification tasks. Future research should explore informity's applications in these domains and develop methods to integrate it with existing entropy-based frameworks. By replacing extropy with informity, researchers can achieve a clearer and more consistent understanding of the interplay between certainty and uncertainty in information theory.

Statements and Declarations

Conflict of Interest

The author declares no conflicts of interest.

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Declarations

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