

Research Article

Modular Closure of Yang–Mills Dynamics: A Falsifiable UV–IR Framework for the Mass Gap, Glueball Correlators, Lattice Benchmarks, and Experimental Continuation

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We formulate a compact and falsifiable ultraviolet-to-infrared effective closure for four-dimensional pure Yang–Mills theory. The construction uses a gauge-invariant collective modular coordinate reconstructed from gradient-flowed scalar and topological densities and a sector-relative Modular-Weierstrass scale. The core constitutive hypothesis is that the leading nonanalytic contribution to the collective Yang–Mills energy density inherits the squared sector-relative Hermitian-bilinear scale, $\hat{\kappa}^2$ with a positive matching coefficient. Near the lemniscatic point, the Weierstrass invariants then imply $\hat{\kappa}^2 \propto \rho^{-2}$. Combined with the leading regular analytic restoring term, this produces a unique finite radial minimum and a positive scalar curvature; a CP-even periodic angular term gives a separate positive pseudoscalar curvature. The distinctive inverse-square relation is formulated as a configuration-level lattice test rather than inferred from glueball masses. Published continuum pure $SU(3)$ spectra provide a quantitative mass-ratio benchmark, while multi- N spectra test regular large- N behavior. The BESIII $X(2370)$ result is used only as a full-QCD continuation benchmark after explicit gluonic–quark mixing. A genuine out-of-sample pseudoscalar prediction requires all modular and mixing inputs to be fixed independently of the target resonance mass. The framework therefore yields a positive infrared quadratic spectral scale within the stated effective closure and a concrete program of falsification; it is not presented as a constructive proof of the Clay Yang–Mills existence-and-mass-gap problem.

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1. Introduction

Pure Yang–Mills theory contains no explicit mass parameter at the classical level, yet its physical gauge-invariant spectrum is expected to possess a strictly positive lowest excitation. High-energy observables are controlled by asymptotic freedom, whereas low-energy calculations reveal finite correlation lengths, confinement, and a tower of glueball states. Wilson’s lattice formulation provides the canonical nonperturbative regulator in which these infrared properties can be computed [1].

The purpose of the present work is not to replace perturbative Yang–Mills theory, lattice gauge theory, or constructive field theory. It is to test whether one gauge-invariant collective coordinate can provide a minimal effective closure linking ultraviolet decoupling, an infrared positive-curvature minimum, glueball correlators, and configuration-level lattice observables.

Two companion studies motivate the scale variable used here. The first introduced a normalized Modular-Weierstrass scale coordinate $\mathcal{K}(\tau)$ and its positive modulus $\kappa = |\mathcal{K}|$ on a near-square branch [2]. The second showed that a scalar lift of the Lorentz spinor map, $B = \lambda A$ with $|\lambda| = \kappa$, leaves projective ratios unchanged while multiplying Hermitian bilinears, including four-momentum densities, by κ^2 [3]. The present manuscript is nevertheless self-contained at the level required for its Yang–Mills derivation: the scale-transfer relations used below are restated explicitly, and no cosmological normalization, Planck-time impulse, or physical-spacetime expansion equation from the companion studies is imported into the Yang–Mills closure.

Flat-space scope. The construction is formulated on a flat-spacetime gauge-theory background. Lorentz–Möbius kinematics refers only to the underlying Minkowski representation used to motivate the bilinear scale transfer; the Euclidean formulation used below is the standard continuation for nonperturbative correlation functions and lattice tests. No spacetime-curvature degree of freedom is introduced. The modular coordinate $\tau[A]$ is an internal, gauge-invariant collective coordinate of the Yang–Mills sector, not a coordinate or curvature variable of physical spacetime.

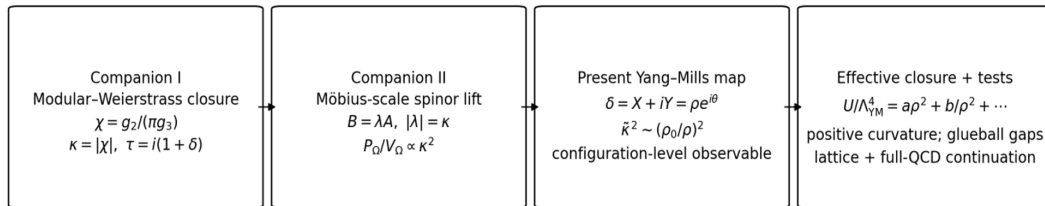
The collective description uses an effective complex modular coordinate reconstructed from gauge-invariant collective observables. Near the square, or lemniscatic, point $\tau = i$, the weight-six elliptic invariant has a simple zero and the sector-relative scale modulus varies inversely with the local modular distance ρ . The squared scale therefore behaves as $\hat{\kappa}^2 \sim \rho^{-2}$. The **single core constitutive hypothesis** of this paper is that the leading nonanalytic contribution to the coarse-grained Yang–Mills energy density is proportional, with a positive matching coefficient, to this sector-relative bilinear scale. The leading

regular analytic response then supplies an opposing ρ^2 term. Their competition produces a finite stable minimum without inserting a Proca mass term and without breaking gauge invariance.

The logical status is deliberately separated into three levels. First, the near-lemniscatic inverse-square scale law follows from the stated Weierstrass expansion and sector-relative normalization. Second, the transfer of $\hat{\kappa}^2$ into a collective Yang–Mills energy-density term is the one constitutive hypothesis of the closure and is directly testable on gauge configurations. Third, minimization, quadratic expansion, correlator asymptotics, parameter-eliminating identities, and the proposed lattice tests follow from the resulting effective action. Published glueball spectra are used only as spectrum-level benchmarks unless the modular parameters are determined independently.

The framework is designed around four requirements:

1. **Gauge consistency.** The elementary action remains pure Yang–Mills; the modular variables are gauge-invariant composite coordinates in an effective action.
2. **Ultraviolet compatibility.** Modular corrections are power suppressed at high momentum, so the universal perturbative beta-function coefficients remain unchanged.
3. **Infrared predictivity.** Positive curvature gives explicit scalar and pseudoscalar mass scales and long-distance exponential correlators inside the closure.
4. **Cross-regime falsifiability.** A single predeclared parameter set must confront configuration-level conditional-energy scaling, masses and residues, topological response, finite-volume behavior, large- N regularity, and full-QCD mixing benchmarks.



The third construction uses only the mathematical scale-transfer ingredients of the companion studies; it does not import their cosmological normalization.

Figure 1. Logical derivation path of the proposed Yang–Mills composite closure. The companion works motivate the scale-transfer structure; the present work introduces the gauge-sector composite map, the single constitutive transfer hypothesis, and the falsifiable lattice and phenomenological consequences.

The main result is therefore intentionally narrow: once the single constitutive transfer is adopted, the minimal radial closure has a finite nonzero minimum and positive curvature, while the same reconstructed coordinate is required to satisfy a direct $1/\rho^2$ configuration-level test. Failure of that relation, or of the parameter-eliminating identities derived below, falsifies the minimal closure without falsifying Yang–Mills theory itself.

2. Yang–Mills scale and the observable meaning of a gap

Let $G = SU(N)$, with Hermitian generators normalized by $\text{tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$, gauge connection $A_\mu = A_\mu^a T^a$, and curvature

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] = F_{\mu\nu}^a T^a. \quad (1)$$

The Euclidean classical action is

$$S_{YM}[A] = \frac{1}{4g_0^2} \int_{\mathbb{R}^4} F_{\mu\nu}^a F_{\mu\nu}^a d^4x. \quad (2)$$

For pure $SU(N)$, the perturbative beta function begins

$$\begin{aligned} k \frac{dg}{dk} &= -b_0 g^3 - b_1 g^5 + O(g^7), \\ b_0 &= \frac{11N}{3(4\pi)^2}, \quad b_1 = \frac{34N^2}{3(4\pi)^4}. \end{aligned} \quad (3)$$

The one-loop and two-loop coefficients are the standard asymptotically free Yang–Mills coefficients [\[4\]\[5\]](#) [\[6\]](#). Dimensional transmutation defines the renormalization-group invariant scale formally by

$$\Lambda_{YM} = k \exp \left[- \int^{g(k)} \frac{dg'}{\beta(g')} \right], \quad (4)$$

where the lower-limit convention and the corresponding multiplicative constant are fixed by the chosen renormalization scheme. The present framework does not replace dimensional transmutation; Λ_{YM} is the only dimensionful scale multiplying the minimal pure-gauge closure.

For a gauge-invariant operator $\mathcal{O}_J$ with quantum numbers J^{PC} , define the connected zero-spatial-momentum Euclidean correlator

$$C_J(t) = \int d^3x \langle \mathcal{O}_J(\mathbf{x}, t) \mathcal{O}_J(\mathbf{0}, 0) \rangle_c. \quad (5)$$

Where reflection positivity and spectral reconstruction hold, the correlator has a positive spectral representation of the form

$$C_J(t) = \sum_n |Z_{Jn}|^2 e^{-M_{Jn}t} + \int_{s_{th}}^{\infty} d\mu \rho_J(\mu) e^{-\mu t}. \quad (6)$$

A mass gap in the physical sector means that there exists $\Delta > 0$ such that the spectrum above the vacuum contains no energies in $(0, \Delta)$. In pure $SU(3)$ Yang–Mills theory ^[7], lattice calculations identify the lightest observed glueball with the scalar channel. An effective action can establish a positive gap only in its own domain by showing positive quadratic curvature and exponentially decaying gauge-invariant correlators. It becomes a solution of the full Clay problem only after the continuum measure, Hilbert-space reconstruction, positivity, and a uniform all-channel lower bound have been established ^[8] ^[9]^[10]. This distinction is maintained throughout.

3. From Lorentz–Möbius kinematics to a gauge-sector scale variable

3.1. Projective invariance and the bilinear scale

Represent a Minkowski four-vector by a Hermitian matrix $P = p_\mu \sigma^\mu$. An $SL(2, \mathbb{C})$ matrix A acts as

$$P \mapsto APA^\dagger. \quad (7)$$

On a spinor ratio or celestial coordinate, the same linear action induces a Möbius transformation ^[3]^[11]^[12].

Introduce a common complex scalar factor

$$B = \lambda A, \quad \lambda = \kappa e^{i\alpha}. \quad (8)$$

The projective ratio is unchanged, whereas the Hermitian bilinear transforms as

$$P \mapsto BPB^\dagger = \kappa^2 APA^\dagger. \quad (9)$$

The global phase $e^{i\alpha}$ cancels from the Hermitian bilinear. The relevant scale transfer is therefore the positive factor κ^2 .

Under the fixed-domain and approximately homogeneous collective projection adopted here, Eq. (9) directly implies a sector-relative scaling of the projected collective density by $\hat{\kappa}^2$. Therefore, for a fixed spatial domain Ω ,

$$P_\Omega^\mu = \int_\Omega d^3x \, p^\mu(x) \simeq \hat{\kappa}^2 V_\Omega \bar{p}_0^\mu. \quad (10)$$

Here the fixed Lorentz part has been absorbed into the reference density $\bar{p}_0^\mu$, and $\hat{\kappa}$ denotes a sector-relative scalar amplitude. The relation used here follows within the present construction from Eq. (9)

under the stated fixed-domain homogeneous projection; Ref. [13] provides only the broader kinematic motivation. In the present Yang–Mills application we define

$$\widehat{\kappa}(\tau) = \frac{\kappa(\tau)}{\kappa(\tau_{ref})}. \quad (11)$$

Thus no absolute cosmological or Planck-scale numerical value of κ enters the Yang–Mills phenomenology. Throughout this paper, $\mathcal{K}$ denotes the complex internal scale coordinate, $\kappa = |\mathcal{K}|$ its positive modulus, and $\widehat{\kappa}$ the sector-relative modulus.

Equation (10) cannot turn a null one-particle momentum into a timelike one: if $P^2 = 0$, then $(\kappa^2 P)^2 = 0$. The scale transfer is therefore applied only to collective gauge-invariant energy and momentum densities. The mass scales derived below belong to composite color-singlet excitations, not to a hard mass assigned to the gauge potential.

3.2. Weierstrass scale near the square lattice

Associate the collective state with the Weierstrass elliptic curve [13][14]

$$E_\tau : y^2 = 4x^3 - g_2(\tau)x - g_3(\tau), \quad \text{Im}\tau > 0. \quad (12)$$

In the fixed normalized-lattice convention,

$$\begin{aligned} g_2(\tau) &= 60 \sum_{(m,n) \neq (0,0)} \frac{1}{(m+n\tau)^4}, \\ g_3(\tau) &= 140 \sum_{(m,n) \neq (0,0)} \frac{1}{(m+n\tau)^6}. \end{aligned} \quad (13)$$

Define the complex Modular-Weierstrass scale coordinate and its positive modulus by

$$\mathcal{K}(\tau) = \frac{g_2(\tau)}{\pi g_3(\tau)}, \quad \kappa(\tau) = |\mathcal{K}(\tau)| = \frac{|g_2(\tau)|}{\pi |g_3(\tau)|}. \quad (14)$$

This ratio is a scale coordinate in the fixed normalized-lattice convention; it is not claimed to be invariant under arbitrary rescaling of an elliptic-curve representative. At the lemniscatic point $\tau = i$, $g_3(i) = 0$ while $g_2(i) \neq 0$. Write

$$\tau = i(1 + \delta), \quad \delta = \rho e^{i\vartheta}, \quad 0 < \rho \ll 1. \quad (15)$$

The weight-six modular form has a simple zero at $\tau = i$ in this local coordinate. Therefore

$$\begin{aligned} g_3(i(1 + \delta)) &= ig'_3(i) \delta + O(\delta^2), \\ \kappa(\tau) &= \frac{K_i}{\rho} [1 + O(\rho)]. \end{aligned} \quad (16)$$

where $K_i = |g_2(i)| / [\pi |g'_3(i)|]$. Relative to a reference displacement $\rho_{ref} > 0$,

$$\hat{\kappa}^2(\rho) = \left(\frac{\rho_{ref}}{\rho} \right)^2 [1 + O(\rho) + O(\rho_{ref})]. \quad (17)$$

The distinction between ρ_{ref} and the dynamically selected minimum ρ_0 is maintained throughout. The former is a fixed, predeclared normalization reference; the latter is selected dynamically by the effective potential.

Lemma 1 (inverse-square scale transfer). Under the fixed-volume, approximately homogeneous projection of Eq. (10), the projected collective-density ratio is $\hat{\kappa}^2$. Combining Eqs. (10), (11), and (17) therefore gives a sector-relative nonanalytic scale proportional to ρ^{-2} to leading order.

Proof. Equation (10) gives $P_{\Omega,r}/V_{\Omega} \propto \hat{\kappa}^2$ for a fixed projection and reference state. Equation (17) gives $\hat{\kappa}^2(\rho) = (\rho_{ref}/\rho)^2 [1 + O(\rho) + O(\rho_{ref})]$. Their composition establishes the inverse-square dependence. The separate step that promotes this scale to a Yang–Mills energy-density contribution is the single constitutive hypothesis stated in Section 5 and tested by Eq. (40). $\square$

4. Gauge-invariant reconstruction of the modular coordinate

A central requirement is that τ not be introduced as an arbitrary elementary field. We define it operationally from finite gauge-invariant composite observables. Let $B_{\mu}(t, x)$ be the Yang–Mills gradient-flow field at flow time $t > 0$, with curvature $G_{\mu\nu}(t, x)$ [15]. With $\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma}$, define

$$\begin{aligned} s_t(x) &= \frac{1}{2} \text{tr} [G_{\mu\nu}(t, x) G_{\mu\nu}(t, x)], \\ q_t(x) &= \frac{1}{16\pi^2} \text{tr} [G_{\mu\nu}(t, x) \tilde{G}_{\mu\nu}(t, x)] = \frac{1}{32\pi^2} G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a. \end{aligned} \quad (18)$$

For $t > 0$, local gauge-invariant flowed observables are ultraviolet finite after the usual parameter renormalization. Introduce dimensionless, reference-subtracted collective coordinates averaged over a four-dimensional resolution cell Ω_k of characteristic scale k^{-1} ,

$$\begin{aligned} X_k[A] &= c_S t_k^2 \frac{1}{V_{4,k}} \int_{\Omega_k} d^4x [s_{t_k}(x) - s_{ref}(t_k)], \\ Y_k[A] &= c_P t_k^2 \frac{1}{V_{4,k}} \int_{\Omega_k} d^4x q_{t_k}(x). \end{aligned} \quad (19)$$

where $t_k = \xi/k^2$, $V_{4,k} = \int_{\Omega_k} d^4x$, c_S and c_P are fixed dimensionless normalization constants, and $s_{ref}(t_k)$ is specified before fitting. These normalization choices are fixed independently of target masses and held fixed across ensembles used for prediction.

$$\begin{aligned} \delta_k[A] &= X_k[A] + iY_k[A] = \rho_k[A] e^{i\vartheta_k[A]}, \\ \tau_k[A] &= i[1 + \delta_k[A]]. \end{aligned} \quad (20)$$

Under CP, $X_k \mapsto X_k$ and $Y_k \mapsto -Y_k$, so $\delta_k \mapsto \bar{\delta}_k$, $\rho_k \mapsto \rho_k$, and $\vartheta_k \mapsto -\vartheta_k$. The radial coordinate is CP even and the angular coordinate is CP odd. The fixed reference subtraction prevents the radial displacement from being forced to vanish by definition and matches the near-square convention of Section 3.2.

The effective fields ρ and ϑ are collective Legendre variables obtained by coupling sources to the composite operators in Eq. (19). The elementary path integral remains over A_μ . Introducing ρ and ϑ is therefore a change of collective description rather than an enlargement of the fundamental particle content.

Composite-map consistency conditions. A valid implementation must demonstrate: (i) $\text{Im}\tau_k[A] > 0$ in the declared fit region; (ii) gauge-equivalent configurations yield the same reconstructed coordinate; (iii) results are stable under predeclared flow-time and cell choices; and (iv) normalized correlation functions of X_k and Y_k have controlled continuum and infinite-volume limits. These are operational validity conditions of the reconstruction, not additional dynamical hypotheses. In addition, modular-equivalent representatives must be reduced to the same predeclared near-lemniscatic coordinate patch before the local distance is evaluated.

5. RG-compatible modular effective action

At coarse-graining scale k , consider the leading-derivative effective average action

$$\begin{aligned} \Gamma_k &= \int d^4x \mathcal{L}_k, \\ \mathcal{L}_k &= \frac{Z_F}{4g^2} F^2 + \frac{f_\rho^2}{2} (\partial\rho)^2 + \frac{f_\vartheta^2 \rho^2}{2} (\partial\vartheta)^2 \\ &\quad + W(k) U_{\text{mod}}(\rho, \vartheta) + \Delta\mathcal{L}_{\text{int}}. \end{aligned} \tag{21}$$

Here $F^2 = F_{\mu\nu}^a F_{\mu\nu}^a$ and all running coefficients are evaluated at k . The term $\Delta\mathcal{L}_{\text{int}}$ denotes residual gauge-composite and higher-derivative interactions not displayed explicitly. In the infrared domain considered below, the collective kinetic normalizations are assumed to be nondegenerate and positive, $f_\rho^2 > 0$ and $f_\vartheta^2 > 0$.

Use the minimal smooth decoupling profile

$$W(k) = \frac{1}{1 + (k/\Lambda_{YM})^4}, \tag{22}$$

so that $W(k) \rightarrow 1$ in the infrared and $W(k) = O[(\Lambda_{YM}/k)^4]$ in the ultraviolet.

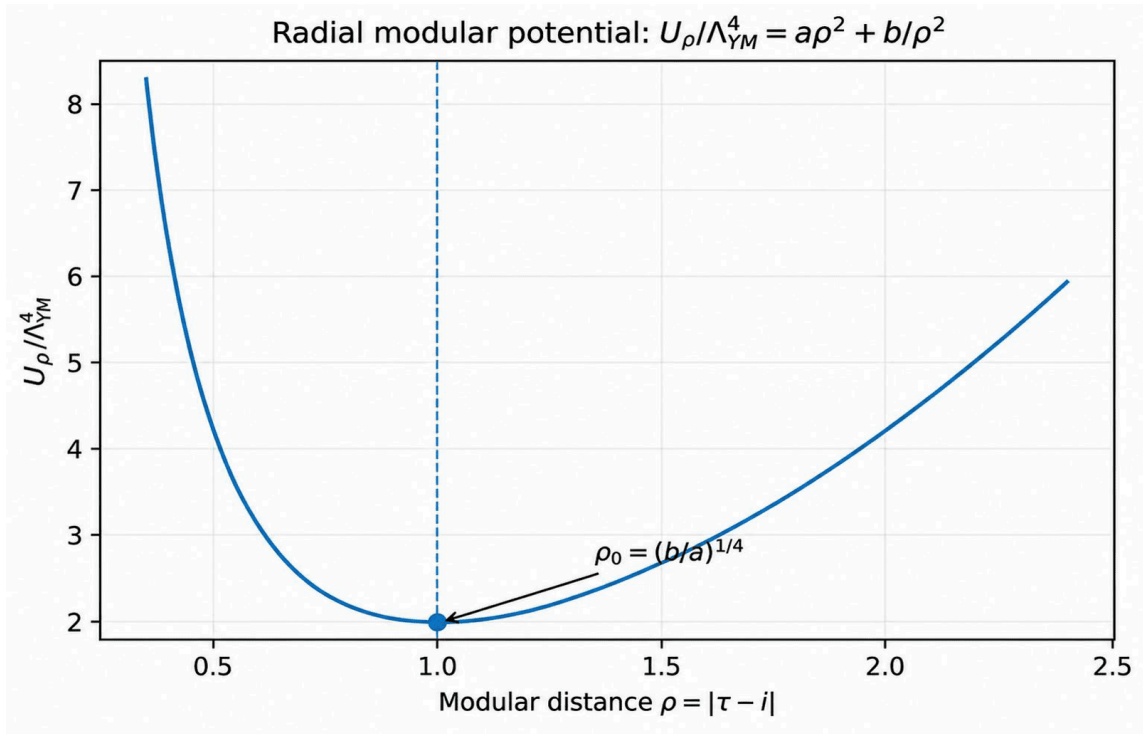


Figure 2. Illustrative radial potential in normalized units. The minimum is rescaled to unity only for visual clarity; the physical near-lemniscatic branch requires $0 < \rho_0 \ll 1$. The curve is not a fit to lattice data.

Equation (17) is an asymptotic near-lemniscatic relation. The minimal closure below retains only its leading ρ^{-2} contribution; the indicated $O(\rho)$ and reference-dependent corrections are treated as higher-order terms and must remain subleading throughout the predeclared fitting window. The leading local potential retained in the minimal closure is

$$U_{mod}(\rho, \vartheta) = \Lambda_{YM}^4 \left(a\rho^2 + \frac{b}{\rho^2} + h[1 - \cos(N_\vartheta \vartheta)] \right), \quad (23)$$

$$a > 0, \quad b > 0, \quad h > 0.$$

The terms play different roles. The $a\rho^2$ term is the leading regular analytic restoring response in the chosen local radial coordinate. The periodic term is the lowest CP-even angular contribution and supplies a local coupling to the topological direction. The b/ρ^2 term contains the only genuinely new Yang–Mills constitutive input of the model.

At the lemniscatic fixed point, the local modular stabilizer reverses the leading displacement, $\delta \mapsto -\delta + O(\delta^2)$. Hence, within the radial truncation adopted here, the leading invariant regular radial response is quadratic, $U_{reg} \propto |\delta|^2 = \rho^2$.

Core modular-closure hypothesis. In the gauge-invariant collective sector reconstructed in Section 4, the leading nonanalytic contribution to the coarse-grained Yang–Mills energy density is proportional, with a positive matching coefficient, to the sector-relative Hermitian-bilinear scale $\widehat{\kappa}^2$. With Eq. (17), this contribution is therefore proportional to ρ^{-2} on the near-lemniscatic branch. The resulting coefficient b absorbs the reference normalization, projection factor, and coarse-grained matching coefficient. This hypothesis is not a consequence of Yang–Mills theory alone; Eq. (40) is designed as its direct configuration-level falsification test.

The integer N_ϑ is a periodicity index of the collective angular coordinate and is not identified with the gauge-group rank N . For the minimal single-branch model one may set $N_\vartheta = 1$ unless an independently reconstructed global structure requires another integer.

To preserve standard ultraviolet physics, require

$$\begin{aligned} Z_F(k, \rho) &= 1 + O\left[\left(\frac{\Lambda_{YM}}{k}\right)^4\right], \\ f_{\rho, \vartheta}(k) &= O(\Lambda_{YM}), \\ \frac{\Delta \mathcal{L}_{int}(k)}{\Lambda_{YM}^4} &= O\left[\left(\frac{\Lambda_{YM}}{k}\right)^4\right], \quad k \gg \Lambda_{YM}. \end{aligned} \tag{24}$$

Provided ρ and ϑ remain composite Legendre variables rather than additional ultraviolet particle degrees of freedom, and all gauge-composite interactions obey the declared power suppression, the universal coefficients b_0 and b_1 remain those of pure Yang–Mills theory. Equation (22) is therefore a decoupling condition of the effective closure, not an independent proof of ultraviolet equivalence. The use of a fourth-power leading suppression is consistent with organizing the first gauge-invariant power correction at canonical dimension four.

6. Modular stabilization and the effective mass gap

Proposition 1 (stable radial minimum). For $a, b > 0$, the radial potential

$$U_\rho(\rho) = \Lambda_{YM}^4 (a\rho^2 + b\rho^{-2}) \tag{25}$$

has a unique global minimum on $\rho > 0$ at

$$\rho_0 = \left(\frac{b}{a}\right)^{1/4}, \tag{26}$$

with

$$U_\rho(\rho_0) = 2\sqrt{ab} \Lambda_{YM}^4, \quad U''_\rho(\rho_0) = 8a\Lambda_{YM}^4 > 0. \tag{27}$$

Proof. Differentiation gives $U'_\rho(\rho) = 2\Lambda_{YM}^4 (a\rho - b\rho^{-3})$, which vanishes at exactly one positive value, Eq. (26). Moreover, $U_\rho \rightarrow +\infty$ as $\rho \downarrow 0$ and as $\rho \rightarrow \infty$. The stationary point is therefore the unique global minimum. Substitution gives Eq. (27). $\square$

The local near-lemniscatic expansion is self-consistent only if $0 < \rho_0 \ll 1$, equivalently $b/a \ll 1$ in the adopted normalization. Parameter sets whose minimum lies outside this domain are not admissible within the minimal near-square closure. The fixed reference in Eq. (17) remains a predeclared normalization choice and is not identified post hoc with the dynamically selected minimum.

Proposition 2 (scalar and pseudoscalar quadratic gaps within the closure). Write the canonically normalized fluctuations as $\rho = \rho_0 + \varphi_\rho/f_\rho$ and $\vartheta = \vartheta_{\min} + \eta/(f_\vartheta\rho_0)$, with $\vartheta_{\min} = 2\pi l/N_\vartheta$, $l \in \mathbb{Z}$. On the principal branch take $\vartheta_{\min} = 0$. Denote the CP-even radial and CP-odd angular quadratic masses by m_S and m_P , respectively. Then

$$\begin{aligned} m_S^2 &= \frac{8a\Lambda_{YM}^4}{f_\rho^2}, \\ m_P^2 &= \frac{N_\vartheta^2 h \Lambda_{YM}^4}{f_\vartheta^2 \rho_0^2}. \end{aligned} \quad (28)$$

The radial result can be summarized without adding a new assumption:

$$\begin{aligned} \hat{\kappa}^2 \propto \rho^{-2} &\Rightarrow U_\rho = \Lambda_{YM}^4 (a\rho^2 + b\rho^{-2}) \\ &\Rightarrow \rho_0 = (b/a)^{1/4} \\ &\Rightarrow m_S^2 = 8a\Lambda_{YM}^4/f_\rho^2 > 0. \end{aligned}$$

The first implication uses the core constitutive hypothesis of Section 5; the remaining steps follow by minimization and quadratic expansion. The radial mode is CP even and naturally overlaps with trF^2 , while the angular mode is CP odd and overlaps with $trF\tilde{F}$.

At quadratic order, the corresponding long-distance Euclidean correlators behave as

$$C_S(t) \sim Z_S e^{-m_S t}, \quad C_P(t) \sim Z_P e^{-m_P t}, \quad t \rightarrow \infty, \quad (29)$$

up to standard spatial-integration prefactors and higher thresholds. The effective closure therefore has a positive scalar spectral scale

$$\begin{aligned} \Delta_{mod} = m_S &= \frac{\sqrt{8a}}{f_\rho} \Lambda_{YM}, \\ f_\rho &= \bar{f}_\rho \Lambda_{YM}. \end{aligned} \quad (30)$$

provided the scalar mode is the lightest gauge-invariant excitation in the theory under consideration. This ordering is a spectrum-level prediction to be tested, not a theorem for every compact simple gauge group.

6.1. Parameter-eliminating identities

Define the model-normalized radial-potential value at the minimum by

$$\mathcal{E}_{mod} = 2\sqrt{ab} \Lambda_{YM}^4 = 2a\rho_0^2 \Lambda_{YM}^4. \quad (31)$$

This is not identified with an absolute Yang–Mills vacuum energy; it is the value of the ρ -dependent part of the effective potential relative to the declared subtraction convention. Eliminating a between Eqs. (28) and (31) gives

$$m_S^2 f_\rho^2 \rho_0^2 = 4\mathcal{E}_{mod}. \quad (32)$$

For the angular sector, use $\theta_{YM} = N_\vartheta \vartheta$ only as a local source coordinate around $\theta_{YM} = 0$ and define

$$\begin{aligned} \mathcal{E}_\vartheta(\theta_{YM}) &= h\Lambda_{YM}^4 [1 - \cos(\theta_{YM})], \\ \chi_{top}^{mod} &\equiv \left. \frac{\partial^2 \mathcal{E}_\vartheta}{\partial \theta_{YM}^2} \right|_{\theta_{YM}=0} = h\Lambda_{YM}^4. \end{aligned} \quad (33)$$

This local curvature is compatible with the standard use of the vacuum angle to define topological susceptibility [\[16\]\[17\]](#); no global large- N vacuum-energy function is assumed. Equation (28) then gives

$$m_P^2 f_\vartheta^2 \rho_0^2 = N_\vartheta^2 \chi_{top}^{mod}. \quad (34)$$

Equations (32) and (34) are more constraining than fitting two masses because the modular-distance distribution, residues, model-normalized radial potential, and topological response can be measured separately on the same ensembles.

For comparison with lattice data, distinguish the full Yang–Mills susceptibility from the contribution represented by the minimal angular closure, $\chi_{top}^{YM} = \chi_{top}^{mod} + \chi_{top}^{other}$. The minimal single-channel saturation limit corresponds to $\chi_{top}^{other} \simeq 0$ over the infrared fit window. A significant residual contribution would falsify the minimal angular saturation, not Yang–Mills theory.

7. Predictivity across energy regimes

7.1. Ultraviolet domain

For $k \gg \Lambda_{YM}$, Eq. (22) power-suppresses the modular potential. The running coupling is therefore the standard asymptotic expression

$$g^2(k) = \frac{1}{2b_0 L} \left[1 - \frac{b_1}{2b_0^2} \frac{\ln L}{L} + O\left(\frac{\ln^2 L}{L^2}\right) \right], \quad L \equiv \ln\left(\frac{k}{\Lambda_{YM}}\right). \quad (35)$$

The proposed theory is consequently not an alternative ultraviolet gauge theory; its new content is an infrared collective closure constrained to decouple at high momentum.

For a renormalized dimension-four gluonic operator $\mathcal{O}_4(\mu)$, define a normalized high-momentum Euclidean quantity $\widehat{\Pi}_G(Q^2)$ after dividing out the leading perturbative Wilson coefficient and kinematic dependence. The operator-product structure can be written schematically as ^[18]

$$\widehat{\Pi}_G(Q^2) = 1 + C_G(Q, \mu) \frac{\langle \mathcal{O}_4(\mu) \rangle}{Q^4} + O\left(\frac{\Lambda_{YM}^6}{Q^6}, \frac{\alpha_s \Lambda_{YM}^4}{Q^4}\right). \quad (36)$$

A modular UV–IR relation requires an independently specified operator-matching coefficient. In the minimal closure, write $\langle \mathcal{O}_4(\mu) \rangle = c_\rho(\mu) \mathcal{E}_{mod}$, where c_ρ is fixed by the normalization of the composite map or by an independent matching calculation. Combining this relation with Eq. (32) gives

$$\widehat{\Pi}_G(Q^2) - 1 = \frac{C_G(Q, \mu) c_\rho(\mu)}{4} \frac{m_S^2 f_\rho^2 \rho_0^2}{Q^4} + O(Q^{-6}). \quad (37)$$

Here m_S denotes the scalar mass in Eq. (30). Equation (37) is conditional: once c_ρ is fixed independently, it links a dimension-four ultraviolet correction to an infrared pole and collective normalization. The matching coefficient must not be adjusted using the held-out ultraviolet observable.

7.2. Crossover domain

At $k \sim \Lambda_{YM}$, neither a finite perturbative expansion nor a purely infrared derivative expansion is sufficient. The reconstructed distribution of $\tau_k[A]$ provides an additional observable. The minimal closure predicts: (i) the mode of the radial distribution approaches a nonzero ρ_0 ; (ii) after subtraction of the regular analytic contribution, conditional collective energy densities follow $\widehat{\kappa}^2(\rho)$ over the declared homogeneous-cell window; and (iii) the curvature extracted from the Jacobian-corrected radial probability potential agrees with the scalar relation in Eq. (28). These statements are directly evaluable on Euclidean lattice ensembles and require no analytic continuation.

7.3. Infrared domain

For $k \ll \Lambda_{YM}$, $W(k) \rightarrow 1$ and the derivative expansion gives isolated low-lying poles plus multiparticle continua. The leading mass ratios are

$$\begin{aligned} \frac{m_P}{m_S} &= \frac{N_\vartheta}{\sqrt{8} \rho_0} \sqrt{\frac{\hbar}{a}} \frac{f_\rho}{f_\vartheta}, \\ \frac{\Delta_{mod}}{\Lambda_{YM}} &= \frac{\sqrt{8a}}{f_\rho}. \end{aligned} \quad (38)$$

The absolute scale comes from Λ_{YM} ; the spectral pattern is encoded in dimensionless collective parameters. A regular large- N organization can be tested through

$$\begin{aligned} a_N &= a_\infty + \frac{a_1}{N^2} + \cdots, \\ \rho_{0,N} &= \rho_{0,\infty} + \frac{r_1}{N^2} + \cdots, \\ \frac{m_{J,N}}{\sqrt{\sigma}} &= R_{J,\infty} + \frac{R_{J,1}}{N^2} + \cdots. \end{aligned} \tag{39}$$

This is a testable regularity ansatz for the effective parameters, consistent with the observed smooth large- N behavior of low-lying pure-gauge spectra [19][20].

8. Lattice realization and quantitative assessment

The lattice is used here as a regulator and measurement platform for the composite coordinate, not as a separate explanatory principle. Published spectra can test mass ordering and large- N regularity immediately; the distinctive inverse-square constitutive law requires a new configuration-level reconstruction and is therefore the stronger prospective test.

8.1. Reconstruction protocol

For each gauge configuration:

1. evolve the links to several gradient-flow times $t_k = \xi/k^2$;
2. calculate s_t and q_t , then form X_k and Y_k using Eq. (19);
3. reconstruct $\delta_k = X_k + iY_k$, $\rho_k = |\delta_k|$, $\vartheta_k = \arg \delta_k$, and $\tau_k = i(1 + \delta_k)$, then evaluate $g_2(\tau_k)$, $g_3(\tau_k)$, and $\hat{\kappa}_k$;
4. estimate a conditional collective energy-density proxy $\varepsilon_k(\rho) = \langle T_{00} \mid \rho_k = \rho \rangle$ or a Euclidean gauge-invariant equivalent appropriate to the ensemble;
5. fit the conditional density and the Jacobian-corrected radial probability potential over a predeclared near-square window;
6. extract f_ρ and f_ϑ from operator residues or momentum-dependent two-point functions; and
7. predict the masses through Eq. (28), rather than fitting the target masses themselves.

The radial extraction is performed either in a predeclared angular window around $\vartheta_{\min}$, or after verifying that angular marginalization introduces no appreciable ρ -dependent distortion over the fitted radial window. For the local flat-coordinate measure $dX dY = \rho d\rho d\vartheta$, define the empirical radial potential by

$$U_{emp}(\rho) = -\frac{1}{V_{4,k}} \ln \left[\frac{P_k(\rho)}{\rho} \right] + C_k,$$

where C_k is an irrelevant additive constant. Over a predeclared near-square fit window, the minimal closure predicts

$$U_{emp}(\rho) \simeq A_k \rho^2 + \frac{B_k}{\rho^2} + C_k, \quad A_k > 0, \quad B_k > 0.$$

Failure to resolve a stable positive- A_k , positive- B_k window after flow-time, continuum, and volume checks would falsify the minimal radial closure.

The decisive configuration-level test of the constitutive transfer is

$$\frac{\varepsilon_k(\rho_1) - \varepsilon_{reg}(\rho_1)}{\varepsilon_k(\rho_2) - \varepsilon_{reg}(\rho_2)} = \frac{\hat{\kappa}^2(\rho_1)}{\hat{\kappa}^2(\rho_2)} \simeq \left(\frac{\rho_2}{\rho_1} \right)^2. \quad (40)$$

Here ε_{reg} is the fitted regular analytic contribution. The reference factor ρ_{ref} cancels from the ratio. Agreement of glueball masses alone is therefore a weaker test than agreement with Eq. (40).

8.2. Continuum and finite-volume organization

For a Wilson-type or $O(a_{lat}^2)$ -improved continuum extrapolation, use the standard Symanzik organization [\[21\]](#)

$$\left. \frac{M_J}{\sqrt{\sigma}} \right|_{a,L} = R_J + c_J a_{lat}^2 \sigma + d_J (a_{lat}^2 \sigma)^2 + \delta_J(L). \quad (41)$$

For a stable isolated state, the large-volume correction is exponentially suppressed. A minimal fit form may be written

$$\delta_J(L) = A_J \frac{e^{-M_{gap}L}}{(M_{gap}L)^{3/2}} + \dots, \quad (42)$$

with channel-dependent coefficient and, where necessary, a more complete Lüscher analysis [\[22\]](#). The modular closure predicts continuum quantities R_J ; it does not modify the regulator corrections. A valid test uses common continuum modular parameters across lattice spacings and volumes.

8.3. Benchmark consistency

Continuum pure $SU(3)$ lattice calculations provide the most direct published nonperturbative spectrum benchmark [\[23\]\[24\]\[25\]](#). Athenodorou and Teper report $M_S/\sqrt{\sigma} = 3.405(21)$ for the lightest scalar channel and $M_P/\sqrt{\sigma} = 5.276(45)$ for the lightest pseudoscalar channel [\[24\]](#). The corresponding ratio is

$$R_{P/S}^{lat} = 1.549 \pm 0.016. \quad (43)$$

Using their r_0 -based scale setting gives illustrative values of approximately $1.653(26) \text{ GeV}$ and $2.561(40) \text{ GeV}$, respectively. Because importing a physical GeV scale into pure gauge theory is convention dependent, Eq. (43) is the preferred benchmark. A genuine out-of-sample test must determine ρ_0 , f_ρ/f_ϑ , h/a , and N_ϑ without using the target pseudoscalar mass.

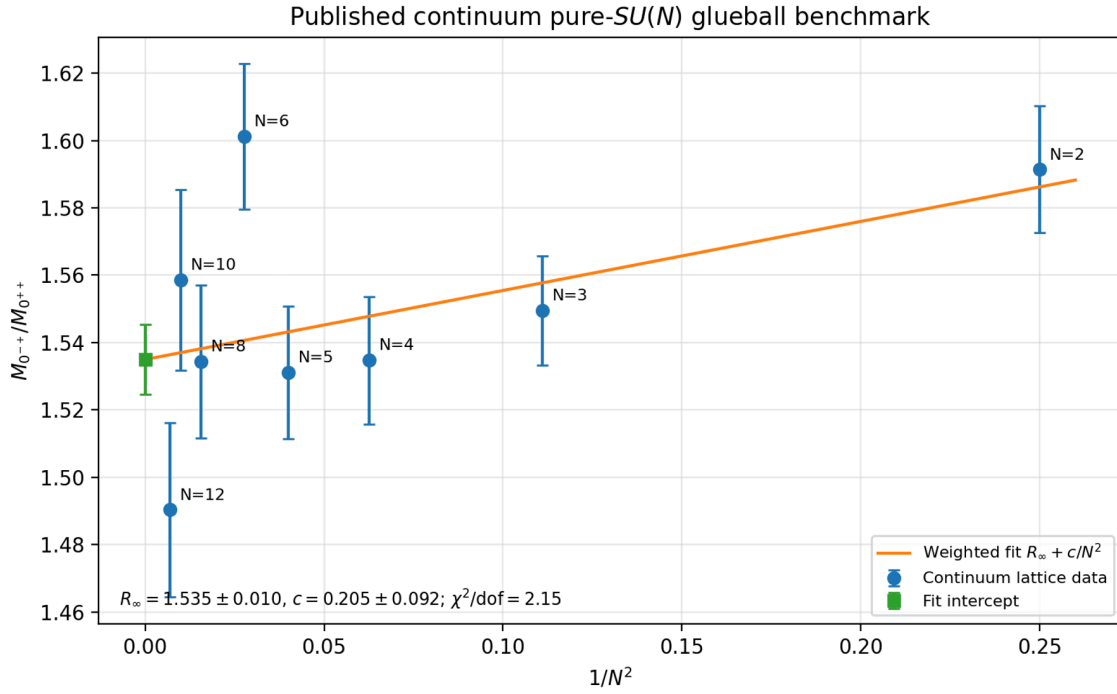


Figure 3. Continuum pure $SU(N)$ lattice benchmark for the pseudoscalar-to-scalar glueball mass ratio as a function of $1/N^2$, using the published $N = 2, 3, 4, 5, 6, 8, 10, 12$ results of Athenodorou and Teper [20]. The weighted line is a spectrum-level test of regular large- N behavior; it does not determine the modular parameters.

Using the published continuum $SU(N)$ spectrum [20], an inverse-variance fit of the scalar masses to $R_S(N) = R_{S,\infty} + c_S/N^2$ gives $R_{S,\infty} = 3.0719(138)$, consistent with the published $3.072(14)$. The analogous pseudoscalar fit gives $R_{P,\infty} = 4.7114(244)$, consistent with $4.711(26)$. Fitting the ratio with $R_{P/S}(N) = R_\infty + c/N^2$ gives $R_\infty = 1.535(10)$ and $c = 0.205(92)$ with $\chi^2/dof \simeq 2.15$. Because this simple estimate neglects cross-channel covariance and the scatter is non-negligible, it is only a consistency demonstration of the regular $1/N^2$ form, not evidence specific to the modular mechanism.

Role	Inputs used for calibration	Held-out predictions
Scale	One predeclared reference scale. If a mass is used to set units, that same mass is calibration rather than prediction.	All remaining dimensionful masses not used for scale setting
Radial sector	$P(\rho)$ shape/curvature and scalar residue f_ρ	Scalar mass (if not used for scale setting); conditional $1/\rho^2$ law; finite-volume behavior
Angular sector	χ_{top}^{mod} (or χ_{top}^{YM} under declared saturation), ρ_0 , f_ϕ , and predeclared N_ϕ	Held-out pseudoscalar mass
Group dependence	A declared subset of N values	Spectra for held-out N
Crossover	$\hat{\kappa}(\rho)$ reconstructed from elliptic invariants	Conditional energy ratio in Eq. (40)

Table 1. Separation of calibration inputs and held-out predictions for a publication-level test.

The closure is rejected if one predeclared parameter set cannot pass the held-out tests within propagated statistical, continuum, finite-volume, and reconstruction uncertainties.

9. Experimental continuation and the X (2370) benchmark

Pure Yang–Mills poles are not directly identical to resonances in full QCD because states with the same quantum numbers mix with quark and multihadron channels. Dynamical-fermion lattice studies likewise show additional mixing structure, especially in the scalar channel [26]. The modular closure supplies a bare gluonic collective pole inside the pure-gauge theory; conventional coupled-channel dynamics must then be included before comparison with experimental resonances.

For a minimal two-state pseudoscalar system, let G denote the bare gluonic basis state and Q a quark-dominated singlet basis state. Use the mass-squared matrix

$$\mathcal{M}_P^2 = \begin{pmatrix} m_G^2 & \mu_P^2 \\ \mu_P^2 & m_Q^2 \end{pmatrix}. \quad (44)$$

Its eigenvalues are

$$M_{\pm}^2 = \frac{m_G^2 + m_Q^2}{2} \pm \frac{1}{2} \sqrt{(m_G^2 - m_Q^2)^2 + 4\mu_P^4}. \quad (45)$$

Label the heavier and lighter eigenvalues by M_H and M_L . Their invariants are

$$M_H^2 + M_L^2 = m_G^2 + m_Q^2, \quad M_H^2 M_L^2 = m_G^2 m_Q^2 - \mu_P^4. \quad (46)$$

The off-diagonal parameter μ_P^2 is a gluonic–quark mixing element. Consequently, the bare modular mass m_G should not in general be identified directly with the observed resonance mass. The mixing angle may be defined by

$$|\tan 2\theta_P| = \frac{2|\mu_P^2|}{|m_Q^2 - m_G^2|},$$

with the sign and quadrant fixed by the chosen eigenstate convention.

Because the experimental state has a finite width, Eqs. (44)–(46) are used only as a real mass-level illustration. A quantitative resonance-pole analysis would require energy-dependent complex self-energies and explicit channel couplings.

BESIII determined the $X(2370)$ quantum numbers to be $J^{PC} = 0^{-+}$ and measured a mass of $2395 \text{ MeV}/c^2$, with a statistical uncertainty of 11 MeV and an asymmetric systematic uncertainty of $+26/-94 \text{ MeV}$; the reported width is 188 MeV with corresponding statistical and systematic uncertainties [27]. A quenched lattice calculation optimized for radiative phenomenology independently obtained a pseudoscalar-gluon mass $2.395(14) \text{ GeV}$ [28]. The mass proximity and matching quantum numbers are useful phenomenological benchmarks, but they do not identify the full-QCD resonance with a pure-Yang–Mills eigenstate.

Prospective out-of-sample pseudoscalar test. Equation (28) gives the bare modular pseudoscalar mass

$$m_G^{mod} \equiv m_P^{mod} = \frac{N_{\vartheta} \Lambda_{YM}^2 \sqrt{\hbar}}{f_{\vartheta} \rho_0} = \frac{N_{\vartheta} \sqrt{\chi_{top}^{mod}}}{f_{\vartheta} \rho_0}.$$

A genuine out-of-sample test must determine ρ_0 , f_{ϑ} , N_{ϑ} , and χ_{top}^{mod} (or χ_{top}^{YM} under an explicitly declared saturation hypothesis), together with the overall Yang–Mills scale where required, from predeclared lattice observables that exclude the target pseudoscalar mass. The quark-dominated mass m_Q and the mixing element μ_P^2 must likewise be constrained from independent full-QCD information. Only after those quantities are fixed should Eq. (45) be used to obtain a predicted mixed eigenvalue and compare it

with the held-out $X(2370)$ mass and production data. If the target mass is used in determining any of these parameters, the result is a phenomenological fit rather than an out-of-sample prediction.

If a gluon-rich interpolating operator $\mathcal{O}_P$ couples dominantly to the bare gluonic basis state, $|\langle 0 | \mathcal{O}_P | Q \rangle| \ll |\langle 0 | \mathcal{O}_P | G \rangle|$, write

$$|X_{phys}\rangle = \cos\theta_P |G\rangle + \sin\theta_P |Q\rangle. \quad (47)$$

Neglecting channel-dependent short-distance corrections, the gluonic residue then satisfies

$$\frac{|\langle 0 | \mathcal{O}_P | X_{phys} \rangle|^2}{|\langle 0 | \mathcal{O}_P | G \rangle|^2} \simeq \cos^2 \theta_P. \quad (48)$$

If the direct quark-basis coupling is not negligible, Eq. (48) must be generalized by retaining the second production amplitude. A simultaneous pole-and-residue analysis is therefore more discriminating than mass proximity alone.

For the scalar sector, the same construction can be extended to a 3×3 mixing matrix involving the bare modular scalar state and nonstrange and strange quarkonia, consistent with standard gluonium phenomenology [\[29\]](#). This extension is not used in the core mass-gap derivation.

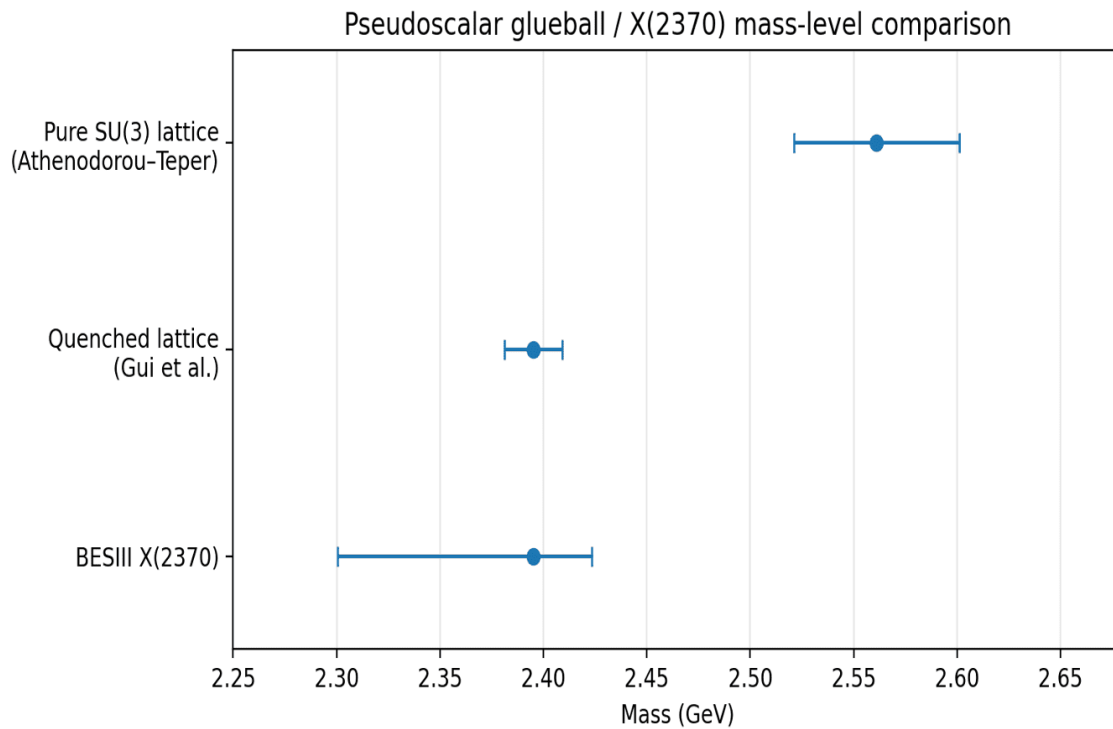


Figure 4. Mass-level comparison of representative pseudoscalar-glueball benchmarks and the BESIII $X(2370)$ result. The pure $SU(3)$ GeV value is scale-setting dependent, the Gui et al. point is a quenched-lattice benchmark, and the BESIII point is a full-QCD resonance. The figure is not a fit of the modular model to the $X(2370)$ mass. A genuine prediction requires m_G , m_Q , and μ_P^2 to be constrained independently of the target mass.

Three levels of empirical confrontation must remain distinct. Pure-Yang-Mills configuration data directly test the composite map and Eq. (40). Published continuum glueball spectra provide spectrum-level consistency tests unless the modular parameters are independently reconstructed. Experimental resonances test only the full-QCD continuation after mixing and cannot by themselves establish the pure-gauge Yang-Mills mass gap.

10. Secondary consequences and extensions

The results in this section are not used in Propositions 1-2 and do not add assumptions to the core mass-gap closure. They indicate secondary tests if the minimal framework survives the direct configuration-level test.

10.1. Correlation-volume scaling

If a correlated glue domain has characteristic volume $V_c = \xi_c^3$, with representative scale factor $\hat{\kappa}_c = \hat{\kappa}(\rho_c)$ and reference collective energy density $\bar{\varepsilon}_c$, the fixed-volume scaling suggests the heuristic relation

$$m_S \sim \hat{\kappa}_c^2 \bar{\varepsilon}_c \xi_c^3, \quad \xi_c \sim m_S^{-1}. \quad (49)$$

Consequently,

$$m_S^4 \sim \hat{\kappa}_c^2 \bar{\varepsilon}_c, \quad (50)$$

up to a dimensionless shape coefficient. This scaling is heuristic rather than part of the proof of Proposition 1. If $\bar{\varepsilon}_c \propto \sigma^2$, then $m_S/\sqrt{\sigma}$ is naturally dimensionless and finite.

10.2. Finite-temperature continuation

At finite temperature allow a , b , and f_ρ to depend on T . The radial screening scale is then

$$m_S^2(T) = \frac{8a(T)\Lambda_{YM}^4}{f_\rho^2(T)}, \quad \rho_0(T) = \left[\frac{b(T)}{a(T)} \right]^{1/4}. \quad (51)$$

Here Λ_{YM} remains the zero-temperature RG-invariant transmutation scale; temperature dependence resides in the effective coefficients. A rapid change in $\rho_0(T)$ or loss of positive curvature would provide a modular diagnostic to compare with conventional deconfinement observables.

10.3. Large- N fluctuations

If the reconstructed coordinates are built from conventionally normalized single-trace densities, large- N factorization motivates the test

$$\text{Var}(\rho)_N = O(N^{-2}), \quad \rho_{0,N} = \rho_{0,\infty} + O(N^{-2}). \quad (52)$$

This supplies a configuration-level test independent of spectral fitting.

11. Falsifiability and mathematical status

The minimal theory is falsified if, after controlled continuum and volume extrapolation, any of the following occurs:

- no stable gauge-invariant composite map $A \mapsto \tau_k[A]$ exists in a predeclared reconstruction scheme;
- the reconstructed sector-relative factor fails the conditional energy law in Eq. (40);

- the Jacobian-corrected radial probability potential has no stable $A_k \rho^2 + B_k / \rho^2$ window with positive curvature;
- Eqs. (32) and (34) fail when masses, residues, the model-normalized radial potential, and topology are measured independently;
- one continuum parameter set cannot describe the declared lattice spacings and volumes, or the predeclared large- N organization fails across held-out gauge groups;
- ultraviolet matching requires an unsuppressed lower-dimensional correction incompatible with the decoupling conditions; or
- full-QCD glueball-candidate residues are incompatible with the mixing structure implied by independently predicted bare gluonic masses.

The present result is an effective-field-theory proposition. It does not prove the existence and mass gap of continuum Yang–Mills theory in the Clay sense. A constructive completion would require, at minimum: a regulator-independent definition of the composite map; proof that the collective description is equivalent to the original continuum measure; reflection positivity or an equivalent Hilbert-space reconstruction; uniform continuum and infinite-volume bounds; an all-channel lower spectral bound; and control of topological and gauge-fixing subtleties without gauge-dependent assumptions [\[8\]\[9\]\[10\]](#). These requirements are not peripheral; they separate a testable physical closure from a completed mathematical proof.

12. Conclusions

We have presented a compact Modular-Weierstrass closure of pure Yang–Mills dynamics that preserves the elementary gauge action, uses the standard transmuted scale Λ_{YM} , and introduces no hard gluon mass. The two companion constructions provide historical motivation for the scale variable. All scale-transfer relations required for the Yang–Mills propositions, however, are restated or derived within the present manuscript, so the core closure and its mathematical consequences do not depend on importing results from those companion studies. Within this self-contained formulation, the Yang–Mills application is reduced to one core constitutive hypothesis: in the reconstructed gauge-invariant collective sector, the leading nonanalytic energy-density contribution is proportional, with a positive matching coefficient, to the sector-relative squared bilinear scale $\widehat{\kappa}^2$.

Near the square elliptic lattice, the Weierstrass expansion gives $\widehat{\kappa}^2 \propto \rho^{-2}$. Its constitutive transfer, combined with the leading regular restoring response, produces

$$U_{mod}(\rho, \vartheta_{\min}) = U_\rho(\rho) = \Lambda_{YM}^4 \left(a\rho^2 + \frac{b}{\rho^2} \right). \quad (53)$$

For $a, b > 0$, this potential has the finite minimum $\rho_0 = (b/a)^{1/4}$ and strictly positive radial curvature. The CP-even radial mode and CP-odd angular mode then yield positive scalar and pseudoscalar quadratic masses inside the effective closure. Published continuum $SU(3)$ data give the benchmark ratio $M_P/M_S = 1.549(16)$, while published multi- N spectra are consistent with a regular $1/N^2$ organization. The $X(2370)$ mass and quantum numbers provide only a full-QCD continuation benchmark after explicit mixing; no out-of-sample experimental pole prediction is claimed here.

The central claim is therefore precise and limited. The manuscript analytically establishes the inverse-square scale transfer, the finite positive-curvature minimum, and the resulting quadratic spectral scales **within the stated closure**. It then identifies Eq. (40) as the decisive configuration-level test that distinguishes the mechanism from a post hoc mass fit. The physical interpretation of κ in terms of momentum-associated spatial response belongs to the companion program and is intentionally not required for the present Yang–Mills derivation. A future cross-scale theory may connect these descriptions quantitatively, but doing so is beyond the scope of this paper.

Appendix A. Local expansion of the Weierstrass scale

Let $\tau = i(1 + \delta)$ with $|\delta| \ll 1$. Since $g_3(i) = 0$ and the weight-six invariant has a simple zero at the lemniscatic point, $g'_3(i) \neq 0$ in the chosen local coordinate. Then

$$\begin{aligned} g_2(i(1 + \delta)) &= g_2(i) + ig'_2(i)\delta + O(\delta^2), \\ g_3(i(1 + \delta)) &= ig'_3(i)\delta + O(\delta^2). \end{aligned} \quad (54)$$

Substitution into Eq. (14) gives

$$\kappa(i(1 + \delta)) = \frac{|g_2(i)|}{\pi |g'_3(i)|} \frac{1}{|\delta|} [1 + O(|\delta|)]. \quad (55)$$

For a reference displacement $\delta_{ref} = \rho_{ref} e^{i\vartheta_{ref}}$,

$$\hat{\kappa}^2 = \frac{\kappa^2(\delta)}{\kappa^2(\delta_{ref})} = \frac{\rho_{ref}^2}{\rho^2} [1 + O(\rho) + O(\rho_{ref})], \quad (56)$$

which establishes Eq. (17). The reference displacement remains a fixed normalization input independent of the dynamically selected minimum; no identification of the two is required.

Appendix B. Derivation of the quadratic masses

Expanding the radial potential around ρ_0 gives

$$U_\rho(\rho_0 + \delta\rho) = U_\rho(\rho_0) + \frac{1}{2}(8a\Lambda_{YM}^4)(\delta\rho)^2 + O(\delta\rho^3). \quad (57)$$

With the canonical radial fluctuation $\varphi_\rho = f_\rho\delta\rho$, the scalar part of Eq. (28) follows. For the angular term, expand about $\vartheta_{\min} = 2\pi l/N_\vartheta$ and let $\Delta\vartheta = \vartheta - \vartheta_{\min}$:

$$h\Lambda_{YM}^4[1 - \cos(N_\vartheta\vartheta)] = \frac{1}{2}N_\vartheta^2 h\Lambda_{YM}^4(\Delta\vartheta)^2 + O((\Delta\vartheta)^4). \quad (58)$$

With $\eta = f_\vartheta\rho_0\Delta\vartheta$, the pseudoscalar mass in Eq. (28) follows.

Appendix C. Pre-registered numerical test

A reproducible implementation should publish the following before inspecting target masses:

- ensemble identifiers, lattice actions, volumes, lattice spacings, and flow-time windows;
- definitions of c_S , c_P , ξ , s_{ref} , and the cell Ω_k ;
- numerical method and error control for g_2 and g_3 ;
- the fit window in ρ for the empirical radial potential;
- priors and covariance treatment for a , b , h , f_ρ , and f_ϑ ;
- calibration observables and held-out observables; and
- a predeclared rejection criterion.

The primary held-out quantities should include the pseudoscalar mass, the coefficient of the dimension-four ultraviolet correction, the conditional ratio in Eq. (40), the shape of the Jacobian-corrected radial probability potential, and at least one gauge group not used in calibration.

Appendix D. Published benchmark data used in Figures 3 and 4

The masses below are transcribed from the cited primary publications to make the spectrum-level demonstrations reproducible. The ratios in Table D1 are calculated from the quoted central values; their displayed uncertainties use standard independent-error propagation and therefore do not include cross-channel covariance. No raw gauge configurations or experimental events were reanalyzed.

N	$M_S/\sqrt{\sigma}$	$M_P/\sqrt{\sigma}$	M_P/M_S
2	3.781(23)	6.017(61)	1.591(19)
3	3.405(21)	5.276(45)	1.549(16)
4	3.271(27)	5.020(46)	1.535(19)
5	3.156(31)	4.832(40)	1.531(20)
6	3.102(32)	4.967(43)	1.601(22)
8	3.099(26)	4.755(58)	1.534(23)
10	3.102(37)	4.835(60)	1.559(27)
12	3.151(33)	4.696(65)	1.490(26)

Table D1. Continuum pure $SU(N)$ lattice masses used in Figure 3 [\[20\]](#).

Source	Mass	Interpretation
Athenodorou-Teper [24]	2.561(40) GeV	Pure $SU(3)$; physical GeV scale setting is conventional
Gui et al. [28]	2.395(14) GeV	Quenched-lattice pseudoscalar glueball
BESIII [27]	2.395 GeV	Full-QCD $X(2370)$ resonance; uncertainties stated in Section 9

Table D2. Pseudoscalar mass-level benchmarks shown in Figure 4.

Statements and Declarations

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This research received no external funding.

Potential Competing Interests

The author declares no conflict of interest.

Ethics

This study was conducted using publicly available data and did not involve human participants or require ethical approval.

Data Availability

No new lattice ensembles or experimental measurements were generated in this study. Numerical benchmark values and plotted points were transcribed from the cited primary lattice and BESIII publications. The configuration-level reconstruction proposed in Section 8 remains prospective and is not claimed to have been performed in the present work.

Author Contributions

Conceptualization, methodology, theoretical development, formal analysis, and writing—original draft and review and editing: Peter Yongtao Wang. The author has read and approved the final version of the manuscript.

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