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Bridging Classical and Computational Physics: Integrating Unsolvable Differential Equations into Undergraduate Education

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Abstract: This manuscript explores the significant gap in undergraduate physics curricula concerning unsolvable differential equations, despite their ubiquity in describing physical systems. Traditional educational frameworks often omit these equations due to their complexity and lack of analytic solutions, leaving computational methods underutilized in academic settings. By implementing computational calculus, this study demonstrates an accessible, straightforward method to handle such equations, supported by nine prototypical examples across classical physics domains. These include the three-body problem, rocket trajectories, electric circuit responses, and more. The approach is not only feasible for inclusion in high school and undergraduate courses but also enhances the conceptual understanding of physics through practical computation, proposing a foundational shift in physics education.

Keywords: Computational physics, Differential equations, Physics education, Numerical methods, Curriculum development

1. The University's Little Secret

Since Newton, the basic paradigm for the analysis of physical systems has been:

1. State the laws of physics governing the system. Laws of physics governing how things change are written as differential equations.
2. Derive a differential equation model of the system from the laws of physics governing it
3. Analyze the differential equation model, with the goal of predicting the performance of the system.

For example, the physical laws governing falling bodies and orbits are covered in high school physics, and the formula for the acceleration of a falling object is derived from these laws; an equation for acceleration is a differential equation. There is no closed-form solution to this differential equation. Lagrange derived an infinite series solution in 1771[1], a modern version is given in Wikipedia[2]:

$$r(t)=\sum_{n=1}^{\infty}\left[\lim_{q\to 0}\left(\frac{x^n}{n!}\frac{d^{n-1}}{dq^{n-1}}\left[r^n\left(\frac{7}{2}\left(\arcsin(\sqrt{q}-\sqrt{q-q^2})\right)^{-\frac{2}{3}}\right]\right)\right]$$

which evaluates to:

$$r(t)=y_0\left(x-\frac{1}{5}x^2-\frac{3}{175}x^3-\frac{23}{7875}x^4-\frac{1894}{3931875}x^5-\frac{3293}{21896875}x^6-\frac{2418092}{62077640625}x^7-\dots\right)$$

$$\textit{with } x=\left[\frac{3}{2}\left(\frac{\pi}{2}-t\sqrt{\frac{2\mu}{r_0^3}}\right)\right]^{\frac{2}{3}},\ \mu=\mathrm{G}\cdot(\mathrm{m}_1+\mathrm{m}_2)\ \text{ for masses } \mathrm{m}_1,\mathrm{m}_2, G=\textit{grav.}\ \text{constant}$$

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This equation is an analytic solution to the 1-D two-body problem, with r(t) = separation at time t. As you might imagine, it requires very advanced calculus to derive, and is beyond the scope of university undergraduate physics. The three-body problem, the motion of three bodies affected only by their mutual gravitational attraction, is analytically unsolvable.

This might surprise you: the differential equation models of most physical systems are analytically unsolvable. This is the university's little secret.

This begs the question - how are unsolvable systems covered in undergraduate physics education? The answer is: they aren't, they aren't mentioned. The limitations of analytic calculus have been internalized in the current curriculum, and unsolvable systems are invisible. Determining orbit position as a function of time, the problem Newton solved in 1687 that marked the beginning of modern math and physics, is invisible in the current classical physics curriculum.

[[Unsolvable differential equations are also invisible in physics education literature, so if you search for 'unsolvable differential equation' in all issues of the AJP, Physics Today, Nature Physics, and Physical Review Physics Education Research, you'll get only one hit, my AJP Letter to the Editor A Revolution in Physics was Forecast in 1989, Why Hasn't It Happened? What Will It Take']]

Computers and computational calculus revolutionized physics in the mid-20th century because they made it possible to analyze unsolvable differential equations. For example, they made planning the Apollo trajectory, a solution to the analytically unsolvable three-body problem, possible.

Unlike Newton's analytic calculus, computational calculus, i.e., the computational methods for calculating solutions to differential equations, is simple and intuitively transparent, and the powerful basic method can be taught to high school science students with no previous exposure to calculus in a single one-hour lecture.

This paper demonstrates the simplicity, ease of use, and extraordinary power of computational methods, with analyses of nine prototypical systems spanning the range of classical physics; all but the space station orbit and the RLC circuit impulse response are analytically unsolvable, the RLC circuit would be unsolvable if it contained a non-linear element.

2. Central Force Motion, Orbits and Rocket Trajectories

2.A. The physics of central force motion

The physical laws governing central force motion are Newton's law of gravity and Newton's second law of motion. The law of gravity is:

$F = G \cdot m_{object} \cdot m_{Earth} / r^2$

where F is the force of gravity, m_{object} and m_{Earth} are the masses of the object and Earth, r is the distance between the centers of the object and Earth, and G is the gravitational constant. The second law of motion is

$F = m \cdot A$

where F is the force acting on the object, m is the object's mass, and A is the object's acceleration.

2.B. The differential equation model for 2-D central force motion

A differential equation model for a physical system consists of a set of state variables that define the state of the system and a differential equation for the rate of change of each state variable.

The equation for the acceleration of a falling/orbiting object in Earth's gravitational field is derived in high school physics from the law of gravity and the second law of motion, it is:
$A = -G \cdot m_{Earth} / r^2$

The state variables for the model for a 2-D falling object are the object's position (distance from the center of the earth) r, and the object's velocity, v (which equals r'). The rate, i.e. differential, equations for the model are:
$r'(t) = v(t)$
$v'(t) =- G \cdot m_{Earth} / r(t)^2$

2.C. Computing trajectories and orbits

Euler's method translates each rate equation into a computational equation. The translation is one-to-one and by rote; the two rate equations above translate to computational equations:
$r(t_{i+1}) = r(t_i) + r'(t_i) \cdot dt = r(t_i) + v(t_i) \cdot dt$
$v(t_{i+1}) = v(t_i) + v'(t_i) \cdot dt = v(t_i) + (- G \cdot m_{Earth} / r(t_i)^2) \cdot dt$

The computational equations translate into programming language statements on a 1-to-1 basis that is for all practical purposes by rote. For more details on programming computational calculus, along with coded examples, see [3],

The state variables for an object in a 2-D orbit are position $\boldsymbol{p}$ and velocity $\boldsymbol{v}$, both 2-D vectors. The model is created by resolving the 2-D gravity vector into x and y

components and calculating the x and y trajectories independently. With $\boldsymbol{p}(t) = (p_x(t), p_y(t))$ and $r(t) = \mathrm{sqrt}(p_x(t)^2 + p_y(t)^2)$ the rate equations are:
$p_x'(t) = v_x(t)$
$v_x'(t) = - (p_x(t) / r(t)) \cdot G \cdot m_{Earth} / r(t)^2$
$p_y'(t) = v_x(t)$
$v_y'(t) = - (p_y(t) / r(t)) \cdot G \cdot m_{Earth} / r(t)^2$.

The corresponding computational equations are:
$p_x(t_{i+1}) = p_x(t_i) + v_x(t_i) \cdot dt$
$v_x(t_{i+1}) = v_x(t_i) + a_x(t_i) \cdot dt = v_x(t_i) - p_x(t_i) / r(t_i) \cdot (G \cdot m_{Earth} / r(t_i)^2) \cdot dt$
$p_y(t_{i+1}) = p_y(t_i) + v_y(t_i) \cdot dt$
$v_y(t_{i+1}) = v_y(t_i) + a_y(t_i) \cdot dt = v_y(t_i) - p_y(t_i) / r(t_i) \cdot (G \cdot m_{Earth} / r(t_i)^2) \cdot dt$.

All that is needed to generate the space station orbit shown in Figure 1 are the relevant parameters: the mass of the earth, and the initial altitude and velocity of the space station. MATLAB programs for the analyses in the paper can be found here.[5]

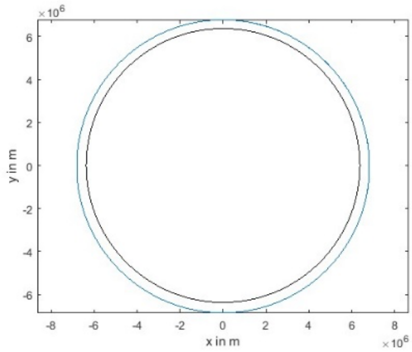


Figure 1 – Calculated space station orbit, with an outline of earth for reference

A rocket launched from Earth toward the Moon is pulled by the Earth's gravity and the Moon's gravity, so the rocket acceleration is calculated using the sum of the gravitational forces of Earth and Moon.

We would like to steer the rocket. The rocket is being modeled as a point mass, and we (as always) are striving for simplicity; we model a guidance boost on the rocket by adding the acceleration due to the guidance boost to the computational equations as shown, where a_{gx} and a_{gy} can be pre-programmed or calculated on the fly,

$v_x(t_{i+1}) = \ldots\ldots\ldots\ldots + a_{gx}(t_i) \cdot dt$
$v_y(t_{i+1}) = \ldots\ldots\ldots\ldots + a_{gy}(t_i) \cdot dt.$

Now we can model a rocket that just misses the moon, and with a slight nudge from a guidance boost put it into orbit around the moon. The rocket and moon trajectories are shown in Figure 2.

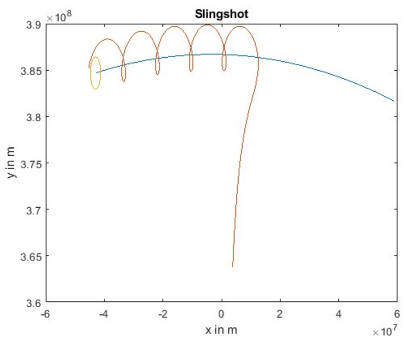


Figure 2. The Apollo trajectory

3. Electric Circuit Analysis

3.A. The physics of electric circuits

The physics of electric circuit analysis is simple and intuitively clear; it consists of three component models and Kirchhoff's laws:

- Resistor model: Ohm's law, $v_R = I_R \cdot R$, the resistor voltage v_R equals the current I_R through the resistor times the resistor's resistance R.
- Capacitor model: $v_C' = I_C / C$, the rate of change of voltage across a capacitor v_C' equals the current flow into the resistor I_C divided by the capacitor's capacitance C.
- Inductor model: $I_L' = v_L / L$, the rate of change of current in an inductor I_L' equals the applied voltage v_L divided by the inductor's inductance L. An intuitive model of an inductor is a bidirectional frictionless turbine, an applied voltage increases/decreases the speed of the turbine, when the applied voltage is zero the turbine spins at a constant speed.
- Kirchhoff's loop law: the voltages across the components in a loop sum to 0.
- Kirchhoff's node law: the currents into a node sum to 0.

3.B. Differential equation models for electric circuits

The model for an electric circuit contains a state variable for each component that is capable of storing energy, that is, each capacitor and each inductor, and a differential equation for the rate of change of each state variable.

The model for the RLC oscillator shown in Figure 4 has state variables for the capacitor voltage, v_C, and the inductor current I_L. This circuit contains no non-linear elements and its impulse response is analytically solvable using the Laplace transform.

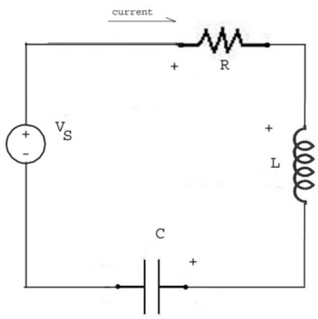


Figure 3. RLC oscillator circuit diagram

From the capacitor model: $v_C'(t) = I_C(t)/ C$

From Kirchoff's voltage law: $V_S(t) – v_R(t) – v_L(t) – v_C(t) = 0$.

Solving for $v_L(t)$ gives $v_L(t) = V_S(t) – v_R(t) – v_C(t) = V_S(t) – I_R(t) \cdot R – v_C(t)$.

From the inductor model: $I_L'(t) = v_L(t)/ L = (V_S(t) – I_R(t) \cdot R – v_C(t)) / L$.

The current in the loop is everywhere the same so $I_C=I_L=I_R=I$ and the equations for $v_C'(t)$ and $I_L'(t)$ above are rate equations for the model. The voltage source $V_S(t)$, the input voltage, and can programmed to any desired input voltage as a function of time.

3.C. Computing electric circuit response

The computational equations for the model are:

$v_C(t_{i+1}) = v_C(t_i) + I(t_i) / C \cdot dt$

$I(t_{i+1}) = I(t_i)) + ((V_S(t_i) – I_R(t_i) \cdot R – v_C(t_i)) / L \cdot dt$

If dt = 0.01, then the input values $V_S(t_{100}) = 100$ and $V_S(t_i) = 0$ for $i \neq 100$ correspond to a unit impulse at t = 1; the circuit response is shown in Figure 4.

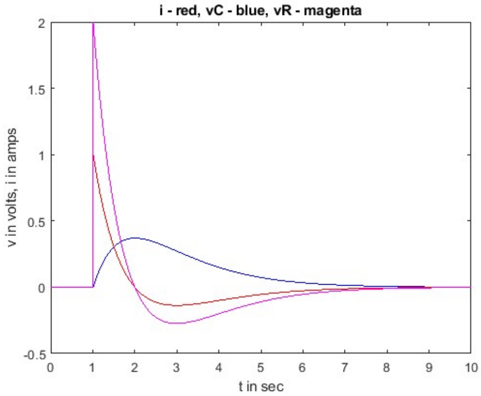


Figure 4. RLC oscillator circuit impulse response

4. 2-D Rigid Body Dynamics

Analysis of 2-D, as opposed to 3-D, rigid body dynamics avoids the complexities of quaternions, the moment of inertia tensor, and Euler's equations of motion, these topics are covered in Appendices III, IV, and V.

4.A. The physics of 2-D rigid body motion

The physics for a 2-dimensional rocket is:

- Newton's 2nd law for translation: **F** = m · ***p***'' where ***p***'' is the acceleration of the object's center of mass, where **F** and ***p*** are 2-dimensional vectors.
- Newton's 2nd law for rotation: Γ = I · α'' where Γ is the applied torque, I is the object's moment of inertia about its center of mass., and α'' is the acceleration of its orientation angle α.

4.B. Differential equation model for a 2-D rigid body rocket

The rocket is modeled (length, mass, thrust) on the Delta IV rocket in the US inventory. The rocket is steered using gimballed engine mounts to control thrust direction. The thrust angle θ can be computed in flight or pre-programmed, see Figure 5.

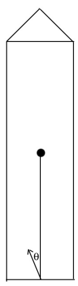


Figure 5. Delta IV rocket model

The forces acting on the rocket are gravity and engine thrust (atmospheric drag is neglected). Thrust is a constant T and is steered using the pre-programmed thrust angle

$\theta(t)$. Thrust is resolved into x and y components using the rocket orientation angle α plus the thrust offset angle θ; the applied torque is $-T\cdot\sin(\theta)\cdot L/2$ where L is the length of the rocket.

4.C. Computing the trajectory of a rocket launched into orbit

The differential equations in the model are translated one to one using Euler's method to obtain the computational equations. The computational equations for the orientation angle α are:

$\alpha(t_{i+1}) = \alpha(t_i) + \alpha'(t_i) \cdot dt$

$\alpha'(t_{i+1}) = \alpha'(t_i) + (-T\cdot\sin(\alpha)\cdot L/2)/I\cdot dt$

where I is the rocket's moment of inertia about its center of mass.

Figure 6 shows the rocket launched from the equator of a spinning earth, inheriting the launch point's lateral velocity and the earth's rotation rate, and steered into orbit.

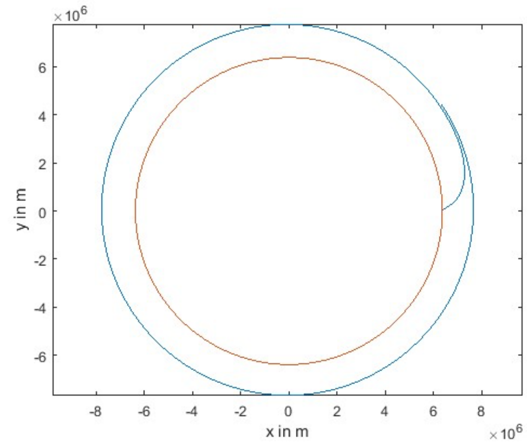


Figure 6. Trajectory for launch to orbit

5. Partial Differential Equations and the Finite Difference Method

The state variables for the physical processes considered thus far depended on a single independent variable, time, and the rate equations for these variables were ordinary differential equations, i.e. differential equations having one independent variable. The systems described in the remainder of the paper will have multiple variables, and the differential equations will be partial differential equations.

Euler's method for computing solutions to ordinary differential equations can be easily extended to compute solutions to partial differential equations, the extended method is known as the finite difference method (FDM).

The FDM substitution for a second order partial derivative $\partial^2 v(t,x,y)/\partial^2 x$ is derived as follows:

The central difference Euler estimate of $\partial^2 v(t,x,y)/\partial^2 x = \partial(\partial(v(t,x,y)/\partial x)/\partial x$ is
$(\partial v(t,x+dx/2,y)/\partial x - \partial v(t,x-dx/2,y)/\partial x) / dx$

The central difference estimate of $\partial v(t,x+dx/2,y)/\partial x$ is $(v(t,x+dx,y) – v(x,y)) / dx$

The central difference estimate of $\partial v(t,x-dx/2,y)/\partial x$ is $(v(t,x,y) – v(t,x-dx,y)) / dx$

Substituting the last two estimates into the first yields
$\partial^2 v(t,x,y)/\partial^2 x \sim (v(,x+dx,y) – v(t,x,y)) / dx - (v(t,x,y) – v(,x-dx,y)) / dx) / dx$
$= (v(t,x+dx,y) - 2\cdot v(t,x,y) + v(t,x-dx,y)) / dx^2$

This is the FDM substitution for a 2nd-order partial derivative. Derivations of the FDM estimates for the other 2nd-order partial derivatives are similar [6].

6. Heat Transfer

6.A. The physics of 2-D heat transfer

In two dimensions the heat flow rate is a vector $\boldsymbol{q} = (q_x, q_x)$. Fourier's heat transfer law in two dimensions is:
$q_x(t,x,y) = -k\cdot \partial T(t,x,y) /\partial x$
$q_y(t,x,y) = -k\cdot \partial T(t,x,y) /\partial y$
where $\boldsymbol{q}$ is the heat flow rate in W/m, k is the material conductivity in (W/m)/(°K/m), and T is temperature in °K.

The relationship between heat energy and temperature is given by
heat energy = c · volume · T
where c is the heat capacity of the material in joules/(volume·°K)

6.B. The partial differential equation model for 2-dimensional time-dependent heat transfer

The model for 2-dimensional time-dependent heat flow is derived as follows: consider the control volume (cv) shown in Figure 7, the heat flow rate into the control volume thru sides A and B is at time t is $q_x(t,x-dx/2,y) - q_x(t,x+dx/2,y)$.

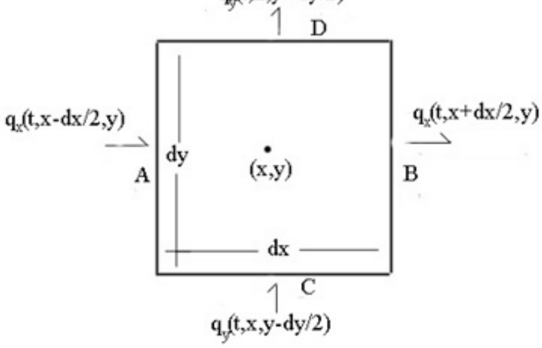


Figure 7. Rate of heat flow in/out of control volume

Using Fourier's heat transfer law, the net flow rate into the control volume through sides A and B is
$q_x(t,x-dx/2,y) - q_x(t,x+dx/2,y) = k\cdot(\partial T(t,x+dx/2,y) /\partial x - \partial T(t,x-dx/2,y) /\partial x)\cdot dy$

Similarly, the heat flow rate into the cv through sides C and D is
$k\cdot(\partial T(t,x,y+dy/x) /\partial y - \partial T(t,x,y-dy/2) /\partial y)\cdot dx$

The temperature in the cv is related to the heat energy in the cv by
T = (1/c) · heat energy / volume = (1/c) · heat energy / dx·dy

So, $\partial T(t,x,y)/\partial t = (1/c) \cdot$ (heat flow rate -> cv) / dx·dy

$= (1/c) \cdot (k\cdot (\partial T(t,x+dx/2,y) /\partial x - \partial T(t,x-dx/2,y) /\partial x) \cdot dy$
$+ k\cdot (\partial T(t,x,y+dy/x) /\partial y - \partial T(t,x,y-dy/2) /\partial y) \cdot dx) / (dx\cdot dy)$

$= (k/c) \cdot (\partial T(t,x+dx/2,y) /\partial x - \partial T(t,x-dx/2,y) /\partial x) / dx$
$+ (\partial T(t,x,y+dy/x) /\partial y - \partial T(t,x,y-dy/2) /\partial y) / dy$

and taking the limit as dx->0 and dy->0 gives the model for 2-dimensional time-dependent heat transfer
$\partial T/\partial t = (k/c) \cdot(\partial^2 T/\partial^2 x+ \partial^2 T /\partial^2 y)$

6.C. Computing 2-D time-dependent heat transfer

To analyze heat transfer in a 2-D rectangular plate the time domain 0-*T* is divided as before into N evenly spaced intervals $0 = t_1$, ... $t_{N+1} = T$, with $\Delta t = T/N$, and the spatial domain 0-X, 0-Y is divided into a uniform grid of points as shown in the diagram, with $0 = x_1$, ... $x_{NX+1} = X$, with $\Delta x = X/N_X$, and $0 = y_1$, ... $y_{NY+1} = Y$, with $\Delta y = Y/N_Y$, as shown in Figure 8.

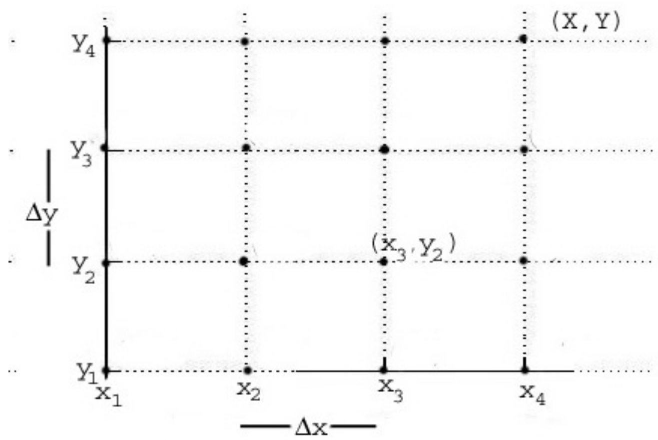


Figure 8. Grid for FDM analysis
The state variables are the temperature values $T(x_i, y_j)$ at the grid points.

The computational equations for T are:
$T^{n+1}(x_i,y_j) = T^n(x_i,y_j) + \Delta t\cdot(k/c) \cdot(\partial^2 T^n(x_i,y_j) /\partial^2 x+ \partial^2 T^n(x_i,y_j) /\partial^2 y)$
following the standard convention of writing $T^n(x_i,y_j)$ for $T(t_n,x_i,y_j)$.

Making the FDM substitutions for the partial derivatives, we have
$T^{n+1}(x_i,y_j) = T^n(x_i,y_j) + \Delta t\cdot(k/c)\cdot[(T^n(x_{i+1},y_j) – 2\cdot T^n(x_i,y_j) + T^n(x_{i-1},y_j))/dx^2$

$+ (T^n(x_i,y_{j+1}) - 2\cdot T^n(x_i,y_j) + T^n(x_i,y_{j-1}))/dy^2]$

With dx = dy this simplifies to
$T^{n+1}(x_i,y_j) = T^n(x_i,y_j) + \Delta t\cdot(k/c)\cdot[(T^n(x_{i+1},y_j) + T^n(x_i,y_{j+1}) - 4\cdot T^n(x_i,y_j) + T^n(x_{i-1},y_j) +$
$+ T^n(x_i,y_{j-1})]/dx^2$

The above equation written in index form is
$T^{n+1}(i,j) = T^n(i,j) + \Delta t\cdot(k/c)\cdot[(T^n(i+1,j) + T^n(i,j+1) - 4\cdot T^n(i,j) + T^n(i-1,j)$
$+ T^n(i,j-1))]/dx^2$

Given the values for $T^n(i,j)$, i.e. the temperature values for the grid at time t_n, the computational equations can be used to calculate $T^{n+1}(i,j)$ for each of the interior points in the grid.

There are three common ways of specifying boundary values, as follows:
-Dirichlet boundary condition – a boundary value is constant, i.e. does not change with time, e.g. $T^{n+1}(i,1) = C$.
-Neumann boundary condition – the value of a partial derivative is specified at the boundary, e.g. $(T^{n+1}(i,2) - T^{n+1}(i,1)) /\Delta y = C$, so $T^{n+1}(i,1) = T^{n+1}(i,2)\cdot C\cdot\Delta y$.
-Robin boundary condition – a linear combination of Dirichlet and Neumann boundary conditions.

It's necessary to specify initial conditions, i.e. temperatures, for all of the grid points at time $t_1 = 0$. For the example shown in Fig. 9 Dirichlet boundary conditions keep the bottom boundary at 0°K, the upper boundary at 5°K, and the interior section at 10°K, Neumann boundary conditions $\partial T/\partial x = 0$ are specified for the left and right borders.

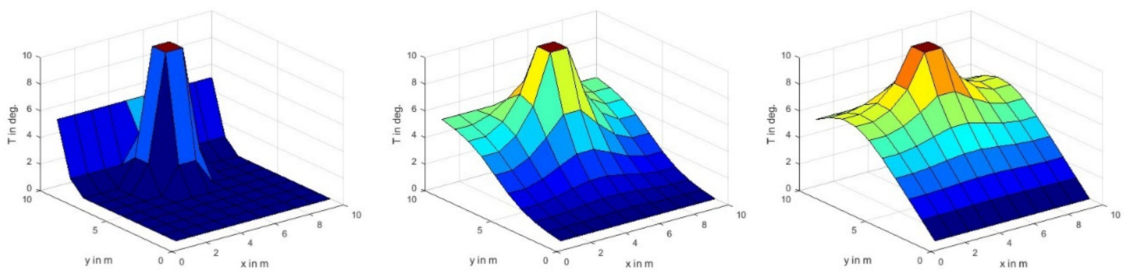


Figure 9. Temperature distribution at t = 0.0, 0.2, and 0.8 seconds
Note: MATLAB plots the graphs directly from the temperature array.

7. Wave Phenomena

7. A. The physics of wave motion

The wave equation for a vibrating 2-D surface can be derived from Newton's 2nd law of motion.

7.B. The partial differential equation model for wave motion

The forces acting on a small control volume in a 2-D surface under constant tension of T newtons/m are shown in Fig. 10 below:

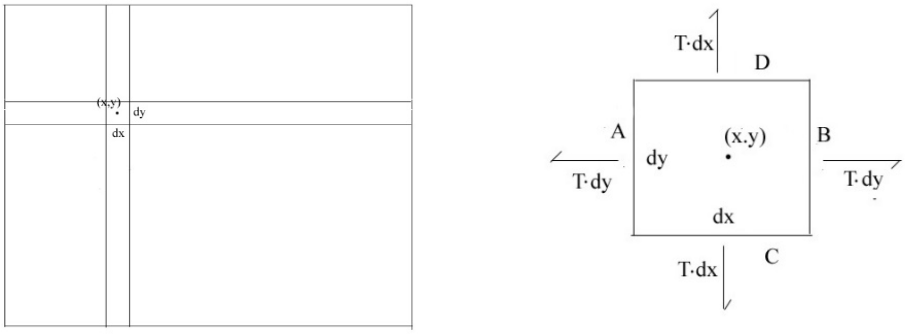


Figure 10 – Forces acting on a control volume in the surface under tension

The displacement function *u*(t,x,y) gives the vertical displacement of the surface from 0. Fig. 11 shows the cv looking head-on at side C.

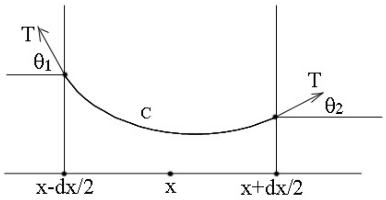


Figure 11. View from front

The vertical forces acting on the control volume on sides A and B are
$T \cdot dy \cdot \sin(\theta_1)$ and $T \cdot dy \cdot \sin(\theta_2)$ respectively.

From the small angle approximation
$\sin(\theta_1) \approx -\partial u(t,x-dx/2, y)/\partial x$, and
$\sin(\theta_2) \approx \partial u(t,x+dx/2, y)/\partial x$
The total vertical force on the control volume from sides A and B is
$T \cdot dy \cdot (\partial u(t,x+dx/2, y+dy/2)/\partial x - \partial u(t,x-dx/2, y+dy/2)/\partial x)$

Similarly, the total vertical force on the control volume from sides C and D is
$T \cdot dx \cdot (\partial u(t,x, y+dy/2)/\partial y - \partial u(t,x, y-dy/2)/\partial t)$

From the 2nd law of motion, the acceleration of the control volume equals the force on the control volume divided by its mass. Thus

$$\frac{\partial^2 u(x,y)}{\partial t^2} = \frac{F}{m} = T\,\frac{\frac{\partial u}{\partial x}(x+dx/2,y) - \frac{\partial u}{\partial x}(x-dx/2,y)dy}{\rho dxdy} + T\,\frac{\frac{\partial u}{\partial y}(x,y+dy/2) - \frac{\partial u}{\partial y}(x,y-dy/2)dx}{\rho dxdy}$$

where □ is the density of the surface in kg/m^2.

Taking the limit as dx -> 0, dy -> 0 yields the differential equation model for 2-D wave phenomena:

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

7.C Computing wave motion

Making the FDM substitutions for the derivatives in the differential equation model and rearranging to solve for $u^{n+1}(i,j)$ gives the computational equation for wave motion.

$u^{n+1}(i,j) = 2 \cdot u^n(i,j) - u^{n-1}(i,j) + (T/\rho)^2 \cdot dt^2 \cdot (u^n(i+1,j) - 2 \cdot u^n(i,j) + u^n(i-1,j)$
$+ u^n(i,j+1) - 2 \cdot u^n(i,j) + u^n(i,j-1))/dx^2$

For the example shown in Fig. 12 the surface displacement was initialized to zero except for a sine wave pulse in the middle. Dirichlet boundary conditions hold u=0 on all boundaries.

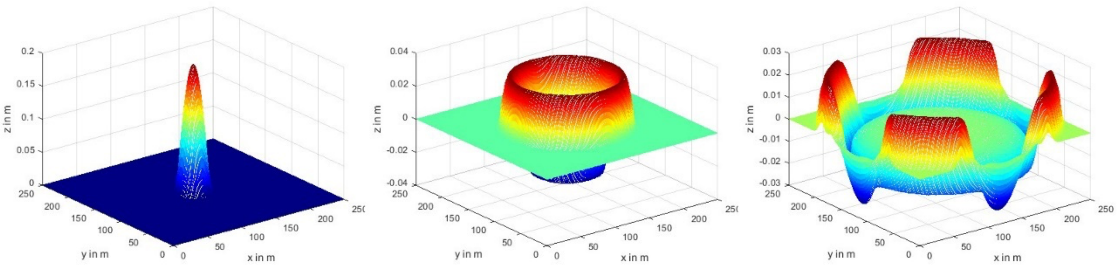


Figure 12: Initial pulse, a wave traveling outward, wave being reflected at the boundary

8. Stress and Strain in 2-D Elastic Materials

8.A. The physics of stress and strain

8.A.1. The 2-D stress tensor

The force applied to the surface of an elastic 2-D object is transmitted to the interior of the object creating stress throughout the object. Stress is force per unit area, the stresses acting on the right side of a vertical surface in a stressed 2-D object are shown in Figure 13:

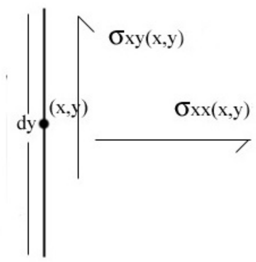


Figure 13. Stress tensor components for a vertical surface

$\sigma_{xx}(x,y)$ = the x-component of stress on a small surface centered at (x,y) with normal vector **i** = (1, 0)
$\sigma_{xy}(x,y)$ = the y-component of stress on a small surface centered at (x,y) with normal vector **i** = (1, 0)
There are four components of stress at a point in a 2-D object. $\sigma_{yy}(x,y)$ and $\sigma_{yx}(x,y)$ are the stresses on the top of a horizontal surface centered at (x,y). The four components make up a rank 2 stress tensor.

8.A.2. The 2-D strain tensor

The strain at a point in a stressed object is defined in terms of the partial derivatives of the displacement function $\Delta(x,y) = (u(x,y), v(x,y))$ as shown in Figure 14.

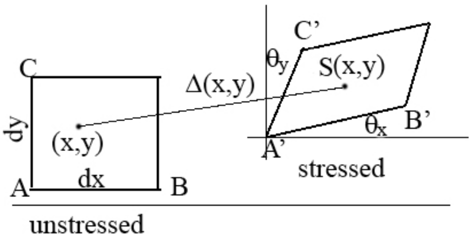


Figure 14. $S(x,y) = (x,y) + \Delta(x,y) = (x,y) + (u(x,y),v(x,y))$

Normal strain in the x direction is defined as the ratio of change in length to length:
$e_{xx} = (B'_x - A'_x - dx) / dx = (S_x(x+dx,y) - S_x(x,y) - dx)/dx = ((u(x+dx,y) - u(x,y))/dx$
Taking the limit as dx -> 0, the normal strain rate $e_{xx} = \partial u/\partial x$. Similarly, $e_{yy} = \partial v/\partial y$
Shear strain corresponds to a change in the angle between the x and y axes:

$\theta_x \approx \sin(\theta_x) \approx (B'_y - A'_y) / dx = (S_y(x,y+dx) - S_y(x,y))/dx) = ((v(x+dx,y) - v(x,y))/dx$

$\theta_y \approx \sin(\theta_y) \approx (C'_x - C_x) / dy = (S_x(x,y+dy - S_y(x,y))/dy) = ((u(x,y+dy) - u(x,y))/dy$
Taking the limit as dx, dy -> 0, $e_{xy}(x,y) = \varepsilon_{yx}(x,y) = \theta_x + \theta_y = \partial v/\partial x + \partial u/\partial y$

The four components of strain up make up the rank 2 strain tensor.

8.A.3. Hooke's Law

The physical laws governing stress and strain in materials are Newton's second law of motion, and Hooke's law relating stress and strain.

A 2-dimensional form of Hooke's law is the following:[7]
E = Young's modulus
ρ = Poisson's ratio

G = shear modulus

ε_{xx}, ε_{yy}, σ_{xx}, σ_{yy}.. are normal strains and stresses

θ, $\sigma_{xy} = \sigma_{yx}$ are shear strain and stress

8.B. The partial differential equation model for a stressed elastic object

8.B.1 The stress equilibrium model

Figure 15 shows the forces acting on a small control volume in a stressed object.

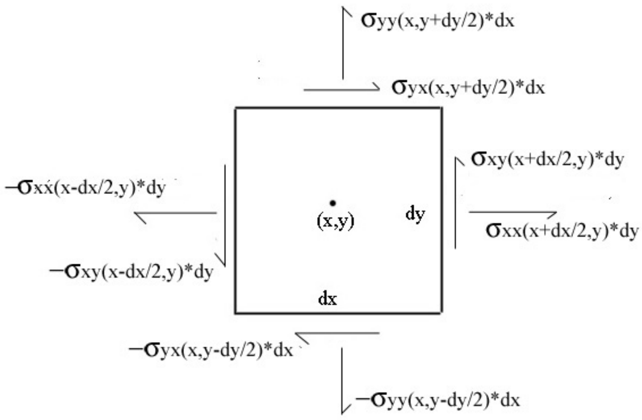


Figure 15. Forces acting on a control volume

The x-component force acting on the cv equals 0, thus

$$0=\frac{(\sigma_{xx}(x+dx/2,y)-\sigma_{xx}(x-dx/2,y))dy+(\sigma_{yx}(x,y+dy/2)-\sigma_{yx}(x,y-dy/2))dx}{dxdy}$$

$$=\frac{\sigma_{xx}(x+dx/2,y)-\sigma_{xx}(x-dx/2,y)}{dx}+\frac{+\sigma_{yx}(x,y+dy/2)-\sigma_{yx}(x,y-dy/2)}{dy}$$

Taking the limit as dx -> 0 and dy -> 0 gives
$\partial\sigma_{xx}/\partial x + \partial\sigma_{yx}/\partial y = 0$.
Similarly $\partial\sigma_{yy}/\partial y + \partial\sigma_{xy}/\partial x = 0$. This is the stress equilibrium model for a stressed elastic object.

Note that $\partial\sigma_{xy}$ acting on the sides of the cv is counter-clockwise and $\partial\sigma_{yx}$ acting on the top and bottom is clockwise, so with dy = dx the torque on the plate as dx -> 0 is

$$T \approx 2\cdot(\sigma_{xy}(x,y)-\sigma_{yx}(x,y))\cdot \mathrm{dx}\cdot \mathrm{dx}$$

The moment of inertia for a square control volume with density d is I= $d\cdot 12\cdot \mathrm{dx}^4$, so the rotational acceleration of the control volume equals T/I -> ∞ unless $\sigma_{xy} = \sigma_{yx}$; thus the stress tensor must be symmetric.

Note that the model describes a system that does not evolve with time, and time is not a state variable in the model.

8.B.2 The differential equation model for displacement

The next step is to derive the model in terms of displacement by using Hooke’s law to substitute for the stress variables in the stress equilibrium model, giving a model in terms of strain variables, and then writing the strain variables using their definitions in terms of derivatives of the displacement function.

Inverting the Hooke’s Law gives the stresses as functions of the strains,

$$\sigma_{xx} = \frac{E}{1-V^2}\varepsilon_{xx} + \frac{E\cdot\rho}{1-\rho^2}\varepsilon_{yy} = A\cdot\frac{\partial u}{\partial x} + A\cdot\rho\cdot\frac{\partial v}{\partial x}$$

$$\sigma_{yy} = \frac{E\cdot\rho}{1-\rho^2}\varepsilon_{sx} + \frac{E}{1-\rho^2}\varepsilon_{yy} = A\cdot\frac{\partial u}{\partial x} + A\cdot\rho\cdot\frac{\partial v}{\partial x}$$

$$\sigma_{xy} = \sigma_{yx} = G\theta = G(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$$

$$with\ \ A = \frac{E}{1-\rho^2}$$

Substituting these expressions into the stress equilibrium model, and taking derivatives, e.g., $\partial(\partial u/\ \partial x)/\partial x = \partial^2 u/\partial x^2$, gives:

$$A\frac{\partial^2 u}{\partial x^2} + G\frac{\partial^2 v}{\partial y^2} + (A\cdot\rho + G)\frac{\partial^2 v}{\partial xy} = 0$$

$$A\frac{\partial^2 u}{\partial y^2} + G\frac{\partial^2 v}{\partial x^2} + (A\cdot\rho + G)\frac{\partial^2 u}{\partial xy} = 0$$

8.C. Computing stress and strain

Making the FDM substitutions and rearranging to solve for u and v gives the computational equations:

$$u(i,j) = 1/(\frac{2A}{dx^2} + \frac{2\mathrm{G}}{dy^2})\begin{bmatrix} \frac{A}{dx^2}(u(i+1,j) + u(i-1,j)) + \frac{G}{dy^2}(u(i,j+1) + u(i,j-1)) + \ldots \\ \ldots + \frac{A\cdot\rho + G}{4dxdy}(v(i+1,j+1) - v(i+1,j-1) - v(i-1,j+1) + v(i-1,j-1)) \end{bmatrix}$$

$$v(i,j) = 1/(\frac{2A}{dx^2} + \frac{2\mathrm{G}}{dy^2})\begin{bmatrix} \frac{A}{dy^2}(v(i+1,j) + v(i-1,j)) + \frac{G}{dx^2}(v(i,j+1) + v(i,j-1)) + \ldots \\ \ldots + \frac{A\cdot\rho + G}{4dxdy}(u(i+1,j+1) - u(i+1,j-1) - u(i-1,j+1) + u(i-1,j-1)) \end{bmatrix}$$

This is a steady-state model, that is, a set of simultaneous linear equations, and the solution is obtained using linear algebra, not calculus. Solutions can be calculated using the Jacobi method, described in Appendix I.

Figure 16 shows two 2x2 stressed plates, both with fixed Dirichlet boundaries on the bottom, one with a downward 1N/m traction along the top, and the other with a 0.2N/m shear traction along the top, specified by Neumann boundary conditions.

Along the sides, there are no tractions. There is vertical strain; from Hooke's law, the ratio of horizontal strain to vertical strain is given by Poisson's ratio ρ. So $\varepsilon_{xx} = \rho \cdot \varepsilon_{yy}$, that is $\partial u/\partial x = \rho \cdot \partial v/\partial y$ creating a Neumann boundary condition for u; vertical displacement is copied from the adjacent node using a Neuman boundary condition $\partial v/\partial x = 0$.

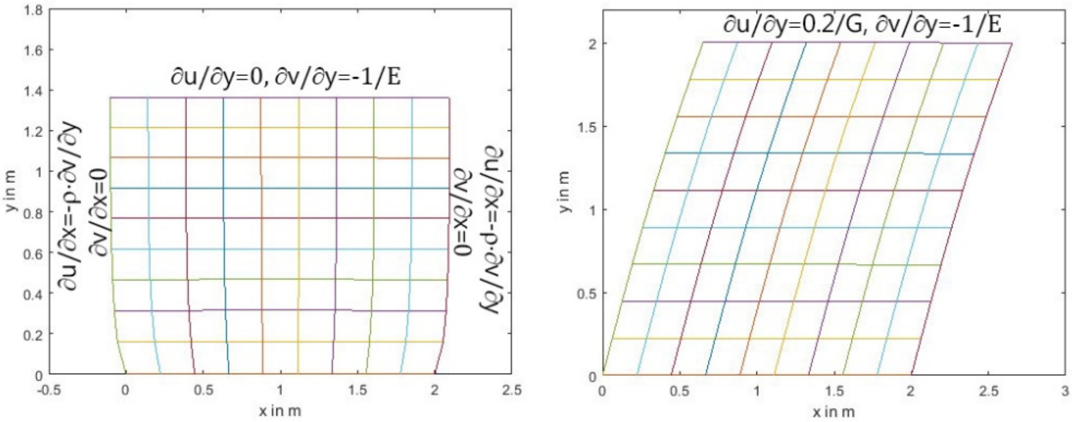


Figure 16. Plates showing effects of normal and shear stress

9. FLUID DYNAMICS

9.A The physics of fluid motion

The laws of physics governing fluid motion are Newton's 2nd law of motion and Newton's law of viscosity.

9.B. The differential equation model of 2-D fluid motion: the Navier-Stokes equations

The state variables are u and v, the x and y velocities of the fluid, and p, the fluid pressure. □ is fluid density and μ is viscosity. The Navier-Stokes equations for incompressible Newtonian fluids in 2-D are

$$\rho(\frac{\partial u}{\partial t}) + \rho(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}) = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

$$\rho(\frac{\partial v}{\partial t}) + \rho(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}) = -\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The first two equations are the momentum equations, and the third is the continuity equation. The equations are derived by analyzing fluid flowing through a fixed control volume (cv) as shown in Fig. 17.

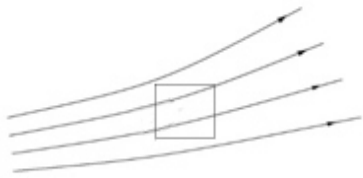


Figure 17. A fixed control volume in a 2-D moving fluid

9.C. The continuity equation

Fig. 18 shows the fluid flow rates in/out of the control volume.

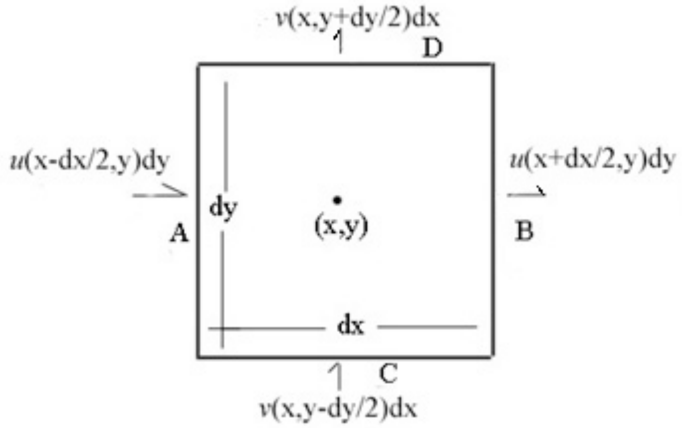


Figure 18. Fluid flow rates in/out of cv

The net flow rate into the cv is 0 since the fluid is incompressible, so we have

$$0 = \frac{u(x - dx/2, y)\cdot dy - u(x + dx/2, y)\cdot dy + v(x, y - dy/2)\cdot dx - v(x, y + dy/2)\cdot dx}{dx \cdot dy}$$

$$= \frac{u(x - dx/2, y) - u(x + dx/2, y)}{dx} + \frac{v(x, y - dy/2) - v(x, y + dy/2)}{dy}$$

The continuity equation follows taking the limit as dx -> 0 and dy -> 0, giving

$$\frac{\partial u(x)}{\partial x} + \frac{\partial v(x)}{\partial y} = 0$$

9.D. The momentum equations

Newton's law of motion applies to a fixed mass, but the fluid in the control volume is constantly changing, so F=M·A can't be applied directly to determine the rate of change of momentum in the control volume. It can be applied to a system control volume that moves with the fluid using the Reynolds Transport Theorem.

Reynolds Transport Theorem: the force applied to a fixed control volume equals the rate of change of momentum of the fluid in the control volume plus the rate of momentum flow out of the control volume.[8, 9]

Let cv be a fixed control volume, and scv be a system control volume consisting of a given set of molecules that is coincident with cv at time t, and that moves with the fluid. The solid line in Figure 19 represents the cv and the dashed line represents the scv, at times t and t+dt; the scv is coincident with the cv at time t.

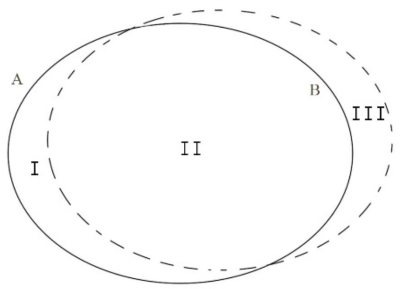


Figure 19 – Fixed and system control volumes at times t and t+Δt

Region I is the area in the cv at time t + dt that is not in the scv, Region II is the area in the cv and the scv at time t + dt, and Region III is the area in the scv that is not in the cv at time t + dt.

The momentum in the scv at t+-dt equals the momentum in Region III plus the momentum in Region II. Thus, it equals the momentum in Region III plus the momentum in the cv minus the momentum in Region I.

$$F = \frac{\partial M_{scv}}{\partial t} = \frac{\partial M_{III}}{\partial t} + \frac{\partial M_{cv}}{\partial t} - \frac{\partial M_{I}}{\partial t}$$

$$\frac{\partial M_{cv}}{\partial t} = \frac{d}{dt}\int_{cv} \rho \cdot u(x,y,t) \cdot dV = \int_{cv} \frac{\partial}{\partial t} \rho \cdot u(x,y,t) \cdot dV \text{ by the Leibniz integral rule}$$

$$\text{as cv-> 0, } \int_{cv} \frac{\partial}{\partial t} \rho \cdot u(x,y,t) \cdot dV = \rho \frac{\partial u}{\partial t}, \text{ so}$$

$$F = \frac{\partial M_{scv}}{\partial t} = \rho \frac{\partial u}{\partial t} + \frac{\partial M_{III}}{\partial t} - \frac{\partial M_{I}}{\partial t}$$

At time t the cv and scv are coincident, so regions I and III are empty and

$$\frac{\partial M_{III}}{\partial t} = \text{the rate of momentum flow out of the cv thru B}$$

$$\frac{\partial M_{I}}{\partial t} = \text{the rate of momentum flow into the cv thru A}$$

$$\frac{\partial M_{III}}{\partial t} - \frac{\partial M_{I}}{\partial t} = \text{net rate of momentum flow out of/into the cv}$$

9.D.1. The stress tensor

Figure 14 above shows stresses acting on a control volume in a stressed elastic object. The same stresses act on a fixed control volume in a moving fluid. Stress in a moving fluid consists of stress caused by hydrostatic pressure p, and deviatoric stress caused by fluid motion and represented by the deviatoric stress tensor **t**. The total stress tensor **s** is

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} -p & 0 \\ 0 & -p \end{bmatrix} + \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix}$$

Stokes assumptions:

1. deviatoric stress is a linear function of strain rates
2. deviatoric stress is 0 when the strain rates are 0
3. the stress to strain rate relation is isotropic, that is, fluid properties are independent of direction

9.D.2. The LHS of the momentum equations

X-momentum flow in/out of the cv. is shown in Fig. 20

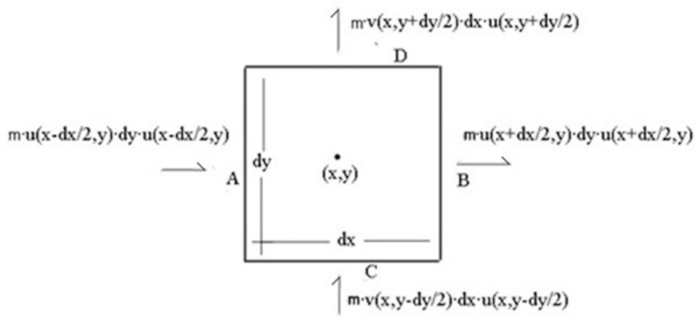


Figure 20. x-momentum flow in/out of the control volume

The net x-momentum flow out of the control volume, divided by volume is

$$\frac{\rho(u(x+dx/2,y)^2 - u(x-dx/2,y)^2)dy}{dxdy} + \frac{\rho(u(x,y+dy/2)\cdot v(x,y+dy/2) - u(x,y-dy/2)\cdot v(x,y-dy/2))dx}{dxdy}$$

Taking the limit as dx -> 0 and dy -> 0 gives

$$\rho(\frac{\partial\left(u(x,y)^2\right)}{\partial x} + \frac{\partial\left(u(x,y)\cdot v(x,y)\right)}{\partial y})$$

$$= \rho(2u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}) = \rho(u\frac{\partial u}{\partial x} + u(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0) + v\frac{\partial u}{\partial y})$$

$$= \rho(u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y})$$

Thus the LHS side of the momentum equations are

$$\rho(\frac{\partial u}{\partial t}) + \rho(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}), \text{and similarly}$$

$$\rho\frac{\partial v}{\partial t} + \rho(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y})$$

9.D.3. The strain rate tensor

Stress and strain are linearly related in elastic objects. In a moving fluid, the strain on the control volume is constantly changing; for fluids, stress is linearly related to strain rate. The strain tensor is given by:

$$\vec{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix}$$

Figure 19 shows the deformation of the scv from time t to t + dt.

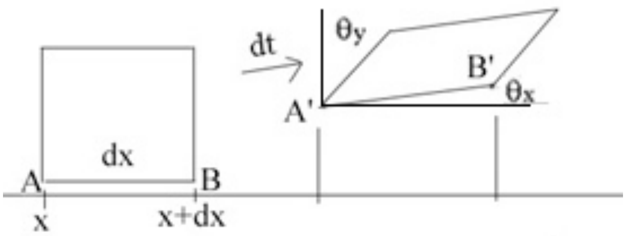


Figure 21: Deformation of the scv due to fluid motion

Normal strain at time t+dt in the x-direction is given by:

e_{xx} = (B'$_x$ – A'$_x$ – dx) / dx = (x + dx +u(x+dx,y) · dt – (x + u(x,y)·dt) - dx) / dx

= (u(x+dx,y)·dt – u(x,y)·dt)/dx = (u(x+dx,y) - u(x,y))/dx · dt

Taking the limit as dx -> 0, the normal strain rate e_{xx}/dt = ∂u/∂x. Similarly, e_{yy}/dt = ∂v/∂y

Shear strain at time t+dt is calculated using the small angle approximation for the sine function:

$\theta_y \approx$ sine(θ_y) ≈ (u(x,y + dy) – u(x,y)) · dt / dy

$\theta_x \approx$ sine(q_x) ≈ (v(x + dx, y)– u(x,y)) · dt / dx

$\theta_y + \theta_x \approx$ (u(x,y + dy) – u(x,y)) /dy · dt + (v(x + dx, y)– u(x,y)) / dx · dt

Taking the limit as dx -> 0 and dy ->0, $\theta_y + \theta_x \approx$ (∂u(x,y) / ∂y + ∂v(x,y) / ∂x) · dt

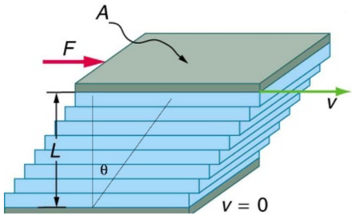
and the shear strain rate is ∂u(x,y) / ∂y + ∂v(x,y) / ∂x

The strain rate tensor is symmetric and is given by:

$$\begin{bmatrix} \dot{\varepsilon}_{xx} & \dot{\varepsilon}_{xy} \\ \dot{\varepsilon}_{yx} & \dot{\varepsilon}_{yy} \end{bmatrix} = \begin{bmatrix} \partial u / \partial x & \dfrac{\dot{\theta}}{2} = \dfrac{\partial u / \partial y + \partial v / \partial x}{2} \\ \dfrac{\dot{\theta}}{2} = \dfrac{\partial u / \partial y + \partial v / \partial x}{2} & \partial v / \partial y \end{bmatrix}$$

9.D.4. The Relationship Between Stress and Strain Rate

For a Newtonian fluid the relationship between shear stress due to fluid motion and the shear strain rate is given by Newton's Law of Viscosity, shown in Figure 23.



shear stress = F/A = $\boldsymbol{\tau}_{yx}$

shear strain = θ

$\boldsymbol{\tau}_{yx} = \mu \cdot d\theta / dt$, where μ = fluid viscosity

Figure 23 Newton's Law of Viscosity

The strain rate tensor is defined by spatial derivatives of the fluid velocity functions. Since the fluid is isotrophic the matrix relating the stress tensor to the strain rate tensor is independent of the coordinate axes, and Newton's law of viscosity is sufficient to determine the entire matrix.

The relationship between the stress tensor and the strain rate tensor is specified by a rank 4 isotropic tensor. From Appendix II and Newton's law of viscosity, we have

$$\begin{bmatrix} \tau_{xx} = 0 \\ \tau_{xy} = \mu \cdot \dot{\theta} \\ \tau_{xy} = \mu \cdot \dot{\theta} \\ \tau_{yy} = 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \lambda & 0 & 0 & \alpha \\ 0 & \beta & \gamma & 0 \\ 0 & \gamma & \beta & 0 \\ \alpha & 0 & 0 & \lambda \end{bmatrix} \end{bmatrix} \begin{bmatrix} \dot{\varepsilon}_{xx} = 0 \\ \dot{\theta}/2 \\ \dot{\theta}/2 \\ \dot{\varepsilon}_{yy} = 0 \end{bmatrix}$$

thus $\beta = \gamma = \mu$. Rotating the basis vectors 45 deg., we have (Appendix I)

$$\begin{bmatrix} \tau_{x'x'} = \lambda \cdot \dot{\varepsilon}_{x'x'} + \gamma \cdot \dot{\varepsilon}_{y'y'} = \mu \cdot \dot{\theta} \\ 0 \\ 0 \\ \tau_{y'y'} = \lambda \cdot \dot{\varepsilon}_{y'y'} + \gamma \cdot \dot{\varepsilon}_{x'x'} = -\mu \cdot \dot{\theta} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \lambda & 0 & 0 & \alpha \\ 0 & \beta & \gamma & 0 \\ 0 & \gamma & \beta & 0 \\ \alpha & 0 & 0 & \lambda \end{bmatrix} \end{bmatrix} \begin{bmatrix} \dot{\varepsilon}_{x'x'} = \dot{\theta}/2 \\ \dot{\theta}'/2 = 0 \\ \dot{\theta}'/2 = 0 \\ \dot{\varepsilon}_{y'y'} = -\dot{\theta}/2 \end{bmatrix}$$

note: $\dot{\varepsilon}_{x'x'} = -\dot{\varepsilon}_{y'y'}$ is the continuity equation, so

$\lambda \cdot \dot{\varepsilon}_{x'x'} + \gamma \cdot \dot{\varepsilon}_{y'y'} = (\lambda - \gamma) \cdot \dot{\theta}/2$ and $\gamma \cdot \dot{\varepsilon}_{x'x'} + \lambda \cdot \dot{\varepsilon}_{y'y'} = -(\lambda - \gamma) \cdot \dot{\theta}/2$

so $(\lambda - \gamma) = 2 \cdot \mu$ and

$$T = \begin{bmatrix} \begin{bmatrix} 2 \cdot \mu & 0 & 0 & \alpha \\ 0 & \mu & \mu & 0 \\ 0 & \mu & \mu & 0 \\ \alpha & 0 & 0 & 2 \cdot \mu \end{bmatrix} \end{bmatrix}$$

thus

$\tau_{xx} = 2 \cdot \mu \cdot \partial u / \partial x$

$\tau_{xy} = \mu \cdot (\partial u / \partial y + \partial v / \partial x)$

$\tau_{xx} = 2 \cdot \mu \cdot \partial v / \partial y$

A geometric derivation of the stress-strain rate relationship from first principles is given in [10].

9.D.5. The RHS of the momentum equations

The stresses acting on the control volume are

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} -p & 0 \\ 0 & -p \end{bmatrix} + \begin{bmatrix} 2 \cdot \mu \cdot \partial u / \partial x & \mu \cdot (\partial u / \partial y + \partial v / \partial x) \\ \mu \cdot (\partial u / \partial y + \partial v / \partial x) & 2 \cdot \mu \cdot \partial v / \partial y \end{bmatrix}$$

Fig. 24 shows the x components of the forces on a control volume.

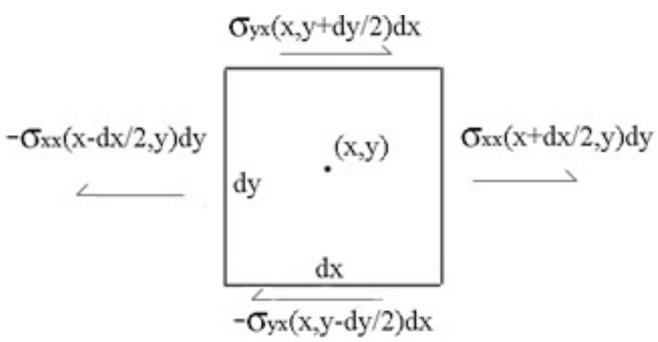


Figure 24 – x-forces on control volume

Thus the sum of the x components of the forces acting on the control volume divided by volume is given by

$$\frac{(\sigma_{xx}(x+dx/2,y)-\sigma_{xx}(x-dx/2,y))dy+(\sigma_{yx}(x,y+dy/2)-\sigma_{xyx}(x,y-dy/2))dx}{dxdy}$$

$$=\frac{\sigma_{xx}(x+dx/2,y)-\sigma_{xx}(x-dx/2,y)}{dx}+\frac{+\sigma_{yx}(x,y+dy/2)-\sigma_{yx}(x,y-dy/2)}{dy}$$

Taking the limit as dx - > 0 and dy - > 0,

$$=\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}$$

Thus the RHS of the x - momentum equation is

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial t_{xx}}{\partial x}+\frac{\partial t_{yx}}{\partial y}$$

$$\frac{\partial\tau_{xx}}{\partial x}=\frac{\partial(2\mu\frac{\partial u}{\partial x})}{\partial x}=2\mu\frac{\partial^2 u}{\partial x^2},\quad \frac{\partial\tau_{yx}}{\partial y}=\frac{\partial\mu(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})}{\partial y}=\mu(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y})$$

so

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}=-\frac{\partial p}{\partial x}+2\mu\frac{\partial^2 u}{\partial x^2}+\mu(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y})$$

$$=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial y^2}+\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y})$$

$$=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial y^2}+\frac{\mu\partial(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0)}{\partial x}$$

$$=-\frac{\partial p}{\partial x}+\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})$$

Similarly, the RHS of the y-momentum equation is

$$=-\frac{\partial p}{\partial y}+\mu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2})$$

9.E. Computing 2-D fluid motion

The FDM substitutions can be applied to the momentum equations to give computational equations for the u and v. However, two problems remain, there is no computational equation for p, and, the computational equations for u and v do not enforce the continuity equation.

An ad hoc method that solves both these problems is to compute u and v at time t^n, and then solve a Poisson equation for p at time t^n that enforces the continuity equation at time t^{n+1}.[11]

9.E.1. A Poisson Equation for p

Start with the Euler estimates for ∂u/∂t and ∂v/∂t at time t^{n+1}

$$u^{n+1} = u^n - dt \cdot (u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}))$$

$$v^{n+1} = v^n - dt \cdot (u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}))$$

Differentiate the 1st equation w.r.t. x and the 2nd w.r.t. y and rearrange:

$$\frac{\partial u^{n+1}}{\partial x} = \frac{\partial u^n}{\partial x} - dt \cdot (\frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} + \frac{\partial(.......)}{\partial x})$$

$$\frac{\partial v^{n+1}}{\partial y} = \frac{\partial v^n}{\partial y} - dt \cdot (\frac{1}{\rho}\frac{\partial^2 p}{\partial y^2} + \frac{\partial(......)}{\partial y})$$

Add the two equations

$$\frac{\partial u^{n+1}}{\partial x} + \frac{\partial v^{n+1}}{\partial y} = \frac{\partial u^n}{\partial x} + \frac{\partial v^n}{\partial y} - dt \cdot (\frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} + \frac{1}{\rho}\frac{\partial^2 p}{\partial y^2} + \frac{\partial(......)}{\partial x} + \frac{\partial(......)}{\partial y})$$

Set the LHS of the equation to 0 to enforce the continuity equation at t_{n+1}, giving

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{\rho}{dt}(\frac{\partial u^n}{\partial x} + \frac{\partial v^n}{\partial y}) - \rho(\frac{\partial(......)}{\partial x} + \frac{\partial(......)}{\partial y})$$

Since ρ/dt is orders of magnitude greater than ρ the last term can be dropped, giving a Poisson equation for p

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \cong \frac{\rho}{dt}(\frac{\partial u^n}{\partial x} + \frac{\partial v^n}{\partial y})$$

9.E.2. Making the FDM substitutions into the computational equations and rearranging to solve for u and v yields:

$$u^{n+1}(i,j) = u^{n+1}(i,j) + dt \cdot [u^n(i,j)\frac{u^n(i,j) - u^n(i-1,j)}{dx} + v^n(i,j)\frac{u^n(i,j) - u^n(i,j-1)}{dy}$$
$$= \frac{-1}{\rho}\frac{p^n(i+1,j) - p^n(i-1,j)}{2dx} + \frac{\mu}{\rho}(\frac{u^n(i+1,j) - 2u^n(i,j) + u^n(i,j+1)}{dx^2}$$
$$+ \frac{u^n(i+1,j) - 2u^n(i,j) + u^n(i,j-1)}{dy^2})]$$

and a similar equation for v.

Making the FDM substitutions into the Poisson equation and rearranging to solve for the steady-state p(i,j) yields a set of simultaneous linear equations:

$$p(i,j) = (p(i+1,j) - p(i-1,j) + p(i,j+1) - p(i,j-1) - dx^2 \cdot b(i,j))/4$$

where

$$b(i,j) = \frac{\rho}{dt}\left(\frac{u(i+1,j) - u(i-1,j)}{2dx} + \frac{v(i,j+1) - v(i,j-1)}{2dy}\right)$$

Solutions can be calculated using the Jacobi method, see Appendix I.

Figure 25 shows a small pipe opening into a larger pipe. The boundary conditions are shown, with the fluid flowing into the pipe from the left with velocity u = 3.

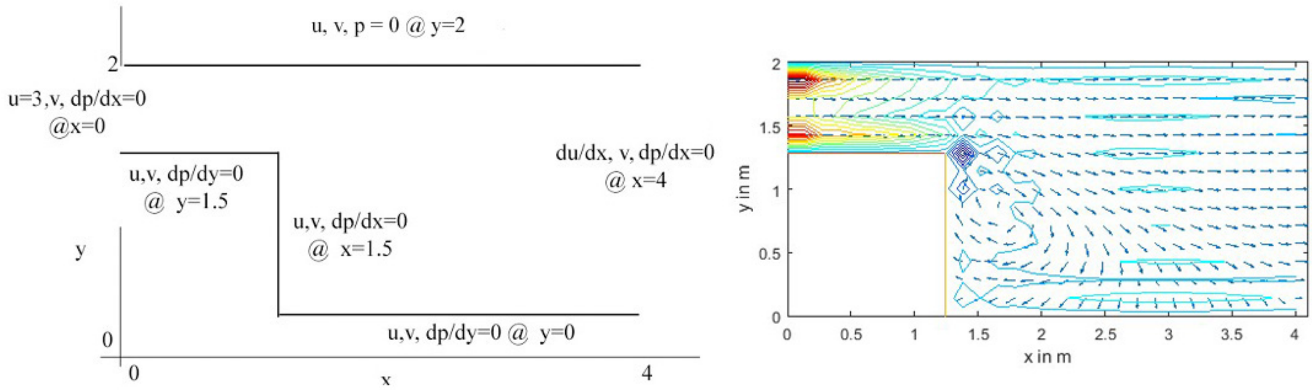


Figure 25. Fluid flow over a backward step, with contour lines for pressure. Contours and 'quivers' are built-in MATLAB graphing features.

10. Electrodynamics

10.A. The physics of electromagnetic waves

The physics of electromagnetic waves is given by Maxwell's 3rd and 4th equations in integral form. Both follow from observations made of simple experiments.

Using an experimental setup diagrammed in Fig. 26, Faraday observed that a changing current in the coil on the left produced a changing magnetic field in the core which extended through the coil on the right and produced a voltage in that coil.

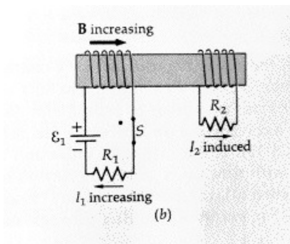


Fig. 26 – Faraday's experiment

Faraday's law of induction states that the induced voltage in a loop equals the rate of change of the magnetic flux through the loop. Expressed as a line integral Maxwell's third equation in integral form is:

$\oint_C \vec{E} \cdot d\vec{s} = \dfrac{d\Phi_B}{dt}$ Where E is the induced electric field in volts/meter and Φ_B is the magnetic flux through the loop in Webers.

Ampere observed that a current in a wire produces a magnetic field around the wire, with magnitude

$$B = \frac{\mu_0 I}{2\pi R}$$

where B is the magnetic flux field strength in Webers·m^{-2}, μ_0 is the magnetic permeability of free space, and I is the current in the wire, see Fig. 27.

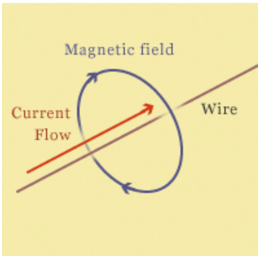


Figure 27. The magnetic field generated by current in a wire

B is constant on the path C consisting of the circle of radius R, so

$$\oint_C \vec{B} \cdot ds = \frac{\mu_0 I}{2\pi R} 2\pi R = \mu_0 I$$

It is easy to show that the integral is path-independent. Maxwell added a term to account for the field generated by changing electric flux Φ_E, created by a capacitor, for example, giving Maxwell's fourth equation in integral form.

$$\oint_C \vec{B} \cdot ds = \mu_0 I + \mu_0 \frac{d\Phi_E}{dt}$$

10.B. The differential equation model for electromagnetic waves

Maxwell's 3rd and 4th equations in differential form are the model for electromagnetic radiation.

To derive Faraday's law in differential form consider an infinitesimal loop in the x,y plane, Fig. 28, along with an increasing magnetic field $B_z(x,y,t)$ directed out of the page. Denote the induced electric field by $\boldsymbol{E}(x,y,t) = (E_x(x,y,t), E_y(x,y,t))$.

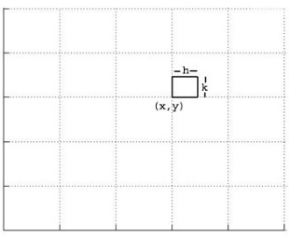


Figure 28. An infinitesimal loop

Φ_Z approximately equals $B_Z \cdot h \cdot k$ in the loop, so, from Faraday's Law in integral form:

$$\oint_C \vec{E} \cdot d\vec{s} = [E_x(x, y+k) - E_x(x,y)] \cdot h + [E_y(x,y) - E_y(x+h, y)] \cdot k = \frac{dB_z(x,y,t)hk}{dt}$$

Divide both sides of the equation by h·k,

$$\frac{[E_x(x, y+k) - E_x(x,y)]}{k} + \frac{[E_y(x,y) - E_y(x+h, y)]}{h} = \frac{dB_z(x,y,t)}{dt}$$

then with h -> 0 and k -> 0 we have:

$$\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = \frac{dB_z}{dt}$$

Similarly

$$\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = \frac{dB_z}{dt}$$

$$\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = \frac{dB_z}{dt}$$

With an E_Z field coming through the loop and out of the page, Φ_E approximately equals $e_0 \cdot E_Z \cdot h \cdot k$, and from Ampere's Circuital Law Circuital Law in integral form, with I = 0:
Dividing by h·k gives

$$\oint_C B \cdot ds = B_x(x,y) \cdot h + B_y(x+h, y) \cdot h - B_x(x, y+k) \cdot k - B_y(x,y) \cdot k \approx \mu_0 e_0 \frac{dE_z}{dt}$$

then with h -> 0 and k -> 0 we have

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 e_0 \frac{dE_z}{dt}$$

Similarly,

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 \varepsilon_0 \frac{dE_x}{dt}$$

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{dE_y}{dt}$$

10.C. Computing 2-D EM waves

We start with an **E** field that is in the z direction and varies with x and y, thus $E_Z(x,y)$ represents our **E** field, and $E_x(x,y) = E_y(x,y) = 0$. From the first set of equations, we have (with I = 0 and H=B/μ):

$$-\mu\frac{\partial H_x}{\partial t}=\frac{dE_z}{dy}$$

$$\mu\frac{\partial H_y}{\partial t}=\frac{dE_z}{dx}$$

and from the 2nd set

$$\varepsilon\frac{\partial E_z}{\partial t}=\frac{dH_y}{dx}-\frac{dH_x}{dy}$$

A perfect electric conductor has zero resistance and infinite conductivity. An electric field cannot exist in a perfect electric conductor. A perfect electric conductor on a boundary is modeled by a Dirichlet boundary $E_z(t,x,y) = 0$. The magnetic field is reflected and the corresponding boundary condition is $\partial H_y(t,x,y) / \partial x = 0$.

Making the FDM substitutions yields the computational equations:

$$H_x^{n+1}(i,j)=H_x^{n-1}(i.j)+\frac{2\cdot dt}{\mu}\cdot\frac{E_z^n(i,j+1)-E_z^n(i,j-1)}{2\cdot dy}$$

$$H_y^{n+1}(i,j)=H_y^{n-1}(i.j)+\frac{2\cdot dt}{\mu}\cdot\frac{E_z^n(i+1,j)-E_z^n(i-1.j)}{2\cdot dx}$$

$$E_z^{n+1}(i,j)=E_z^{n-1}(i,j)+\frac{2\cdot dt}{\varepsilon}\cdot\left(\frac{H_y^n(i+1,j)-H_y^n(i-1,j)}{2\cdot dx}-\frac{H_x^n(i,j+1)-H_x^n(i,j-1)}{2\cdot dy}\right)$$

Fig. 29 shows a 2-D wave before and after hitting a boundary with an opening.

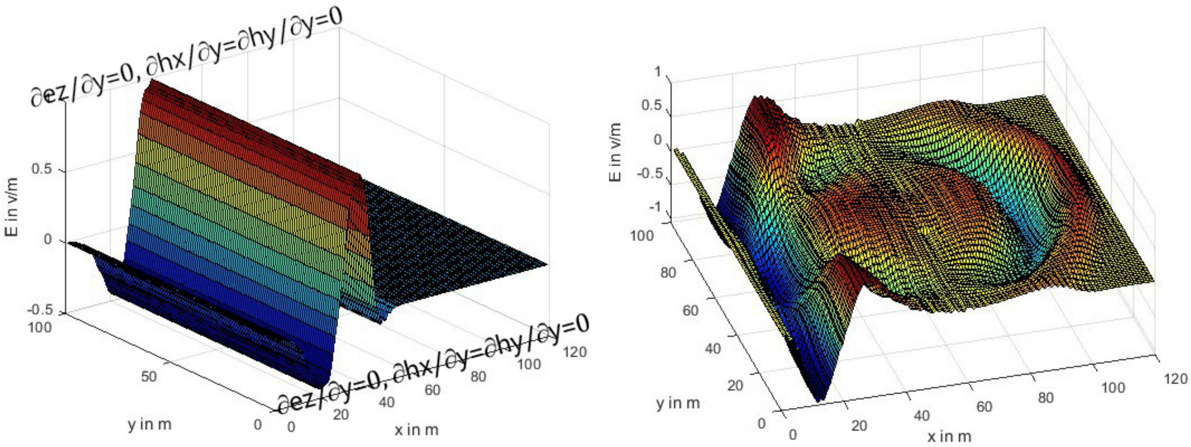


Fig. 29. EM wave, MATLAB draws the graphs from the E-field array

10.D. The Yee algorithm

The state-of-the-art computational method for electromagnetic waves is the Yee algorithm [12]. The Zee algorithm is a modified form of the FDM method. Fig. 30 shows the 1-D Yee (and FDM) algorithm dependencies for calculating E_z.[13]

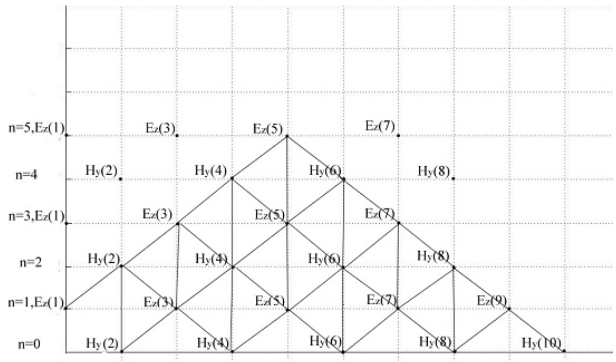


Figure 30 – FDM/Yee algorithm dependencies

The 2-D Yee algorithm is identical to the 2-D FDM, except that it only calculates $E_z^n(i,j)$ for n, i, and j all odd, $H_x^n(i,j)$ for n even, i odd and j even, $H_y{}^n(i,j)$ for n even, i even, and j odd, and is hence 2·2·2 = 8 times faster than the FDM while calculating fewer points with equal accuracy.

11. Conclusions

We covered an astounding amount of physics in this short paper. This was possible because:

- The laws of physics expressed as differential equations are simple, intuitively clear, and concise.
- The model derivations given for all but the stress and strain and fluid dynamics projects were just a few lines long. The fluid dynamics model derivation was longer and more complex, but required no difficult math or calculus beyond the basic definitions of differential and integral calculus, the product rule, the chain rule, and the Leibniz integral rule[14].
- Once the models were derived, the application of Euler's method or the finite difference method for computing solutions was routine.

Differential equations have extraordinary analytic and explanatory power, as we've seen in this short paper. But it takes computers and computational calculus to unlock this power. Introducing students early to modeling with differential equations, computational calculus, and computers, and focusing on the analysis of real physical systems that they make possible, will lead to the development of a new and radically different curriculum for classical physics and engineering education.

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