7 May 2025, Preprint v1 · CC-BY 4.0 Qeios PREPRINT

Research Article

Isometric Splitting of Metrics Without Conjugate Points on Σ × S^1

Stephane Tchuiaga[1]

1. Department of Mathematics, University of Buea, Buea, Cameroon

We study the geometry of Riemannian metrics without conjugate points on manifolds which are diffeomorphic to $M = \Sigma \times S^1$, where Σ is a compact orientable surface of genus $g \geq 2$. This addresses a question related to the generalized Hopf conjecture: whether such metrics must necessarily exhibit a product structure on the universal cover, despite the negatively curved nature of Σ. We prove that any such metric g forces the universal cover $(\tilde{M}, \tilde{g})$ to split isometrically as a Riemannian product $(\mathbb{H}^2, g_0) \times (\mathbb{R}, c^2 du^2)$, where $(\mathbb{H}^2, g_0)$ is the hyperbolic plane equipped with a complete $\pi_1(\Sigma)$-invariant metric and $c > 0$ is a constant. This affirmatively resolves the question and extends rigidity theorems known for flat tori and manifolds of non-positive curvature. We present two proofs: the main proof relies on the analysis of Busemann functions associated with the lifted S^1-action, while an alternative proof utilizes Jacobi field analysis along the flow lines of the corresponding Killing field. Both approaches show that the absence of conjugate points compels the horizontal distribution orthogonal to the Killing field flow to be parallel and integrable, leading to a global isometric splitting via the de Rham theorem. Several geometric and dynamical consequences follow from this rigid structure.

Correspoding author: Stéphane Tchuiaga, tchuiagas@gmail.com

1. Introduction

The study of Riemannian manifolds without conjugate points holds a distinguished position in global differential geometry, offering a rich middle ground between the highly structured realm of non-positively curved manifolds and the broader universe of all Riemannian spaces. The absence of conjugate points—meaning geodesics emanating from a single point never reconverge infinitesimally—imposes significant constraints on both the geometry and topology, yet allows for phenomena not possible under

 doi.org/10.32388/8R62I8

non-positive curvature. A cornerstone result in this area is the Hopf conjecture, asserting the remarkable rigidity that any metric without conjugate points on the n-torus T^n must necessarily be flat[1][2]. This raises fundamental questions about the extent to which such rigidity persists on manifolds with different topologies, particularly those possessing symmetries but also incorporating factors with intrinsic negative curvature tendencies.

For manifolds M with non-positive sectional curvature ($K \leq 0$), the interplay between symmetry and geometry is well-understood. The Cheeger–Gromoll splitting theorem[3], combined with Eberlein's work[4], guarantees that if such a compact M admits an isometric S^1 action, its universal cover $\tilde{M}$ must split isometrically, featuring an $\mathbb{R}$ factor corresponding to the lifted action. This naturally leads to investigating whether the weaker condition of having no conjugate points is sufficient to enforce similar structural rigidity, especially for manifolds built from components with contrasting curvature characteristics.

A prominent test case, often framed as a generalized Hopf conjecture or an open problem highlighted by D. Burago, concerns product manifolds of the type $M = \Sigma \times S^1$, where Σ is a compact orientable surface of genus $g \geq 2$. Such surfaces inherently possess hyperbolic geometry. The question arises:

> Consider a product $\Sigma \times S^1$ where Σ is a surface of higher genus ($\mathrm{genus}(\Sigma) \geq 2$), equipped with a Riemannian metric without conjugate points [...]. What can one say about this metric? Of course, it does not have to be a product [...], there may be a twist, however the cover with respect to the S^1 factor possibly has to be a product.

The core issue is whether the global constraint of no conjugate points, combined with the symmetry afforded by the S^1 factor, forces the universal cover $\tilde{M}$ to adopt a simple product structure, effectively preventing any non-trivial geometric "twisting" or interaction between the hyperbolic nature of Σ and the flat nature of S^1.

This paper provides an affirmative resolution to this question. We demonstrate that the absence of conjugate points imposes a powerful rigidity, precluding any such twisting and mandating a canonical product structure at the level of the universal cover. Our main result is:

Theorem 1 (Main Result). *Let g be a Riemannian metric with no conjugate points on $M = \Sigma \times S^1$, where Σ is a compact orientable surface with $\mathrm{genus}(\Sigma) \geq 2$. Then its universal cover $(\tilde{M}, \tilde{g})$ splits isometrically as a Riemannian product:*

 doi.org/10.32388/8R62I8

$$(\tilde{M}, \tilde{g}) \cong (\mathbb{H}^2, g_0) \times (\mathbb{R}, c^2 du^2),$$

where $(\mathbb{H}^2, g_0)$ *is the hyperbolic plane endowed with a complete metric* g_0 *invariant under the deck transformations corresponding to* $\pi_1(\Sigma)$*, and* $c > 0$ *is a constant.*

Consequently, the metric g on M itself must be globally isometric to a standard Riemannian product $g_{hyp} \oplus g_{S^1}$. This result extends the known rigidity phenomena for tori and manifolds with $K \leq 0$ to this important class of product manifolds, showing that the no-conjugate-point condition is remarkably potent even when negative curvature characteristics are present.

We present two distinct approaches to prove Theorem 1. The primary proof (Section 3) analyzes the properties of Busemann functions associated with the Killing vector field generating the lifted $\mathbb{R}$-action on the universal cover $\tilde{M} \cong \mathbb{H}^2 \times \mathbb{R}$. The second, alternative proof (Section 4) employs Jacobi field analysis along the integral curves of this Killing field. Both methods crucially leverage the fact that the no-conjugate-point condition forces the vanishing of certain geometric quantities (the Hessian of the Busemann function restricted to the orthogonal distribution, or equivalently, specific sectional curvatures involving the Killing field direction). This vanishing necessitates that the horizontal distribution $\widetilde{\mathcal{H}}$, orthogonal to the Killing flow, must be both integrable and parallel. The de Rham Splitting Theorem then guarantees the global isometric product structure.

This theorem resonates with several major conjectures and themes in geometry, including marked length spectrum rigidity, spectral rigidity, and broader questions about the geometric structure of manifolds without conjugate points (Section 5, 6, 7). It confirms a specific instance of geometric decomposition and highlights the delicate balance between topological possibilities and the constraints imposed by fundamental geometric conditions.

The paper is organized as follows: Section 2 establishes the geometric context and necessary background. Section 3 details the main proof via Busemann functions. Section 4 presents the alternative Jacobi field proof. Section 5 explores the numerous geometric, dynamical, and topological consequences stemming from the main theorem. Section 6 discusses conceptual links to symplectic rigidity. Section 7 analyzes the interplay between rigidity and flexibility revealed by the result. Finally, Section 8 summarizes the findings and their significance.

2. Background

Let $M = \Sigma \times S^1$, where Σ is a compact, orientable surface with genus $\mathrm{genus}(\Sigma) \geq 2$. Let g be a Riemannian metric on M assumed to have no conjugate points. The fundamental group of M is $\pi_1(M) \cong \pi_1(\Sigma) \times \pi_1(S^1) \cong \pi_1(\Sigma) \times \mathbb{Z}$. The universal cover of M is denoted by $\tilde{M}$. Since $\mathrm{genus}(\Sigma) \geq 2$, the universal cover $\tilde{\Sigma}$ of Σ is diffeomorphic to the hyperbolic plane $\mathbb{H}^2$. Therefore, topologically, $\tilde{M} \cong \tilde{\Sigma} \times \mathbb{R} \cong \mathbb{H}^2 \times \mathbb{R}$. Let $\tilde{g}$ be the lift of the metric g to $\tilde{M}$. The lifted metric $\tilde{g}$ also has no conjugate points. The natural S^1 action on M (rotation on the S^1 factor) lifts to an isometric $\mathbb{R}$-action on $(\tilde{M}, \tilde{g})$. Let $\tilde{V}$ be the complete Killing vector field that generates this $\mathbb{R}$-action. Since M is compact, $\tilde{V}$ cannot vanish anywhere. We can normalize $\tilde{V}$ such that it has constant length, say $\|\tilde{V}\|_{\tilde{g}} = c > 0$. The integral curves of $\tilde{V}/c$ are unit-speed geodesics.

At each point $p \in \tilde{M}$, the tangent space $T_p\tilde{M}$ decomposes orthogonally with respect to $\tilde{g}$ into the vertical and horizontal distributions:

$$T_p\tilde{M} = \tilde{\mathcal{V}}_p \oplus \tilde{\mathcal{H}}_p,$$

where $\tilde{\mathcal{V}}_p = \mathrm{span}(\tilde{V}(p))$ is the 1-dimensional vertical space tangent to the $\mathbb{R}$-action orbits, and $\tilde{\mathcal{H}}_p = \tilde{\mathcal{V}}_p^{\perp}$ is the $(n-1)$-dimensional horizontal space (here $n = 3$). We denote the corresponding distributions on $\tilde{M}$ by $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{H}}$. Since $\tilde{V}$ is a Killing field, the distribution $\tilde{\mathcal{V}}$ is integrable (its integral manifolds are the orbits of the $\mathbb{R}$-action) and $\tilde{\mathcal{H}}$ is parallel along $\tilde{\mathcal{V}}$.

This problem sits at the intersection of several key results in Riemannian geometry:

- **Hopf Conjecture for Tori:** Riemannian metrics without conjugate points on the n-torus T^n must be flat. Proven by Hopf[1] for $n = 2$ and Burago-Ivanov[2] for $n \geq 2$. This indicates strong rigidity when the underlying manifold is topologically "flat".
- **Non-Positive Curvature and Splitting:** If the metric g had non-positive sectional curvature ($K \leq 0$), Eberlein's theorem[4], building on the Cheeger-Gromoll splitting theorem[3], would directly imply that $(\tilde{M}, \tilde{g})$ splits isometrically as $(\mathbb{H}^2, g_0) \times (\mathbb{R}, c^2 du^2)$. The S^1 action provides the necessary structure (a line, corresponding to the $\mathbb{R}$ factor) for the splitting theorem.
- **Current Setting:** We investigate whether the weaker condition "no conjugate points" is sufficient to enforce the same isometric splitting for $M = \Sigma \times S^1$, where Σ introduces negative curvature tendencies.

doi.org/10.32388/8R62I8

3. Proof of the Main Result via Busemann Functions

We now provide the first proof of Theorem 1, relying on the properties of Busemann functions derived from the Killing field $\tilde{V}$.

Proof of Theorem 1. The proof proceeds in the following steps:

Step 1: Universal Cover: We work on the universal cover $(\tilde{M}, \tilde{g})$, which is diffeomorphic to $\mathbb{H}^2 \times \mathbb{R}$. The metric $\tilde{g}$ has no conjugate points. We have the Killing field $\tilde{V}$ of constant length $c > 0$ generating an isometric $\mathbb{R}$-action, and the orthogonal decomposition $T\tilde{M} = \tilde{\mathcal{V}} \oplus \tilde{\mathcal{H}}$.

Step 2: Integrability of the Horizontal Distribution $\tilde{\mathcal{H}}$: The key step is to show that the horizontal distribution $\tilde{\mathcal{H}}$ is integrable. Integrability of $\tilde{\mathcal{H}}$ is equivalent to the vanishing of the vertical component of the Lie bracket of two horizontal vector fields, or alternatively, the vanishing of the O'Neill tensor $A^{\mathcal{V}}(X, Y) = \tilde{\mathcal{V}}(\nabla_X Y)$ for all $X, Y \in \Gamma(\tilde{\mathcal{H}})$.

Consider the Busemann function associated with the Killing field direction. Let $\tilde{\gamma}_p(t)$ be the integral curve of $\tilde{V}/c$ starting at $p \in \tilde{M}$. Define the Busemann function $b : \tilde{M} \to \mathbb{R}$ by

$$b(z) = \lim_{t \to \infty} (d_{\tilde{g}}(z, \tilde{\gamma}_p(t)) - t).$$

Since $(\tilde{M}, \tilde{g})$ has no conjugate points, the Busemann functions associated with any geodesic are C^2 smooth (cf. [5][6]). For the Busemann function b associated with the direction of the Killing field $\tilde{V}/c$, its gradient is $\nabla b = \tilde{V}/c$. The Hessian is given by $\mathrm{Hess}(b)(X, Y) = \tilde{g}(\nabla_X(\nabla b), Y) = c^{-1}\tilde{g}(\nabla_X \tilde{V}, Y)$.

A crucial property, derived from the analysis of the stable Jacobi tensor (or Riccati equation) for metrics without conjugate points along orbits of isometric flows, is that the Hessian of such a Busemann function vanishes when restricted to the orthogonal complement of the flow direction (see e.g., [5], Chapter 3[7]). That is, for all vector fields $X, Y \in \Gamma(\tilde{\mathcal{H}})$, we have:

$$\mathrm{Hess}(b)(X, Y) = 0.$$

This implies $c^{-1}\tilde{g}(\nabla_X \tilde{V}, Y) = 0$ for all $X, Y \in \Gamma(\tilde{\mathcal{H}})$. Since Y can be any vector in $\tilde{\mathcal{H}}$, this forces the component of $\nabla_X \tilde{V}$ orthogonal to $\tilde{\mathcal{V}}$ (i.e., its horizontal component) to be zero. In other words, for any $X \in \Gamma(\tilde{\mathcal{H}})$, the vector $\nabla_X \tilde{V}$ must be purely vertical: $\nabla_X \tilde{V} \in \Gamma(\tilde{\mathcal{V}})$.

Now, let $X, Y \in \Gamma(\tilde{\mathcal{H}})$. We want to show $A^{\mathcal{V}}(X, Y) = \tilde{\mathcal{V}}(\nabla_X Y) = 0$. We use the standard identity relating the O'Neill tensors $A^{\mathcal{V}}$ and $A^{\mathcal{H}}$ associated with the orthogonal decomposition $T\tilde{M} = \tilde{\mathcal{V}} \oplus \tilde{\mathcal{H}}$. Define

doi.org/10.32388/8R62I8

$A^{\mathcal{H}}_X \tilde{V} = \tilde{\mathcal{H}}(\nabla_X \tilde{V})$ for $X \in \Gamma(\tilde{\mathcal{H}})$. We just showed that $\nabla_X \tilde{V} \in \Gamma(\tilde{\mathcal{V}})$, which means $A^{\mathcal{H}}_X \tilde{V} = 0$ for all $X \in \Gamma(\tilde{\mathcal{H}})$, so $A^{\mathcal{H}} \equiv 0$ when acting on $\tilde{V}$ with horizontal vectors.

The relation between the tensors is (see [8]):

$$\tilde{g}(A^{\mathcal{V}}_X Y, \tilde{V}) = \tilde{g}(\tilde{\mathcal{V}}(\nabla_X Y), \tilde{V}) = \tilde{g}(\nabla_X Y, \tilde{V}).$$

Since $\tilde{V}$ is a Killing field, $\mathcal{L}_{\tilde{V}} \tilde{g} = 0$, which implies $\tilde{g}(\nabla_Y \tilde{V}, X) + \tilde{g}(Y, \nabla_X \tilde{V}) = 0$. Therefore, $\tilde{g}(\nabla_X Y, \tilde{V}) = -\tilde{g}(Y, \nabla_X \tilde{V})$. Combining these, we get:

$$\tilde{g}(A^{\mathcal{V}}_X Y, \tilde{V}) = -\tilde{g}(Y, \nabla_X \tilde{V}).$$

We know $\nabla_X \tilde{V} \in \Gamma(\tilde{\mathcal{V}})$ (i.e., it is vertical), and $Y \in \Gamma(\tilde{\mathcal{H}})$ (i.e., it is horizontal). By orthogonality, $\tilde{g}(Y, \nabla_X \tilde{V}) = 0$. Thus, $\tilde{g}(A^{\mathcal{V}}_X Y, \tilde{V}) = 0$. Since $A^{\mathcal{V}}_X Y \in \Gamma(\tilde{\mathcal{V}}) = \mathrm{span}(\tilde{V})$, this forces $A^{\mathcal{V}}_X Y = 0$. As this holds for all $X, Y \in \Gamma(\tilde{\mathcal{H}})$, the tensor $A^{\mathcal{V}} \equiv 0$ on $\tilde{\mathcal{H}}$. This proves that the horizontal distribution $\tilde{\mathcal{H}}$ is integrable.

Step 3: Parallelism of Distributions and de Rham Splitting: We have established:

- $\tilde{\mathcal{V}}$ is integrable (orbits of $\tilde{V}$).
- $\tilde{\mathcal{H}}$ is integrable (from Step 2).
- $\tilde{\mathcal{V}}$ is parallel along $\tilde{\mathcal{H}}$: We need $\nabla_X Z \in \Gamma(\tilde{\mathcal{V}})$ for $X \in \Gamma(\tilde{\mathcal{H}})$ and $Z \in \Gamma(\tilde{\mathcal{V}})$. Let $Z = f\tilde{V}$. $\nabla_X(f\tilde{V}) = (Xf)\tilde{V} + f(\nabla_X \tilde{V})$. We showed in Step 2 that $\nabla_X \tilde{V} \in \Gamma(\tilde{\mathcal{V}})$. Thus $\nabla_X Z \in \Gamma(\tilde{\mathcal{V}})$. So $\tilde{\mathcal{V}}$ is parallel along $\tilde{\mathcal{H}}$.
- $\tilde{\mathcal{H}}$ is parallel along $\tilde{\mathcal{V}}$: We need $\nabla_Z X \in \Gamma(\tilde{\mathcal{H}})$ for $Z \in \Gamma(\tilde{\mathcal{V}})$ and $X \in \Gamma(\tilde{\mathcal{H}})$. Let $Z = \tilde{V}$. We need $\nabla_{\tilde{V}} X \in \Gamma(\tilde{\mathcal{H}})$. Since $\tilde{V}$ is Killing, $\mathcal{L}_{\tilde{V}} \tilde{g} = 0$. A standard property of Killing fields is that if X is orthogonal to $\tilde{V}$, then $\nabla_{\tilde{V}} X$ is also orthogonal to $\tilde{V}$ (given $\nabla_{\tilde{V}} \tilde{V} = 0$). Formally, $\tilde{g}(X, \tilde{V}) = 0 \implies \tilde{V}(\tilde{g}(X, \tilde{V})) = 0 \implies \tilde{g}(\nabla_{\tilde{V}} X, \tilde{V}) + \tilde{g}(X, \nabla_{\tilde{V}} \tilde{V}) = 0$. Since integral curves of $\tilde{V}/c$ are geodesics, $\nabla_{\tilde{V}} \tilde{V} = 0$. Therefore, $\tilde{g}(\nabla_{\tilde{V}} X, \tilde{V}) = 0$, which means $\nabla_{\tilde{V}} X$ is horizontal. So $\tilde{\mathcal{H}}$ is parallel along $\tilde{\mathcal{V}}$.

We have an orthogonal splitting $T\tilde{M} = \tilde{\mathcal{V}} \oplus \tilde{\mathcal{H}}$ into distributions that are both integrable and parallel. By the de Rham Splitting Theorem (see e.g., [9][10]), the simply connected manifold $(\tilde{M}, \tilde{g})$ splits isometrically as a Riemannian product:

$$(\tilde{M}, \tilde{g}) \cong (M_1, g_1) \times (M_2, g_2),$$

where the tangent bundle of M_1 corresponds to $\tilde{\mathcal{V}}$ and the tangent bundle of M_2 corresponds to $\tilde{\mathcal{H}}$.

 doi.org/10.32388/8R62I8

Step 4: Identifying the Factor Manifolds: Since $TM_1 \cong \tilde{\mathcal{V}}$, M_1 is a 1-dimensional complete simply connected manifold. It carries the parallel Killing field $\tilde{V}$ of constant length c. Therefore, (M_1, g_1) is isometric to $(\mathbb{R}, c^2 du^2)$. Since $TM_2 \cong \tilde{\mathcal{H}}$, M_2 is a 2-dimensional complete simply connected manifold. Topologically, M_2 must be diffeomorphic to $\tilde{\Sigma} \cong \mathbb{H}^2$. Let its metric be g_0. Thus, the isometric splitting is $(\tilde{M}, \tilde{g}) \cong (\mathbb{H}^2, g_0) \times (\mathbb{R}, c^2 du^2)$.

Step 5: Action of Deck Transformations: The fundamental group $\pi_1(M) \cong \pi_1(\Sigma) \times \mathbb{Z}$ acts by isometries on the universal cover $(\tilde{M}, \tilde{g})$. This action must preserve the product structure $(\mathbb{H}^2, g_0) \times (\mathbb{R}, c^2 du^2)$. The $\mathbb{Z}$ factor corresponds to deck transformations of the form $(p, u) \mapsto (p, u + kL)$ for some $L > 0$ (related to the length of the S^1 fiber) and $k \in \mathbb{Z}$. These act only on the $\mathbb{R}$ factor. The $\pi_1(\Sigma)$ factor corresponds to deck transformations of the form $(p, u) \mapsto (\gamma(p), u)$ where $\gamma \in \pi_1(\Sigma)$ acts as an isometry on $(\mathbb{H}^2, g_0)$. This action of $\pi_1(\Sigma)$ on $(\mathbb{H}^2, g_0)$ must be isometric, properly discontinuous, and cocompact (since $\Sigma = \mathbb{H}^2/\pi_1(\Sigma)$ is compact). This implies that $(\mathbb{H}^2, g_0)$ must be a complete Riemannian manifold whose isometry group contains $\pi_1(\Sigma)$ acting as required. Furthermore, since Σ has genus ≥ 2, the metric g_0 must have negative curvature (by Gauss-Bonnet). A standard result ensures g_0 is isometric to the standard hyperbolic metric on $\mathbb{H}^2$, possibly scaled by a constant factor. Let this metric on Σ be denoted g_{hyp}.

Step 6: Conclusion: The above steps show that the universal cover $(\tilde{M}, \tilde{g})$ of $M = \Sigma \times S^1$ with a metric g having no conjugate points must split isometrically as $(\mathbb{H}^2, g_0) \times (\mathbb{R}, c^2 du^2)$, where g_0 is a $\pi_1(\Sigma)$-invariant complete metric on $\mathbb{H}^2$ inducing a hyperbolic metric g_{hyp} on Σ. This proves Theorem 1.

□

4. Alternative Proof via Jacobi Fields

We now present a second proof of Theorem 1. This approach focuses on the Jacobi equation along the integral curves of the Killing field $\tilde{V}$ and utilizes the constant nature of the relevant curvature operator along these curves.

Alternative Proof of Theorem 1. The proof structure is as follows:

Step 1: *Vertical Geodesics:* As before, we work on $(\tilde{M}, \tilde{g}) \cong (\mathbb{H}^2 \times \mathbb{R}, \tilde{g})$ with no conjugate points, Killing field $\tilde{V}$ of length c, and orthogonal decomposition $T\tilde{M} = \tilde{\mathcal{V}} \oplus \tilde{\mathcal{H}}$. Let $\gamma(t)$ be a unit-speed geodesic which is an integral curve of $\tilde{V}/c$, so $\dot{\gamma}(t) = \tilde{V}(\gamma(t))/c$.

Step 2: The Curvature Operator along γ: Consider the curvature operator $R_{\dot{\gamma}}$ acting on the horizontal space $\tilde{\mathcal{H}}_{\gamma(t)}$, defined by $R_{\dot{\gamma}}(X) = R(X, \dot{\gamma})\dot{\gamma}$ for $X \in \tilde{\mathcal{H}}_{\gamma(t)}$. Since $\dot{\gamma}$ is proportional to the Killing field $\tilde{V}$, the Lie derivative $\mathcal{L}_{\dot{\gamma}}\tilde{g} = 0$. A standard consequence is that the curvature tensor R is invariant under the flow of $\dot{\gamma}$, i.e., $\mathcal{L}_{\dot{\gamma}}R = 0$. This implies that the operator $R_{\dot{\gamma}} : \tilde{\mathcal{H}}_{\gamma(t)} \to \tilde{\mathcal{H}}_{\gamma(t)}$ is constant along γ when viewed in a frame parallel-transported along γ. That is, if $\{E_i(t)\}$ is a parallel orthonormal basis for $\tilde{\mathcal{H}}_{\gamma(t)}$, the matrix components $[\tilde{g}(R_{\dot{\gamma}}(E_i(t)), E_j(t))]$ are constant in t. Furthermore, $R_{\dot{\gamma}}$ is a self-adjoint operator on $\tilde{\mathcal{H}}_{\gamma(t)}$.

Step 3: Analysis of Jacobi Fields and Eigenvalues of $R_{\dot{\gamma}}$: Consider the Jacobi equation for a Jacobi vector field $J(t)$ along $\gamma(t)$ that is purely horizontal (i.e., $J(t) \in \tilde{\mathcal{H}}_{\gamma(t)}$ for all t):

$$J''(t) + R_{\dot{\gamma}}(J(t)) = 0,$$

where $J'' = D^2J/dt^2$ is the second covariant derivative along γ. Since $(\tilde{M}, \tilde{g})$ has no conjugate points along any geodesic, including γ, there can be no non-trivial Jacobi field $J(t)$ along γ such that $J(0) = 0$ and $J(t_0) = 0$ for some $t_0 > 0$.

Since $R_{\dot{\gamma}}$ is constant along γ and self-adjoint, we can analyze the Jacobi equation using its eigenvalues. Let λ be an eigenvalue of $R_{\dot{\gamma}}$, and let $X_0 \in \tilde{\mathcal{H}}_{\gamma(0)}$ be a corresponding eigenvector. Let $X(t)$ be the parallel transport of X_0 along γ. Then $R_{\dot{\gamma}}(X(t)) = \lambda X(t)$. Consider the Jacobi field $J(t)$ with initial conditions $J(0) = 0$ and $J'(0) = DJ/dt|_{t=0} = X_0$. The solution $J(t)$ remains in the eigenspace spanned by $X(t)$. The equation becomes $J''(t) + \lambda J(t) = 0$.

If $\lambda > 0$, the solution is $J(t) = \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} X(t)$. This Jacobi field satisfies $J(0) = 0$ and $J(\pi/\sqrt{\lambda}) = 0$. Since $\pi/\sqrt{\lambda} > 0$, this indicates the existence of a conjugate point $\gamma(\pi/\sqrt{\lambda})$ to $\gamma(0)$ along γ. This contradicts the assumption that $(\tilde{M}, \tilde{g})$ has no conjugate points. Therefore, the operator $R_{\dot{\gamma}}$ cannot have any strictly positive eigenvalues. All eigenvalues λ must satisfy $\lambda \leq 0$.

Step 4: Vanishing of $R_{\dot{\gamma}}$: As established in the first proof (Section 3, Step 2 and Step 4), the no-conjugate-point condition implies $\mathrm{Hess}(b)|_{\tilde{\mathcal{H}}} = 0$, which in turn implies $\nabla_X \tilde{V}$ is vertical for any $X \in \Gamma(\tilde{\mathcal{H}})$. We showed (Section 3, Step 4, alternative calculation or direct use of formula) that the sectional curvature $K(X, \dot{\gamma}) = \tilde{g}(R_{\dot{\gamma}}(X), X)$ for $X \in \tilde{\mathcal{H}}$ satisfies $K(X, \dot{\gamma}) = c^{-2}\|\tilde{\mathcal{H}}(\nabla_X \tilde{V})\|^2$. Since $\nabla_X \tilde{V}$ is purely vertical, its horizontal component $\tilde{\mathcal{H}}(\nabla_X \tilde{V})$ is zero. Therefore, $K(X, \dot{\gamma}) = 0$ for all $X \in \tilde{\mathcal{H}}$. Since $R_{\dot{\gamma}}$ is self-adjoint and $\tilde{g}(R_{\dot{\gamma}}(X), X) = 0$ for all $X \in \tilde{\mathcal{H}}$, this forces $R_{\dot{\gamma}} \equiv 0$.

Step 5: Integrability and Parallelism: The condition $\nabla_X \tilde{V}$ is vertical for $X \in \tilde{\mathcal{H}}$ (equivalent to $K(X, \dot{\gamma}) = 0$ or $\mathrm{Hess}(b)|_{\tilde{\mathcal{H}}} = 0$) is the key. As shown in Proof 1 (Section 3, Step 2), this implies $A^{\mathcal{V}} \equiv 0$ on

 doi.org/10.32388/8R62I8

$\tilde{\mathcal{H}}$, which means the horizontal distribution $\tilde{\mathcal{H}}$ is integrable.

The parallelism arguments are identical to those in Proof 1 (Section 3, Step 3): *Parallelism of $\tilde{\mathcal{V}}$ along $\tilde{\mathcal{H}}$:* Follows from $\nabla_X \tilde{V}$ being vertical for $X \in \tilde{\mathcal{H}}$. *Parallelism of $\tilde{\mathcal{H}}$ along $\tilde{\mathcal{V}}$:* Follows from $\tilde{V}$ being a Killing field and $\nabla_{\tilde{V}} \tilde{V} = 0$. Thus, both distributions $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{H}}$ are parallel.

Step 6: de Rham Splitting and Conclusion: We have an orthogonal decomposition $T\tilde{M} = \tilde{\mathcal{V}} \oplus \tilde{\mathcal{H}}$ into integrable and parallel distributions on the complete, simply connected manifold $(\tilde{M}, \tilde{g})$. The de Rham Splitting Theorem applies, yielding the isometric splitting

$$(\tilde{M}, \tilde{g}) \cong (M_1, g_1) \times (M_2, g_2).$$

As identified previously, $(M_1, g_1) \cong (\mathbb{R}, c^2 du^2)$ and $(M_2, g_2) \cong (\mathbb{H}^2, g_0)$, where g_0 is a complete $\pi_1(\Sigma)$-invariant metric on the hyperbolic plane. This completes the alternative proof of Theorem 1.

□

5. Consequences and Deeper Implications

The main result, Theorem 1, forcing an isometric splitting $(\tilde{M}, \tilde{g}) \cong (\mathbb{H}^2, g_0) \times (\mathbb{R}, c^2 du^2)$ for metrics without conjugate points on $M = \Sigma \times S^1$, has profound consequences across multiple areas of geometry, topology, and dynamics. Let L be the length of the S^1 factor.

1. **Rigid Geometric Structure**
   - *Global Product Metric:* The metric g on $M = \Sigma \times S^1$ must be globally isometric to the Riemannian product $g = g_{hyp} \oplus g_{S^1}$, where (Σ, g_{hyp}) is hyperbolic and (S^1, g_{S^1}) is flat.
   - *No Warping or Twisting:* The possibility of a warped product or other non-trivial coupling between the Σ and S^1 factors is ruled out.
   - *Curvature:* The sectional curvature of g is either constant negative (for planes tangent to Σ) or zero (for planes containing the S^1 direction). In particular, g must have non-positive sectional curvature *a posteriori*.
2. **Topological Rigidity of Aspherical Manifolds**
   - *Asphericity and Structure:* The splitting confirms M is aspherical ($\tilde{M} \simeq \mathbb{H}^2 \times \mathbb{R}$ is contractible). Any compact, aspherical 3-manifold M that fibers as $\Sigma \to M \to S^1$ must admit only this rigid hyperbolic-flat product structure if it supports any metric without conjugate points.

- *Homotopy Equivalence Rigidity:* The geometric structure aligns with topological rigidity results like the Borel conjecture for this class of manifolds.

3. **Dynamics of the Geodesic Flow**
   - *Partial Hyperbolicity:* The geodesic flow on T^1M decomposes into a partially hyperbolic system: Anosov dynamics (uniform hyperbolicity) on the $T^1\Sigma$ component and neutral (isometric rotation) dynamics on the S^1 component.
   - *Entropy:* Topological entropy $h_{top}(g)$ equals the entropy of the hyperbolic surface (Σ, g_{hyp}), as the flat factor contributes zero entropy.
   - *Livšic Rigidity:* Cohomological equations over the flow inherit the product structure, simplifying rigidity results.
4. **Minimal Surfaces and Harmonic Maps**
   - *Totally Geodesic Submanifolds:* The factors $\Sigma \times \{pt\}$ and $\{pt\} \times S^1$ are totally geodesic. Minimal surfaces tend to align with this product structure.
   - *Harmonic Map Decomposition:* Harmonic maps into M decompose into harmonic maps into the factors Σ and S^1.
5. **Stable Norms and Large-Scale Geometry**
   - *Pythagorean Stable Norm:* The stable norm on $\pi_1(M) \cong \pi_1(\Sigma) \times \mathbb{Z}$ satisfies $\|\alpha\|_s^2 = \|\gamma\|_{s,\Sigma}^2 + (kL)^2$ for $\alpha = (\gamma, k)$.
   - *Euclidean Structure on H_1:* The stable norm endows $H_1(M, \mathbb{R})$ with a Euclidean structure.
   - *Asymptotic Cones:* Gromov-Hausdorff limits of $(\tilde{M}, d_{\tilde{g}}/R)$ as $R \to \infty$ are products involving $\mathbb{R}$-trees and a line factor.
6. **Impossibility of Non-Trivial Sasakian Structures**
   - *Argument by Contradiction:* Assume $(M = \Sigma \times S^1, g_{Sas})$ is Sasakian with Reeb field ξ, and g_{Sas} has no conjugate points. By Theorem 1, g_{Sas} must be isometric to a product $g_{prod} = g_{hyp} \oplus g_{S^1}$. Under this isometry, the unit Killing field ξ must correspond to a unit Killing field $\tilde{\xi}$ for g_{prod}, which must be parallel to the S^1 factor and satisfy $\nabla^{g_{prod}}\tilde{\xi} = 0$.
   - *Sasakian Condition Violation:* The fundamental Sasakian equation $\nabla_X^{g_{Sas}}\xi = -\phi X$ translates to $\nabla_{\tilde{X}}^{g_{prod}}\tilde{\xi} = -\tilde{\phi}\tilde{X}$. Since $\nabla^{g_{prod}}\tilde{\xi} = 0$, this requires $\tilde{\phi}\tilde{X} = 0$. But ϕ (and hence $\tilde{\phi}$) must be a non-zero tensor acting as an almost complex structure on the contact distribution $\mathcal{D} = \ker(\eta)$, leading to a contradiction.
   - *Conclusion:* No Sasakian metric on $M = \Sigma \times S^1$ (with $\text{genus}(\Sigma) \geq 2$) can be free of conjugate points. The geometric constraints of Sasakian structures (requiring a specific type of "twist") are

incompatible with the rigidity imposed by the absence of conjugate points on this topology.

7. **Spectral Geometry and Inverse Problems**
   - *Laplacian Spectrum Splitting:* The spectrum of the Laplace-Beltrami operator Δ_g on (M, g) is the superposition $\{\lambda_i + \mu_k\}$, where $\{\lambda_i\}$ is the spectrum of $\Delta_{g_{hyp}}$ on Σ and $\{\mu_k = (2\pi k/L)^2 \cdot c^2\}$ is the spectrum of $\Delta_{g_{S^1}}$ on S^1.
   - *Inverse Spectral Rigidity:* The spectrum of Δ_g determines both the hyperbolic metric g_{hyp} (up to isometry, via results on MLS/spectrum for surfaces) and the length L of the S^1 factor, hence determining g up to isometry.
   - *Marked Length Spectrum Rigidity:* The MLS of (M, g) (lengths of closed geodesics corresponding to elements of $\pi_1(M)$) also determines g up to isometry.
8. **Group Actions and Cohomology**
   - *Isometry Group:* The isometry group $\mathrm{Iso}(M, g)$ is restricted to a subgroup of $\mathrm{Iso}(\Sigma, g_{hyp}) \times O(2)$, where $\mathrm{Iso}(\Sigma, g_{hyp})$ is finite.
   - *Cohomological Constraints:* The product structure influences the calculation of group cohomology, bounded cohomology, and L^2-cohomology of $\pi_1(M)$.
9. **Relation to $K \leq 0$ Rigidity**
   - *Strengthening of $K \leq 0$ Result:* While splitting is known for $K \leq 0$ metrics with an S^1 action, Theorem 1 shows the weaker "no conjugate points" condition is sufficient here. This implies such metrics must, in fact, satisfy $K \leq 0$.
10. **Contrast with Non-positive Curvature**
    - *Beyond $K \leq 0$:* The Cheeger-Gromoll theorem guarantees splitting for compact manifolds with $K \leq 0$ admitting a line in the universal cover (like the $\mathbb{R}$ factor here). Our result shows this splitting occurs for $M = \Sigma \times S^1$ under the weaker hypothesis of no conjugate points. It implies a posteriori that such a metric must satisfy $K \leq 0$ (since it's a product of factors with $K \leq 0$), even though this was not assumed a priori. The no-conjugate-point condition prevents the existence of any regions with positive curvature that might otherwise exist in a non-product metric.

Connections: Stable Norms and Entropy

The isometric splitting established by Theorem 1 has clear implications for stable norms and topological entropy, as detailed in Section 5 (Points 6 and 3).

The Pythagorean structure of the stable norm is a direct *consequence* of the isometric splitting forced by the no-conjugate-point condition. Similarly, the topological entropy being solely determined by the hyperbolic factor (Σ, g_{hyp}) highlights the dynamical rigidity imposed by the condition; the flat S^1 factor does not contribute to the complexity growth rate of geodesics. The geometry is rigid in the sense that there is no "mixing" or interaction between the horizontal ($\mathbb{H}^2$) and vertical ($\mathbb{R}$) dynamics that could potentially alter the entropy beyond what is expected from the pure product structure.

6. Connection to Symplectic Rigidity

While Theorem 1 concerns Riemannian geometry and the absence of conjugate points, and Gromov's celebrated non-squeezing theorem arises from symplectic geometry, they share deep thematic connections rooted in geometric rigidity. We briefly explore these parallels.

1. **Thematic Parallel: Rigidity of Structure:** Both theorems exemplify how a fundamental geometric structure imposes global constraints that override topological flexibility.
   - *Splitting Theorem:* Forces a metric on $\Sigma \times S^1$ to adopt a rigid product structure $g_{hyp} \oplus g_{S^1}$, prohibiting twists or warping that might seem topologically permissible. The condition 'no conjugate points' acts as the obstruction.
   - *Non-Squeezing Theorem*[11]*:* Restricts symplectic embeddings (a large symplectic ball $B^{2n}(R)$ cannot be symplectically embedded into a narrow cylinder $Z^{2n}(r) = B^2(r) \times \mathbb{R}^{2n-2}$ if $R > r$), even though volume preservation alone would allow such "squashing". The symplectic form itself acts as the obstruction.

This reflects a shared philosophy: geometric conditions (absence of conjugate points / symplectic structure) can impose powerful obstructions to seemingly plausible configurations or deformations.

2. Geodesic Flows and Symplectic Dynamics: The splitting theorem dictates a specific structure for the geodesic flow, while non-squeezing governs Hamiltonian dynamics.
   - *Splitting Impact:* The geodesic flow on $T^1(\Sigma \times S^1)$ decomposes into a partially hyperbolic system (Anosov on $T^1\Sigma$, isometric on S^1). This rigid dynamical structure has fixed entropy determined by Σ.
   - *Non-Squeezing Context:* Hamiltonian flows preserve the symplectic structure, and non-squeezing reflects this rigidity, often measured by symplectic capacities which obstruct embeddings.

doi.org/10.32388/8R62I8

Connection Hypothesis: The rigidity enforced by the splitting theorem on the geodesic flow might impose constraints on symplectic invariants if $T * M$ is considered. For instance, the trivial dynamics along the S^1 factor could potentially bound certain capacities related to embeddings along that direction.

3. Contact Geometry as a Potential Bridge: If $\Sigma \times S^1$ carries a contact structure η with Reeb field ξ aligned with S^1, Theorem 1 (as seen in Section 5) implies this structure cannot arise from a metric without conjugate points unless it's the trivial product. Contact non-squeezing theorems restrict embeddings related to Reeb dynamics. The forced product structure might imply specific (perhaps trivial) non-squeezing results in the contact setting relative to the Reeb flow defined by the S^1 factor.

2. Symplectic Capacities and Geometric Measures: Non-squeezing is intrinsically linked to symplectic capacities (like Gromov width or cylindrical capacity). The splitting theorem fixes geometric quantities like volume growth ($\mathbb{H}^2 \times \mathbb{R}$ has exponential growth dominated by $\mathbb{H}^2$) and the length spectrum. How these Riemannian invariants interact with symplectic capacities on $T^*(\Sigma \times S^1)$ is an intriguing question. For example, does the length L of the S^1 factor play a role analogous to a capacity for certain symplectic questions?
3. **Open Questions and Speculation:** While a direct technical deduction of one theorem from the other is not apparent, the conceptual parallels invite exploration:
   - Can the rigid metric splitting $(\Sigma, g_{hyp}) \times (S^1, g_{S^1})$ be used to deduce specific bounds or properties related to symplectic embeddings into $T^*(\Sigma \times S^1)$ or related spaces?
   - Conversely, does the presence or absence of conjugate points have implications for the flexibility or rigidity observed in symplectic or contact geometry on the same manifold?

In conclusion, the connection lies in a shared paradigm: fundamental geometric structures impose potent global rigidity, limiting topological flexibility and shaping dynamics in profound ways. While operating in different contexts (Riemannian vs. Symplectic), both Theorem 1 and Gromov's non-squeezing theorem underscore this central theme in modern geometry

7. Rigidity vs. Flexibility

The interplay between rigidity (as exemplified by Theorem 1) and flexibility in geometry is subtle and profound. Our result highlights how the 'no conjugate points' condition dramatically curtails geometric freedom for metrics on $M = \Sigma \times S^1$.

doi.org/10.32388/8R62I8

1. **Flexibility Lost: Metric Rigidity.** Theorem 1 eradicates flexibility in the global metric structure.
    - *No Warping or Twisting:* The metric is forced into the rigid product form $g = g_{hyp} \oplus g_{S^1}$. Warped products like $g = g_{hyp} + f^2 g_{S^1}$ or metrics with cross-terms coupling Σ and S^1 directions are disallowed, as they would inevitably introduce conjugate points.
    - *Curvature Constraints:* The factors must be hyperbolic (g_{hyp}, constant curvature -1 up to scale) and flat (g_{S^1}). There is no room for variable curvature within factors or mixing of curvatures between them. This stark rigidity mirrors Hopf's theorem for flat tori, where the no-conjugate-points condition similarly eliminates all metric flexibility.
2. **Flexibility Retained: Parameters and Moduli.** Despite the structural rigidity, limited flexibility persists in discrete parameters defining the specific product metric:
    - *Hyperbolic Moduli Space:* The hyperbolic metric g_{hyp} on Σ is not unique; it corresponds to a point in the (g_{0-6})-dimensional Teichmüller or moduli space of Σ. Varying this point changes the conformal structure and length spectrum of the Σ factor while preserving the product structure and the absence of conjugate points.
    - *Circle Length:* The constant c (determining the length L of the S^1 factor) is a free parameter. Rescaling c changes the relative size of the factors but maintains the isometric product structure.

Thus, the space of metrics without conjugate points on $\Sigma \times S^1$ is parametrized by the moduli space of Σ and the positive real number c.

3. **Flexibility Regained: Relaxing Conditions.** If the 'no conjugate points' assumption is dropped, the geometric possibilities expand dramatically.
    - *Warped Products:* General warped products $g = g_{hyp} + f^2 g_{S^1}$ become permissible, allowing interaction between the factors and typically introducing conjugate points unless f is constant.
    - *More General Metrics:* Arbitrary Riemannian metrics, potentially exhibiting complex geodesic behavior, positive curvature regions, and diverse topological features (like non-trivial Sasakian structures incompatible with Theorem 1), become available. Conjugate points act as "gatekeepers" restricting access to this broader landscape.
4. **Geometric Analysis Implications.**
    - *Deformation Stability:* The result suggests stability: small perturbations of a product metric $g = g_{hyp} \oplus g_{S^1}$ that remain within the class of metrics without conjugate points must themselves correspond to nearby product metrics (nearby point in moduli space and nearby c).

doi.org/10.32388/8R62I8

- *Inverse Problems Simplified:* Recovering the metric from dynamical data (like MLS or spectrum) is simplified by the rigidity. The inverse problem reduces to determining the finite set of parameters (moduli of Σ, length L) defining the specific product metric.

5. **Flexibility in Higher Dimensions?** For product manifolds like $\Sigma \times T^n$ $(n \geq 2)$, one might conjecture that a metric without conjugate points forces a splitting $\tilde{M} \cong \mathbb{H}^2 \times \mathbb{R}^n$. While the $\mathbb{H}^2$ factor seems rigidly determined, the flat $\mathbb{R}^n$ factor retains flexibility corresponding to the moduli space of flat metrics on T^n. Investigating this remains an open direction.

6. **Hodge Theory Simplification**
   - *Splitting of the Laplacian:* As noted above, Theorem 1 forces the metric g to be a product $g_{hyp} \oplus g_{S^1}$, which implies the Laplace-Beltrami operator splits: $\Delta_g = \Delta_{g_{hyp}} + \Delta_{g_{S^1}}$.
   - *Decomposition of Harmonic Forms:* Hodge theory identifies de Rham cohomology classes $H^k_{dR}(M)$ with spaces of harmonic k-forms $\mathcal{H}^k(M, g) = \ker(\Delta_g|_{k-forms})$. Due to the Laplacian splitting, the space of harmonic forms on M decomposes according to the product structure. Specifically, harmonic k-forms on M can be constructed via tensor products and sums of harmonic forms on (Σ, g_{hyp}) and (S^1, g_{S^1}). For instance, $\mathcal{H}^1(M, g) \cong \mathcal{H}^1(\Sigma, g_{hyp}) \oplus \mathcal{H}^1(S^1, g_{S^1})$, and $\mathcal{H}^2(M, g) \cong \mathcal{H}^2(\Sigma, g_{hyp}) \oplus (\mathcal{H}^1(\Sigma, g_{hyp}) \otimes \mathcal{H}^1(S^1, g_{S^1}))$.
   - *Rigidity of Hodge Structure:* The theorem implies that the Hodge structure (the space of harmonic forms representing cohomology classes) is rigidly determined by the choice of hyperbolic metric on Σ(from its moduli space) and the length L of the S^1 factor. The absence of conjugate points prevents any other metric structure, and thus any other Hodge structure, from arising on $M = \Sigma \times S^1$. This simplifies computations and structural analysis within Hodge theory for this class of manifolds.
   - *Quantum Field Theory and Compactification:* If M serves as a model for spacetime or an internal space in QFT or string theory, the theorem implies that requiring the absence of conjugate points severely restricts the background geometry to the simple product $g_{hyp} \oplus g_{S^1}$. This simplifies mode analysis and potentially affects calculations related to vacuum energy or particle interactions.

7. *Relevance to String Theory and Kaluza-Klein Compactifications:* The manifold $M = \Sigma \times S^1$ serves as a simple model for spacetimes with compactified dimensions, a central concept in String Theory and Kaluza-Klein (KK) theory. Theorem 1 dictates that if such a background geometry is required to lack

doi.org/10.32388/8R62I8

conjugate points (perhaps for stability or well-behaved dynamics), then the metric *must* be the rigid product $g = g_{hyp} \oplus g_{S^1}$. This has specific consequences:

- **Kaluza-Klein Spectrum:** In KK theory, the geometry of the compact internal manifold determines the properties (like masses) of fields in the lower-dimensional effective theory. The eigenvalues $\{\mu_n = (2\pi n/L)^2\}$ of the Laplacian on the S^1 factor correspond directly to the squared masses of the Kaluza-Klein modes arising from compactification on S^1. Theorem 1 ensures this simple, evenly spaced (in n^2) mass spectrum is the only possibility under the no-conjugate-point condition.
- **String Modes:** In string theory, the modes of a string propagating on M would decompose. Excitations along the S^1 factor correspond to the KK modes with energies related to μ_n. Excitations involving the Σ factor would have energies related to λ_k, reflecting the hyperbolic geometry. The total energy levels would combine these, influenced by the chaotic nature of Σ and the simple periodicity of S^1.
- **Simplified Background Dynamics:** The absence of conjugate points and the forced product structure simplify the analysis of field propagation (scalar fields, gravitons, gauge fields) on this background. The d'Alembertian separates, $\Box_g = \Box_{g_{hyp}} + \Box_{g_{S^1}}$, facilitating mode decomposition and analysis in dimensional reduction. The predictable geometry avoids complexities like gravitational lensing or caustics within the background itself.
- **Constraint on Models:** If a theoretical model requires a factor of $\Sigma \times S^1$ and imposes a condition like "no conjugate points" on the metric, Theorem 1 severely restricts the allowable background geometry to the simple, non-warped product. This limits the possibilities for generating gauge fields or other phenomena through geometric twisting or warping in the compact dimensions.

Concrete Example: Quantum Particle on $\Sigma \times S^1$.

Consider the canonical example of a free quantum particle of mass m constrained to move on $(M = \Sigma \times S^1, g = g_{hyp} \oplus g_{S^1})$. Its quantum states are described by wavefunctions $\psi(x, \theta)$ satisfying the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\Delta_g \psi(x, \theta) = E\psi(x, \theta).$$

Theorem 1 guarantees that the metric g must take this simple product form if it lacks conjugate points. Consequently, the Laplacian separates: $\Delta_g = \Delta_{g_{hyp}} + \Delta_{g_{S^1}}$. This allows for separation of variables in the Schrödinger equation, yielding solutions of the form $\psi_{k,n}(x, \theta) = \phi_k(x) \cdot \chi_n(\theta)$, where:

doi.org/10.32388/8R62I8

- $\phi_k(x)$ is an eigenfunction of the Laplacian on the hyperbolic surface (Σ, g_{hyp}) with eigenvalue λ_k: $-\Delta_{g_{hyp}}\phi_k = \lambda_k\phi_k$. The properties of these eigenfunctions and eigenvalues $\{\lambda_k\}$ are intimately linked to the classical chaotic geodesic flow on Σ and are studied in quantum chaos.
- $\chi_n(\theta)$ is an eigenfunction of the Laplacian on the circle (S^1, g_{S^1}) of length L. These are typically $e^{in(2\pi\theta/L)}$ with eigenvalues $\mu_n = (2\pi n/L)^2$ for $n \in \mathbb{Z}$.

Substituting the separated wavefunction into the Schrödinger equation yields the total energy eigenvalues:

$$E_{k,n} = \frac{\hbar^2}{2m}(\lambda_k + \mu_n) = \frac{\hbar^2}{2m}\left(\lambda_k + \left(\frac{2\pi n}{L}\right)^2\right).$$

This explicitly demonstrates the additive nature of the energy spectrum, directly reflecting the product geometry forced by the absence of conjugate points. The quantum dynamics combines features inherited from the hyperbolic chaos on Σ (via λ_k) and the simple periodicity on S^1 (via μ_n). The absence of conjugate points in the metric g ensures that wave packet propagation, particularly components associated with Σ, exhibits characteristic spreading without geometric refocusing phenomena. This provides a clear example of how the geometric rigidity established by Theorem 1 dictates the fundamental structure of quantum states and energy levels on M.

In summary, Theorem 1 starkly illustrates how a global geodesic condition ('no conjugate points') can act as a powerful "geometric police," drastically reducing the allowable metric structures on $\Sigma \times S^1$ from a potentially vast, flexible space to a tightly constrained family of rigid products parametrized by discrete data. This tension between underlying topological flexibility and imposed geometric rigidity is a central theme in modern geometry, evident in fields ranging from Thurston's geometrization program to Gromov's h-principle. Here, the absence of conjugate points draws a sharp line: either rigid splitting or the potential for complex, flexible geometries where geodesics can refocus.

8. Concluding Remarks

The principal achievement of this paper, encapsulated in Theorem 1, is the extension of rigidity theorems to the class of Riemannian metrics without conjugate points on manifolds diffeomorphic to $M = \Sigma \times S^1$, where Σ is a higher-genus surface. The result provides a definitive answer to the question posed in the introduction: such metrics exhibit a strong structural rigidity imposed by the no-conjugate-point condition combined with the S^1 symmetry. Specifically, the universal cover $(\tilde{M}, \tilde{g})$ must split isometrically as a product $(\mathbb{H}^2, g_0) \times (\mathbb{R}, c^2 du^2)$. Consequently, the metric g on M itself must be globally isometric to a standard Riemannian product metric $(\Sigma, g_{hyp}) \times (S^1, g_{S^1})$.

The potential for a geometrically "twisted" or warped product structure, alluded to in the motivating question, is explicitly ruled out. This underscores the powerful constraints imposed by the absence of conjugate points, forcing a simple, non-warped product structure even when one factor (Σ) carries inherent negative curvature. This finding reinforces the theme that global geodesic properties, even weaker than non-positive curvature, can lead to significant structural rigidity, particularly in the presence of symmetry. The numerous consequences detailed in Section 5, ranging from topological and dynamical rigidity to spectral properties and the impossibility of non-trivial Sasakian structures without conjugate points, illustrate the profound impact of this geometric rigidity.

Notes

Mathematics Subject Classification (2020): Primary 53C20; Secondary 53C22, 53C24, 53C12, 37D40.

Acknowledgements

The author would like to express his deep gratitude to the distinguished Professor Dmitri Burago for suggesting a list of open problems in 2021, including the one addressed in this paper.

References

1. a, b *Hopf E. "Closed surfaces without conjugate points." Proc Nat Acad Sci U S A.* ***34*** *(1948): 47–51.*
2. a, b *Burago D, Ivanov S (1994). "Riemannian tori without conjugate points are flat." Geom Funct Anal.* ***4*** *(3): 259–269.*
3. a, b *Cheeger J, Gromoll D (1971/72). "The splitting theorem for manifolds of nonnegative curvature." J. Differential Geometry.* ***6****: 119–128.*
4. a, b *Eberlein P. Structure of manifolds of nonpositive curvature. Global differential geometry and global analysis (Berlin, 1979), pp. 169–209, Lecture Notes in Math., 838, Springer, Berlin-New York, 1981.*
5. a, b *Eschenburg JH. "Horospheres and the stable part of the geodesic flow." Math Z.* ***153*** *(1977), no. 3, 237–251.*
6. ^ *Heintze E, Im Hof H-C (1977). "Geometry of horospheres." J. Differential Geom..* ***12*** *(4): 481–491.*
7. ^ *Gromoll D, Walschap G. Metric foliations and curvature. Progress in Mathematics, 268. Birkhäuser Verlag, Basel, 2009.*

8. ^*O'Neill B. "The fundamental equations of a submersion." Michigan Math. J.* **13** *(1966): 459–469.*

9. ^*Cheeger J, Ebin DG. Comparison theorems in Riemannian geometry. North-Holland Mathematical Library, Vol. 9. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1975.*

10. ^*Petersen P. Riemannian geometry. Third edition. Graduate Texts in Mathematics, 171. Springer, Cham, 2016.*

11. ^*Gromov M. Pseudoholomorphic curves in symplectic manifolds. Invent Math.* **82** *(1985), no. 2, 307–347.*

Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.

doi.org/10.32388/8R62I8