Research Article

Admissible History Deformations and Central 2-Torsion Residues: The $A_5/2I$ Case

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A history-sensitive observable can be lost even when a chosen coarse endpoint or quotient is unchanged, so the relevant conservation question is not merely whether two states agree but which admissible deformations preserve the observable. We formulate this question algebraically using a deformation setup $\mathscr{S} = (F, G, Q, \phi, \mathrm{proj}, \mathrm{lab}, R_p, \mathrm{Adm})$, where the kernel $K = \ker(\mathrm{proj})$ is central and has exponent at most two, while R_p records declared relation-preserving deformations. For $D(R_p) = R_p^2[F, R_p]$, the residue is annihilated on $D(R_p)$, and its restriction to R_p therefore descends to a canonical homomorphism $\bar{\rho} : R_p/D(R_p) \longrightarrow K$.

The quotient carrier is commutative of exponent two. Independently of Case A, we prove the exact criterion $\rho(Hw) = \rho(H) \Leftrightarrow \rho(w) = 1$, and hence $\mathrm{ConservationOnAllowed}(A) \iff A \subseteq \ker\phi$.

The exact kernel $\ker\phi$ is setup-specific, whereas $D(R_p)$ depends only on (F, R_p). A canonical quotient setup realizes equality of its residue kernel with $D(R_p)$, showing that Case A is the largest setup-independent, residue-blind subgroup guaranteed safe from (F, R_p) alone. On the bit domain $R_p \cap \mathrm{Adm}$, the residue-changing sector is exactly the nontrivial-residue part of Case B. We then specialize to $SL(2,5) \to PSL(2,5) \cong A_5$. The original torus witness is local to an embedded V_4/Q_8 carrier, but a third primitive label globalizes the history alphabet: its labels generate all of $PSL(2,5) \cong A_5$ while the same commutator retains residue $-I$ and remains an explicit Case-B witness. We further prove that, for commuting pairs in the concrete A_5 base, failure of simultaneous commuting liftability is equivalent to Klein-four type. The associated central-cover commutator pairing is independent of lift choice, binary-valued on the commuting locus, and has nontrivial support exactly on the Klein-four locus. The paper separates these algebraic results from downstream physical realization: it does not claim a physical history bit, a cosmological initial bit, or a topological $H^2(T^2;\mathbb{Z}_2)$ identification.

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 doi.org/10.32388/8WFJ48

1. Introduction

1.1. The problem: preservation under history deformation

Many reduction procedures replace a detailed trajectory by a coarser object: an endpoint, a state label, a quotient state, or a task-specific sufficient representation. Such reductions are often appropriate, but they change the mathematical question whenever the observable of interest depends on how a trajectory was realized rather than only on its final coarse description. The present paper studies a deliberately narrow version of that problem.

A **finite resolved history** is treated here as an ordered finite trajectory whose elementary steps have been specified at the resolution relevant to the model. The paper does not propose that histories are universally more fundamental than states, nor does it claim that endpoint equality is never sufficient. The operational point is only that, for a relation-sensitive observable, endpoint equality need not by itself certify equality of the observable. One must therefore ask which changes of history are guaranteed to leave the observable invariant.

Our central object is not the entire history space but a **history difference or deformation**. If a chosen algebraic encoding represents a history by H and a deformation by w, the deformed history is written

$$H' = Hw.$$

The conservation problem becomes: for which w is a residue associated with the history guaranteed to satisfy

$$\rho(Hw) = \rho(H)?$$

This shift from absolute histories to admissible deformations is the organizing idea of the paper.

There is an established foundational literature in which histories, rather than instantaneous states alone, carry the primary descriptive burden. Sum-over-histories and decoherent-histories approaches to quantum cosmology organize alternatives as classes of spacetime histories [1][2]; Sorkin's quantum-measure program likewise takes a history space as the natural domain of the generalized measure [3]. In causal-set theory, sequential growth makes the construction history explicitly discrete while discrete general covariance constrains which label-dependent distinctions are physically irrelevant [4]. Topological-geon work shows, in a different setting, that nontrivial topology and mapping-class-group structure can leave quantum-sector information not reducible to a naive local configuration label [5]. These precedents motivate the question posed here, but they do not supply the present central-extension residue mechanism. Our aim is narrower: to isolate, within a finite algebraic deformation interface, a class of history changes that is guaranteed to preserve a declared central residue and a complementary route by which a nontrivial residue can occur.

1.2. Why separate the algebraic core from physical realization

The motivating research program ultimately asks whether a binary history sector can be realized by an actual local dynamics related to the $A_5/2I$ structure. That physical question is not settled here. In particular, the actual state space, primitive move alphabet, physical nullity rule, admissibility domain, and local dynamics Φ have not all been frozen and passed through the physical NONDEG/PRES gates described later.

doi.org/10.32388/8WFJ48

It is nevertheless useful to isolate the algebra that any such physical realization would have to satisfy. Doing so has two advantages. First, the mathematical preservation theorem can be cited and tested independently of whether the downstream physical branch succeeds. Second, a result-blind physical protocol can be frozen before the target binary verdict is inspected, reducing the risk that the algebraic definitions are tuned after the desired sector has been observed.

The resulting architecture is therefore intentionally modular:

$$\boxed{\begin{array}{c}\text{finite-history semantics} \longrightarrow \text{parametric deformation algebra} \\ \longrightarrow A_5/2I \text{ binary case} \longrightarrow \text{open physical realization test}\end{array}}$$

The first three layers can be developed without claiming the fourth.

The $A_5/2I$ specialization is particularly useful because the A_5 base has trivial abelianization, while its binary central cover can nevertheless retain a nontrivial central relation residue:

$$A_5^{ab} = 1, \ker(2I \to A_5) \cong \mathbb{Z}_2.$$

This is only an algebraic motivation for the specialization; it does not assert a physical state bit or a cosmological initial bit.

1.3. Contributions

The mathematical contribution is organized into four linked pillars.

1. Declared-domain architecture. We separate the normal relation subgroup R_p, the residue-admissibility subgroup Adm, and the actual dynamical family $\mathrm{Allowed}$. These encode different questions—declared relation-nullity, the domain on which the central residue is interpreted, and physically/dynamically executable changes—and are not identified by definition.

2. Exact preservation and universal safe-sector sharpness. For

$$D(R_p) = R_p^2[F, R_p],$$

the residue is trivial on $D(R_p)$, while the exact setup-specific criterion is

$$\rho(Hw) = \rho(H) \Leftrightarrow \rho(w) = 1.$$

Consequently,

$$\mathrm{ConservationOnAllowed}(A) \Leftrightarrow A \subseteq \ker \phi.$$

The universal comparison class then gives

$$D(R_p) = \bigcap_{\mathscr{S}' \in \mathfrak{S}(F, R_p)} \ker \phi_{\mathscr{S}'}.$$

Thus $D(R_p)$ is the maximal setup-independent, residue-blind safe sector determined by (F, R_p), while $\ker \phi_{\mathscr{S}}$ is the exact setup-specific preservation kernel.

doi.org/10.32388/8WFJ48

3. Setup-independent carrier and exact Case-B refinement. The residue restriction descends to the setup-independent carrier

$$\mathscr{C}_{hist}(R_p) = R_p/D(R_p),$$

together with a setup-induced residue homomorphism

$$\bar{\rho}_{\mathscr{S}} : \mathscr{C}_{hist}(R_p) \to K_{\mathscr{S}}.$$

The carrier is commutative of exponent two. On the bit domain, Case B splits exactly into

$$B_0 = \{w : CaseB(w) \wedge \rho(w) = 1\}, B_1 = \{w : CaseB(w) \wedge \rho(w) \neq 1\},$$

and residue change occurs exactly on B_1, not on Case B as a whole.

4. Full-A_5 realization and exact Klein-four commutator support. In

$$SL(2,5) \to PSL(2,5) \cong A_5,$$

a three-letter history labeling generates the full A_5 base while the explicit nontrivial residue witness remains supported on an embedded V_4/Q_8 subsystem. For commuting downstairs pairs,

$$\neg\text{CommutingLiftExists}(a,b) \Leftrightarrow \text{KleinPair}(a,b).$$

The associated central-cover commutator pairing

$$\kappa(a,b) = [A,B]$$

is lift-choice independent, binary-valued on the commuting locus, and satisfies

$$\kappa(a,b) = -I \Leftrightarrow \text{KleinPair}(a,b).$$

Hence

$$\text{supp}_{\text{Comm}}\kappa = \text{Klein-four locus}.$$

A freeze-compatible/result-blind interface and Lean formalization support these mathematical claims as claim-control mechanisms rather than as mathematical novelty in themselves. The commutator pairing itself is standard central-extension structure; the paper-specific result is the exact determination of its nontrivial support in the frozen $A_5/2I$ realization.

Figure 1 summarizes the relation between the declared domains, the universal and setup-specific preservation structures, the quotient carrier, and the concrete $A_5/2I$ realization.

doi.org/10.32388/8WFJ48

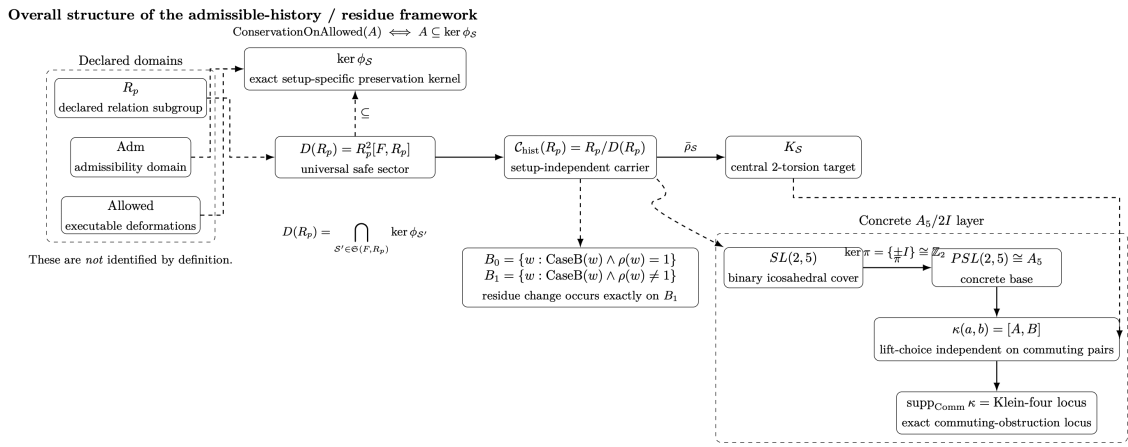

Figure 1. Architecture of the admissible-history deformation framework. The diagram separates the setup-independent safe subgroup and carrier from the setup-specific preservation kernel and residue map, and shows their concrete specialization to $SL(2,5) \to PSL(2,5) \cong A_5$.

1.4. *Claims deliberately not made*

The following implications are not asserted:

$$\boxed{\begin{array}{c}\text{central 2-torsion} \not\Rightarrow \mathbb{Z}_2 \text{ carrier} \not\Rightarrow \text{algebraic nondegeneracy} \\ \not\Rightarrow \text{physical nondegeneracy} \not\Rightarrow \text{cosmological initial bit}\end{array}}$$

More concretely, this paper does **not** claim:

- that the universe possesses a physical binary history sector;
- that an algebraic $A_5/2I$ witness is a cosmological initial condition;
- that the presentation-level torus relation has here been formalized as a class in $H^2(T^2; \mathbb{Z}_2)$ or as a principal-bundle obstruction;
- that every Case-B deformation changes the residue;
- that the proposed later compression problem is a new replacement for bisimulation, lumpability, causal states, or partition refinement.

These exclusions are part of the theorem statement, not merely caveats added after the fact.

1.5. *Organization*

Section 2 introduces the minimal history and deformation semantics. Sections 3 and 4 give the abstract algebra, the exact flip criterion, the quotient residue carrier, and the preservation theorems. Section 5 gives a freeze-compatible free-group interface. Section 6 gives the concrete $A_5/2I$ case, separates the local V_4/Q_8 witness from full-A_5 label generation, and combines both in one fixed mathematical instantiation. Section 7 records auxiliary encoding and representative-choice results, including a counterexample to an overstrong closure-only invariance claim. Section 8 states the downstream

doi.org/10.32388/8WFJ48

physical realization protocol without reporting a physical PASS. Section 9 positions the work relative to central-extension theory, gain graphs, state aggregation, causal states, histories-based quantum cosmology, causal-set growth, topological-geon sectors, spin-foam move invariance, cosmic topology based on the binary icosahedral group, blind analysis, and non-Abelian defect topology in ordered media. Section 10 records formal traceability and provenance. Sections 11 and 12 discuss interpretation, limitations, and conclusions. Appendices A and B provide the formal source inventory and the preregistration/freeze record.

2. Finite histories, differences, and deformation semantics

2.1. Finite resolved histories

Let a resolved history be a finite ordered sequence

$$H = (e_1, e_2, \ldots, e_n),$$

or equivalently a finite state-transition trajectory when states are explicitly included. The word *resolved* means only that the primitive events have been fixed at the level used by the model. A different resolution may produce a different encoding; Section 7 addresses one aspect of this dependence.

Two distinctions are fundamental.

First,

$$\boxed{\text{history equality} \neq \text{endpoint equality}}$$

as a matter of representation: two distinct ordered trajectories may share an endpoint even if the model later decides that the distinction is irrelevant to a particular readout. This is not a universal collision theorem; it is a refusal to define history equality by endpoint equality before the readout has been specified.

Second,

$$\boxed{\text{history} \neq \text{witness}}.$$

A binary or central residue is a function of a history or history difference under additional algebraic data. It is not the history itself.

2.2. From absolute histories to deformations

The abstract algebra of the paper begins only after a representation of history differences has been selected. Suppose histories in a common comparison domain admit a right action by a group F, or are encoded so that a relative deformation can be represented by an element of F. We then write

$$H' = Hw. \tag{2.1}$$

Equation (2.1) is a bookkeeping convention for the comparison problem. It does **not** assert that the set of all physically possible histories is itself a group, nor that every $w \in F$ is physically realizable. The physically allowed subset is introduced only later.

This distinction lets us ask an algebraic question before a physical one:

> Given a deformation w, which structural conditions force a residue to be unchanged under $H \mapsto Hw$?

The answer will be expressed in terms of the relation subgroup R_p, the admissibility subgroup Adm, and a central extension carried by $G \to Q$.

2.3. Branch-independent history semantics and branch-specific residue

The minimal history semantics of this section is independent of A_5. The $A_5/2I$ structure enters only as one later adapter. This separation is important because the same finite-history bookkeeping could, in principle, be paired with a different target observable or a different algebraic carrier.

Schematically,

$$\boxed{\text{finite resolved history} + \text{branch-specific deformation/residue adapter}}$$

is the structure used here. The present paper develops one such adapter in which a central exponent-two kernel controls relation-sensitive residues, and then specializes it to the binary icosahedral cover.

2.4. Null relations, admissibility, and physical allowedness are distinct

Three subsets of deformation space must not be conflated.

1. Relation/null sector R_p. These are deformations declared to be relation-preserving in the algebraic setup. Normality and downstairs label-nullity are imposed in Section 3.

2. Admissibility sector Adm. This is the domain on which the residue is regarded as a canonical candidate for the Case A–D analysis. In the freeze-compatible free-group interface, a conservative benchmark is generated by all squares and commutators.

3. Physically allowed deformations. These depend on an actual dynamics Φ and are not fixed by the abstract theorem. In particular,

$$\mathrm{Allowed} \subseteq R_p \cap \mathrm{Adm}$$

is an additional modeling condition, not an algebraic tautology.

The preservation theorem can therefore be used in two different ways. If an allowed family is proved to lie inside Case A, conservation follows directly. If an allowed deformation changes the residue and is independently known to lie in $R_p \cap \mathrm{Adm}$, the change localizes to Case B. Neither inference identifies the allowed family by itself.

2.5. The binary language is reserved for the specialization

The abstract kernel

doi.org/10.32388/8WFJ48

$$K = \ker(\mathrm{proj})$$

is central of exponent at most two, but it may be trivial or have more than two elements. Accordingly, Sections 3–5 use the language **central two-torsion residue** rather than *bit*.

Only after the concrete specialization

$$SL(2,5) \longrightarrow PSL(2,5) \cong A_5$$

does the kernel become

$$\{\pm I\} \cong \mathbb{Z}_2,$$

at which point *binary residue* is appropriate. Even there, binary carrier nontriviality and algebraic nondegeneracy remain distinct from physical realization.

3. Abstract deformation setup

3.1. Histories as algebraic deformations

The formal object studied in this paper is not a complete physical history by itself, but a **difference or deformation** between two histories. If a history is represented algebraically by an element H of a group F, an inserted deformation is represented by $w \in F$, and the deformed history is written

$$H' = Hw.$$

The group F is intentionally left abstract in this section. In the freeze-compatible instantiation of Section 5, it will be a free group generated by primitive moves, but none of the preservation results below requires freeness. This separation is useful because the algebraic conservation problem depends only on the deformation structure and not on a particular encoding of histories.

We consider groups F, G, Q and homomorphisms

$$\phi : F \to G, \mathrm{proj} : G \to Q, \mathrm{lab} : F \to Q,$$

satisfying

$$\mathrm{proj}(\phi(f)) = \mathrm{lab}(f) \qquad (f \in F).$$

The map *lab* records the downstairs label of a deformation, while ϕ is an upstairs lift compatible with the projection $proj$. We also choose two subgroups

$$R_p, \mathrm{Adm} \leq F.$$

The subgroup R_p specifies the relation-preserving or null sector relevant to the deformation problem. The subgroup Adm specifies the domain on which a residue is regarded as an admissible candidate for the binary-sector analysis developed later.

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Definition 3.1 (deformation setup). A **deformation setup** is a tuple

$$\mathscr{S} = (F, G, Q, \phi, \mathrm{proj}, \mathrm{lab}, R_p, \mathrm{Adm})$$

such that:

1. $R_p \lhd F$;

2. $R_p \leq \ker(lab)$;

3. Adm contains every square $g^2, g \in F$;

4. Adm contains every commutator

$$[a, b] := aba^{-1}b^{-1};$$

5. if

$$K := \ker(\mathrm{proj}),$$

then K is central in G;

6. every $k \in K$ satisfies $k^2 = 1$.

The compatibility condition (3.1) is part of the setup.

The formal implementation is `SGB1R1DeformationSetupV1`.

Remark 3.1 (what is not assumed). Definition 3.1 does **not** assume that F is free, that either proj or lab is surjective, that K is nontrivial, or that $K \cong \mathbb{Z}_2$. In particular, an arbitrary central subgroup of exponent at most two is allowed, including the trivial group and groups of the form $(\mathbb{Z}_2)^m$.

This point is important for the logical organization of the paper. The abstract preservation theorems are statements about central two-torsion; the genuinely binary case appears only after the $A_5/2I$ specialization in Section 6.

3.2. Algebraic and physical readings of R_p

The two conditions on R_p play distinct roles.

$$\boxed{R_p \lhd F} \tag{G0}$$

is the algebraic normality condition. In a physical application, however, the substantive content of G0 is not merely that a normal subgroup exists. It is the separate modeling decision that the physically null deformations are represented by the chosen R_p. This physical identification is not derived by the abstract algebra.

The second condition

$$\boxed{R_p \leq \ker(\mathrm{lab})} \tag{G1}$$

states that every declared null deformation is label-null downstairs. G1 is what makes the central kernel K relevant on R_p.

The formal interface preserves this distinction: normality of a normal closure is discharged algebraically, while the choice of relation generators that are to represent physically null deformations belongs to the later physical preregistration layer.

doi.org/10.32388/8WFJ48

3.3. The residue map and its carrier

For every $w \in F$, define

$$\rho_{\mathscr{S}}(w) := \phi(w) \in G. \tag{3.2}$$

Thus the residue map is globally G-valued. It is not, in general, a map from all of F to K.

The relation sector is different.

Proposition 3.2 (null-sector carrier). If $r \in R_p$, then

$$\rho_{\mathscr{S}}(r) \in K. \tag{3.3}$$

Proof. Because $R_p \leq \ker(\mathrm{lab})$, we have $\mathrm{lab}(r) = 1$. By compatibility,

$$\mathrm{proj}(\rho_{\mathscr{S}}(r)) = \mathrm{proj}(\phi(r)) = \mathrm{lab}(r) = 1.$$

Hence $\rho_{\mathscr{S}}(r) \in \ker(\mathrm{proj}) = K$. □

The corresponding formal declaration is `SGB1R1DeformationSetupV1.residue_mem_ker`.

Firewall 3.1. The statement

$$\rho_{\mathscr{S}}(w) \in K$$

is **not** asserted for arbitrary $w \in F$. The central residue interpretation applies to the relation/null sector R_p. This distinction prevents a global G-valued lift from being silently reinterpreted as a globally binary history label.

3.4. Multiplicativity

The residue inherits multiplicativity from ϕ:

$$\rho_{\mathscr{S}}(ab) = \rho_{\mathscr{S}}(a)\rho_{\mathscr{S}}(b), a, b \in F. \tag{3.4}$$

This elementary property is the mechanism behind all preservation statements in Section 4. The question is therefore reduced to identifying a subgroup on which $\rho_{\mathscr{S}}$ is forced to be trivial.

4. Case-A denominator and preservation theorems

Most later statements in this section are consequences or refinements of three primitive ingredients:

$$\boxed{\text{Case-A annihilation} + \text{multiplicative exact preservation} + \text{canonical quotient sharpness.}} \tag{4.0}$$

The first gives $D(R_p) \subseteq \ker \phi_{\mathscr{S}}$; the second gives $\rho(Hw) = \rho(H) \Leftrightarrow \rho(w) = 1$; the third constructs a comparison setup whose kernel is exactly $D(R_p)$. Case-A preservation, the Case-B localization statements, the B_0/B_1 split, and the allowed-family conservation criteria are organized consequences of this three-part spine.

4.1. The Case-A subgroup

Define the generating set

$$\mathscr{G}_A = \{r^2 : r \in R_p\} \cup \{[f, r] : f \in F, r \in R_p\}. \tag{4.1}$$

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Definition 4.1 (Case-A denominator). The **Case-A subgroup** is

$$D(R_p) := \langle \mathscr{G}_A \rangle = \big\langle \{r^2 : r \in R_p\} \cup \{[f,r] : f \in F, r \in R_p\} \big\rangle. \tag{4.2}$$

We sometimes use the suggestive notation

$$D(R_p) = R_p^2[F, R_p],$$

but the arguments below do not require a formal Hopf-formula identification. In the Lean development this subgroup is `caseASubgroup`.

The two families of generators in (4.1) are selected because they are annihilated by the two defining properties of K: exponent two annihilates null squares, and centrality annihilates commutators with null elements.

4.2. Denominator annihilation

Theorem 4.2 (Case-A annihilation). For every $w \in D(R_p)$,

$$\boxed{\rho_{\mathscr{S}}(w) = 1.} \tag{4.3}$$

Proof. Let $r \in R_p$. By Proposition 3.2, $\rho_{\mathscr{S}}(r) \in K$. Since K has exponent at most two,

$$\rho_{\mathscr{S}}(r^2) = \rho_{\mathscr{S}}(r)^2 = 1. \tag{4.4}$$

Next let $f \in F$ and $r \in R_p$. Again $\rho_{\mathscr{S}}(r) \in K$, and $K \leq Z(G)$. Therefore $\rho_{\mathscr{S}}(r)$ commutes with $\rho_{\mathscr{S}}(f)$, and

$$\rho_{\mathscr{S}}([f,r]) = [\rho_{\mathscr{S}}(f), \rho_{\mathscr{S}}(r)] = 1. \tag{4.5}$$

Thus every generator of $D(R_p)$ lies in $\ker \phi$. Since $\ker \phi$ is a subgroup, it contains the subgroup generated by those elements, hence all of $D(R_p)$. $\square$

Formal counterpart. `SGB1R1DeformationSetupV1.residue_caseA`.

The theorem is deliberately elementary. Its role is not to introduce a new homological formula, but to isolate the minimal algebraic mechanism that is sufficient for residue preservation.

4.3. Preservation under Case-A deformations

Theorem 4.3 (Case-A preservation). For every $H \in F$ and $w \in D(R_p)$,

$$\boxed{\rho_{\mathscr{S}}(Hw) = \rho_{\mathscr{S}}(H).} \tag{4.6}$$

Proof. By multiplicativity and Theorem 4.2,

$$\rho_{\mathscr{S}}(Hw) = \rho_{\mathscr{S}}(H)\rho_{\mathscr{S}}(w) = \rho_{\mathscr{S}}(H). \qquad \square$$

Formal counterpart. `SGB1R1DeformationSetupV1.caseA_preserves_residue`.

Theorem 4.3 is the basic conservation statement. It says that a history may be modified by an arbitrary element of $D(R_p)$ without changing its residue. The theorem does not state that all physically allowed deformations belong to $D(R_p)$; that is a separate downstream question.

4.4. Exact flip criterion and canonical relation-residue carrier

Case-A preservation is a structural sufficient criterion, but multiplicativity gives a stronger exact statement.

Theorem 4.4 (exact preservation and flip criterion).

For every $H, w \in F$,

$$\boxed{\rho_{\mathscr{S}}(Hw) = \rho_{\mathscr{S}}(H) \Leftrightarrow \rho_{\mathscr{S}}(w) = 1.} \tag{4.6a}$$

Equivalently,

$$\boxed{\rho_{\mathscr{S}}(Hw) \neq \rho_{\mathscr{S}}(H) \Leftrightarrow \rho_{\mathscr{S}}(w) \neq 1.} \tag{4.6b}$$

The verdict is independent of the base history H.

Formal counterparts. `SGB1R1DeformationSetupV1.residue_mul_eq_base_iff` and `SGB1R1DeformationSetupV1.residue_mul_ne_base_iff`.

Theorem 4.4 clarifies the role of Case A. $D(R_p)$ is not defined to be the full preservation kernel. For a fixed setup, the exact preservation kernel is setup-specific.

Exact family-level consequence

For any declared family $A \subseteq F$,

$$\boxed{\mathrm{ConservationOnAllowed}(A) \Leftrightarrow \forall w \in A, \rho_{\mathscr{S}}(w) = 1 \Leftrightarrow A \subseteq \ker \phi.} \tag{4.6c}$$

The forward implication follows by evaluating conservation at $H = 1$; the reverse implication follows pointwise from Theorem 4.4.

Formal counterparts. `conservationOnAllowed_iff_residue_trivial` and `conservationOnAllowed_iff_subset_phi_ker`.

Unlike the setup-specific exact kernel $\ker \phi$, the subgroup $D(R_p)$ is determined from F and the declared relation subgroup R_p alone. It can therefore be frozen and audited without first evaluating the target residue.

Universal sharpness of the Case-A sector

Let $D = D(R_p)$. Using $D \lhd F$ and $D \leq R_p$, form

$$F \longrightarrow F/D \longrightarrow F/R_p. \tag{4.6d}$$

Take the upstairs group to be F/D, the downstairs group to be F/R_p, the maps to be the natural quotient homomorphisms, and $\mathrm{Adm} = \top$. Since $[F, R_p] \leq D$, the kernel R_p/D is central in F/D; since $R_p^2 \leq D$, every kernel element has square one.

For this canonical setup,

$$\boxed{\rho_{univ}(w) = 1 \Leftrightarrow w \in D(R_p).} \tag{4.6e}$$

doi.org/10.32388/8WFJ48

Let $\mathfrak{S}(F, R_p)$ denote the class of deformation setups that share the same ambient deformation group F and declared normal relation subgroup R_p. The admissibility subgroup need not be held fixed in this comparison, because the residue homomorphism itself is independent of Adm.

Then the sharpness statement can be written as

$$\boxed{D(R_p) = \bigcap_{\mathscr{S}' \in \mathfrak{S}(F,R_p)} \ker \phi_{\mathscr{S}'}.} \tag{4.6f}$$

The inclusion from left to right is Theorem 4.2 applied to every comparison setup. The reverse inclusion follows because the canonical quotient setup itself belongs to the comparison class and satisfies

$$\ker \phi_{univ} = D(R_p).$$

Thus $D(R_p)$ is exactly the largest subgroup guaranteed residue-trivial from (F, R_p) alone, without inspecting setup-specific residue data.

Formal counterparts. Lean proves the two ingredients separately: universal Case-A annihilation for arbitrary setups, and equality with the kernel in the canonical witness setup via `universalSafeSetup`, `universalSafeResidue_eq_one_iff_caseA`, `caseASubgroup_eq_universalSafePhi_ker`, and `universallySafeSubgroup_le_caseA`. The displayed intersection identity is their immediate mathematical corollary.

No Hopf formula, H_2, cohomology, or topological identification is used in this sharpness argument.

Proposition 4.5 (setup-independent relation-residue carrier).

The subgroup $D(R_p)$ is normal in F. Since $D(R_p) \leq R_p$, define the setup-independent carrier

$$\boxed{\mathscr{C}_{hist}(R_p) := R_p / D(R_p).} \tag{4.6g}$$

This quotient depends only on (F, R_p). Once a deformation setup $\mathscr{S}$ is fixed, the residue restriction to R_p has image in $K_{\mathscr{S}} = \ker(\mathrm{proj}_{\mathscr{S}})$, annihilates $D(R_p)$, and therefore induces a setup-dependent homomorphism

$$\boxed{\bar{\rho}_{\mathscr{S}} : \mathscr{C}_{hist}(R_p) \longrightarrow K_{\mathscr{S}}.} \tag{4.6h}$$

The carrier is canonical from (F, R_p); the induced residue map is canonical only after $\mathscr{S}$ is fixed.

Moreover, for all $x, y \in \mathscr{C}_{hist}(R_p)$,

$$x^2 = 1, xy = yx. \tag{4.6i}$$

Thus the carrier is elementary-abelian-2 in the group-theoretic sense; no scalar-action structure is required here.

Formal counterparts. `caseASubgroup_normal`, `caseAInRp`, `relationResidueKernelHom`, `HistoryResidueCarrier`, `inducedResidueHom`, `historyResidueCarrier_sq_one`, and `historyResidueCarrier_comm`.

If T4/ALG-NONDEG holds, then the setup-induced map is nontrivial:

$$T4 \Longrightarrow \bar{\rho}_{\mathscr{S}} \neq 1. \tag{4.6j}$$

Formal counterpart. `inducedResidueHom_ne_one_of_gateT4`.

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The converse is not asserted in the fully abstract setup: a nontrivial relation residue may exist in $R_p \setminus \mathrm{Adm}$. The quotient construction is direct group theory and is not identified here with a Hopf-formula homology group.

4.5. A necessary condition for residue change

The useful converse is a contrapositive, not a biconditional.

Theorem 4.6 (change requires leaving Case A). For $H, w \in F$,

$$\rho_{\mathscr{S}}(Hw) \neq \rho_{\mathscr{S}}(H) \quad \Longrightarrow \quad \boxed{w \notin D(R_p)}. \tag{4.7}$$

Proof. If $w \in D(R_p)$, Theorem 4.3 gives equality of the two residues. □

Formal counterpart. `SGB1R1DeformationSetupV1.flip_implies_not_caseA`.

This direction is essential. The reverse implication is not claimed: leaving $D(R_p)$ does not by itself force a change of residue.

4.6. Four-way classification of deformations

The Case-A subgroup becomes useful when combined with nullity and admissibility.

Definition 4.7 (Cases A–D). For $w \in F$, define

$$\begin{aligned} \mathrm{CaseA}(w) &\Longleftrightarrow w \in D(R_p), \\ \mathrm{CaseB}(w) &\Longleftrightarrow w \in R_p \cap \mathrm{Adm} \text{ and } w \notin D(R_p), \\ \mathrm{CaseC}(w) &\Longleftrightarrow w \in R_p \text{ and } w \notin \mathrm{Adm}, \\ \mathrm{CaseD}(w) &\Longleftrightarrow w \notin R_p. \end{aligned} \tag{4.8}$$

The interpretation is summarized in Table 1.

Case	Algebraic condition	Interpretation
A	$w \in D(R_p)$	relation-preserving deformation with guaranteed trivial residue
B	$w \in R_p \cap \mathrm{Adm} \setminus D(R_p)$	admissible null deformation not covered by the preservation theorem
C	$w \in R_p \setminus \mathrm{Adm}$	null deformation outside the declared admissible residue domain
D	$w \notin R_p$	not a declared relation-preserving deformation

Table 1.

Because R_p is normal and Adm contains all squares and commutators, every Case-A element is automatically both null and admissible:

$$D(R_p) \leq R_p \cap \mathrm{Adm}. \tag{4.9}$$

doi.org/10.32388/8WFJ48

Formal counterpart of (4.9). `SGB1R1DeformationSetupV1.caseASubgroup_le_bitDomain`, where `bitDomain := Rp ⊓ Adm`; its proof uses `caseA_mem_Rp` and `caseA_mem_Adm`.

Proposition 4.8 (exhaustiveness and exclusion).

Every $w \in F$ belongs to exactly one of Cases A–D.

The exhaustive part is formalized as `SGB1R1DeformationSetupV1.cases_exhaustive`; the pairwise exclusions are formalized by the corresponding `caseA_not_caseB`, `caseA_not_caseC`, `caseA_not_caseD`, `caseB_not_caseC`, `caseB_not_caseD`, and `caseC_not_caseD` theorems.

The classification is intentionally asymmetric with respect to residue change. Case A is guaranteed safe; Case B is the only admissible null region in which a change can occur, but membership in Case B is not sufficient for a change.

4.7. *Case-B localization of a residue change*

Corollary 4.9 (Case-B localization). Let $H, w \in F$. Assume

$$w \in R_p \cap \mathrm{Adm}. \tag{4.10}$$

If

$$\rho_{\mathscr{S}}(Hw) \neq \rho_{\mathscr{S}}(H),$$

then

$$\boxed{\mathrm{CaseB}(w).} \tag{4.11}$$

Proof. By Theorem 4.4, $w \notin D(R_p)$. Together with (4.10), this is exactly the definition of Case B. □

Formal counterpart. `SGB1R1DeformationSetupV1.flip_implies_caseB`.

Firewall 4.1 (one-way implication)

The converse of Corollary 4.9 is not asserted:

$$\boxed{\mathrm{CaseB}(w) \not\Longrightarrow \rho_{\mathscr{S}}(Hw) \neq \rho_{\mathscr{S}}(H).} \tag{4.12}$$

Thus Case B should be read as the region in which residue change is *not ruled out by the Case-A theorem*, not as a synonym for a flipping deformation.

A sharper refinement is available. Define

$$B_0 = \{w : \mathrm{CaseB}(w), \rho_{\mathscr{S}}(w) = 1\}, B_1 = \{w : \mathrm{CaseB}(w), \rho_{\mathscr{S}}(w) \neq 1\}. \tag{4.12a}$$

Then Case B is the disjoint union of B_0 and B_1. On the bit domain $R_p \cap \mathrm{Adm}$, Theorem 4.4 gives

$$\boxed{\rho_{\mathscr{S}}(Hw) \neq \rho_{\mathscr{S}}(H) \Leftrightarrow w \in B_1.} \tag{4.12b}$$

Thus B_0 contains Case-B differences that preserve the residue, whereas B_1 is exactly the active flip sector. In the gate terminology, a T3 witness is a B_0 witness, while a T4 witness is a B_1 witness.

doi.org/10.32388/8WFJ48

Formal counterparts. `CaseB0`, `CaseB1`, `caseB_iff_caseB0_or_caseB1`, `caseB0_not_caseB1`, `flip_iff_caseB_and_nontrivial_residue`, and `flip_iff_caseB1`.

Figure 2 records the exact Case-B split and emphasizes the claim firewall that Case B alone does not imply a residue change.

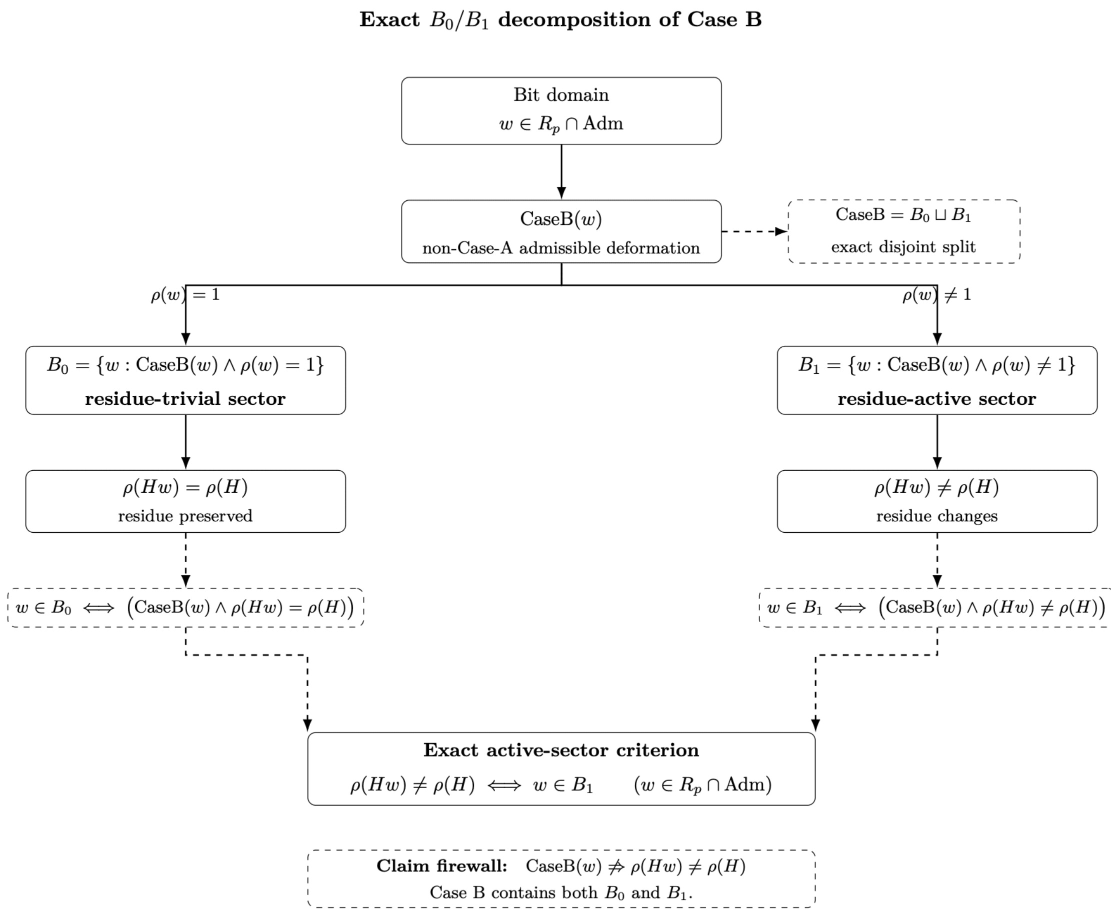

Figure 2. Exact B_0/B_1 decomposition of Case-B deformations on $R_p \cap \mathrm{Adm}$. Only B_1 is residue-active; Case B alone does not imply a residue change.

4.8. Conservation for an allowed deformation family

Let $\mathrm{Allowed} \subseteq F$ be a declared family of allowed differences.

Theorem 4.10 (allowed-deformation conservation). If

$$\forall w \in \mathrm{Allowed}, w \in D(R_p), \tag{4.13}$$

then for every $H \in F$ and every $w \in \mathrm{Allowed}$,

$$\boxed{\rho_{\mathscr{S}}(Hw) = \rho_{\mathscr{S}}(H).} \tag{4.14}$$

 doi.org/10.32388/8WFJ48

Formal counterpart. `SGB1R1DeformationSetupV1.residue_conserved_of_allowed_caseA`.

Theorem 4.10 is deliberately stronger in hypotheses than the exact criterion (4.6c): it replaces the setup-specific test $\mathrm{Allowed} \subseteq \ker \phi$ by the residue-blind sufficient condition $\mathrm{Allowed} \subseteq D(R_p)$. By the universal sharpness result, this is the largest subgroup containment guaranteed safe from (F, R_p) alone without inspecting the residue homomorphism.

4.9. Gate consequences: algebraic nondegeneracy versus conservation

The formal package also isolates the exact interface between the existence of a nontrivial residue and conservation under a declared dynamics. Define the **bit domain**

$$\mathscr{B}_{\mathscr{S}} := R_p \cap \mathrm{Adm}. \tag{4.15}$$

In Lean this is `bitDomain := Rp ⊓ Adm`. Equation (4.9) is therefore the inclusion $D(R_p) \leq \mathscr{B}_{\mathscr{S}}$.

The nonvacuous domain gate T2 and the algebraic nondegeneracy gate T4 are

$$T2: \quad \mathscr{B}_{\mathscr{S}} \neq \{1\}, T4: \quad \exists w \in \mathscr{B}_{\mathscr{S}}, \rho_{\mathscr{S}}(w) \neq 1. \tag{4.16}$$

The formal theorem `gateT2_iff_exists_ne_one` rewrites T2 as the existence of a nonidentity element in the bit domain. T4 is exactly the algebraic nondegeneracy condition used in this paper. It has three immediate consequences in the formal package: a T4 witness is outside Case A (`gateT4_witness_not_caseA`), T4 implies T2 (`gateT4_implies_gateT2`), and T4 realizes two distinct residue values (`gateT4_two_residues`). Since a T4 witness already lies in $R_p \cap \mathrm{Adm}$, it is therefore a Case-B element.

A complementary gate T3 is useful for the one-way firewall. It requires a witness in the same bit domain that lies outside Case A but nevertheless has trivial residue. The theorem `gateT3_witness_caseB` shows that such a witness is Case B. Thus, the formal gate layer contains an explicit algebraic schema in which

$$\boxed{CaseB(w) \quad \text{and} \quad \rho_{\mathscr{S}}(w) = 1} \tag{4.17}$$

can coexist. This is the gate-level counterpart of Firewall 4.1: Case-B membership is not itself a residue-flip criterion.

Now define conservation on a declared family $\mathrm{Allowed} \subseteq F$ by

$$\mathrm{ConservationOnAllowed}(\mathrm{Allowed}) :\Longleftrightarrow \forall H \in F, \forall w \in \mathrm{Allowed}, \rho_{\mathscr{S}}(Hw) = \rho_{\mathscr{S}}(H). \tag{4.18}$$

The GateT1T4 module proves the following exact compatibility statement.

Corollary 4.11 (allowed-family consequence).

By the exact family-level criterion (4.6c), conservation on a declared family is equivalent to

$$\mathrm{Allowed} \subseteq \ker \phi.$$

Hence, if w_* is a T4/ALG-NONDEG witness with $\rho_{\mathscr{S}}(w_*) \neq 1$, then every conserving allowed family must exclude w_*.

This does not conflict with PHYS-NONDEG: nondegeneracy concerns the existence of distinct sectors, while PRES constrains which allowed paths can connect them; see Section 8.3.

4.10. Nondegeneracy ladder

The preservation theorems are valid even when the residue carrier is trivial. We therefore distinguish four logically different levels:

$$\begin{aligned} STRUCT: &\ K \leq Z(G), k^2 = 1 \forall k \in K, \\ CARRIER: &\ K \neq 1, \\ ALG\text{-}NONDEG: &\ \exists w \in R_p \cap \mathrm{Adm} \text{ with } \rho_{\mathscr{S}}(w) \neq 1, \\ PHYS\text{-}NONDEG: &\ \begin{array}{c}\text{actual physical eligible histories/deformations} \\ \text{realize distinct residue sectors.}\end{array} \end{aligned} \tag{4.20}$$

These implications must not be collapsed:

$$\boxed{CARRIER \not\Rightarrow ALG\text{-}NONDEG \not\Rightarrow PHYS\text{-}NONDEG.} \tag{4.21}$$

Section 6 establishes STRUCT, CARRIER, and an explicit ALG-NONDEG witness in the $A_5/2I$ case. PHYS-NONDEG is outside the scope of the present paper.

5. Freeze-compatible free-group interface

5.1. Primitive alphabet and free deformation group

The abstract setup of Section 3 can be instantiated from a primitive alphabet X. Let

$$F_X := F(X)$$

be the free group on X. The role of X is operational: its elements are the primitive deformation symbols whose status is frozen before the residue outcome is inspected.

A label assignment

$$\lambda : X \to Q \tag{5.1}$$

extends uniquely to a homomorphism

$$\mathrm{lab}_\lambda : F(X) \to Q. \tag{5.2}$$

Likewise, a letterwise lift

$$L : X \to G \tag{5.3}$$

extends uniquely to

$$\phi_L : F(X) \to G. \tag{5.4}$$

These are `sgB1R1LabHomV1` and `sgB1R1PhiHomV1` in the formal development.

The free-group construction is used here as a freeze-compatible interface, not as an assumption in Theorems 4.2–4.10 or Proposition 4.11. Once a deformation setup has been constructed, the abstract results no longer depend on freeness.

5.2. Frozen relation generators and the null subgroup

Let

$$R_{gen} \subseteq F(X)$$

be a frozen set of declared relation generators. Define

$$\boxed{R_p := \text{normalClosure}(R_{gen}).} \tag{5.5}$$

This is `sgB1R1RpV1`.

The algebraic statement that R_p is normal is automatic from (5.5). The physical interpretation is not. The statement that the physically null deformations are *exactly* those generated normally by the frozen set R_{gen} is a model-selection assertion and is not proved by the group-theoretic construction.

This distinction is useful because it prevents a physical nullity assumption from being hidden inside an algebraic lemma.

5.3. Canonical admissibility benchmark

For a group F, let

$$\mathscr{A}_{gen}(F) = \{g^2 : g \in F\} \cup \{[a, b] : a, b \in F\}. \tag{5.6}$$

Define

$$\boxed{\text{Adm}_{\min}(F) := \langle \mathscr{A}_{gen}(F) \rangle.} \tag{5.7}$$

The formal implementation is `sgB1R1AdmV1`. It is the canonical minimal subgroup satisfying the abstract admissibility assumptions of Definition 3.1.

For a free group, this subgroup is naturally described as $F^2[F, F]$, equivalently the kernel of mod-two abelianization, but the formally verified preservation results in this paper do not depend on that identification. We therefore use (5.7) as the primary definition.

The canonical mathematical benchmark $\text{Adm}_{\min}$ must not be silently identified with an eventual physical admissibility subgroup Adm_{phys}. The latter remains part of the physical preregistration problem.

5.4. Generator-level reduction of G1

A practical advantage of the free-group instantiation is that G1 can be checked on the frozen relation generators rather than on the full normal closure.

Proposition 5.1 (G1 from frozen generators). Assume

$$lab_\lambda(r) = 1 \qquad \text{for every } r \in R_{gen}. \tag{5.8}$$

Then

$$R_p \leq \ker(\text{lab}_\lambda). \tag{5.9}$$

Proof idea. The kernel of a homomorphism is normal. Since it contains every element of R_{gen}, it contains the normal closure of R_{gen}. □

Formal counterpart. `sgB1R1G1FromGeneratorsV1`.

 doi.org/10.32388/8WFJ48

Thus, the algebraic part of G1 is reduced to a generator-level test. No enumeration of the entire normal closure is required.

5.5. Letterwise reduction of compatibility

Similarly, compatibility of the lifted word homomorphism with the downstairs labeling reduces to the primitive letters.

Proposition 5.2 (compatibility from letters). If

$$\mathrm{proj}(L(x)) = \lambda(x) \qquad (x \in X), \tag{5.10}$$

then

$$\mathrm{proj}(\phi_L(f)) = \mathrm{lab}_\lambda(f) \qquad (f \in F(X)). \tag{5.11}$$

Proof idea. Both sides of (5.11) are homomorphisms from the free group $F(X)$ to Q, and they agree on every free generator. □

Formal counterpart. `sgB1R1CompatFromLettersV1`.

5.6. Construction of the fixed deformation setup

Suppose the following data have been fixed:

- a primitive alphabet X;
- a frozen relation set $R_{gen} \subseteq F(X)$;
- a label assignment $\lambda : X \to Q$;
- a projection $\mathrm{proj} : G \to Q$;
- a letter lift $L : X \to G$;
- the letterwise covering condition (5.10);
- the generator-level nullity condition (5.8);
- centrality and exponent-two hypotheses for $\ker(\mathrm{proj})$.

Then these data determine a deformation setup in the sense of Definition 3.1 by setting

$$\phi = \phi_L, lab = \mathrm{lab}_\lambda, R_p = \mathrm{normalClosure}(R_{gen}), \mathrm{Adm} = \mathrm{Adm}_{\min}. \tag{5.12}$$

The formal constructor is `sgB1R1SetupOfPreregistrationV1`.

Consequently, all results of Section 4 apply automatically after the finite list of interface conditions has been discharged.

5.7. What the formal freeze-compatible interface does and does not certify

The interface is preregistration-ready but separates algebraic verification from any procedural claim that a physical preregistration has already occurred.

It **does** certify the following mathematical reductions:

1. label-nullity of the frozen relation generators implies G1 for the full normal closure;
2. letterwise lift compatibility implies global compatibility on the free group;

3. the canonical admissibility subgroup contains all squares and commutators;
4. the resulting data instantiate the Case A–D deformation setup.

It **does not** certify the following procedural or physical assertions:

1. that the alphabet X was selected before examining residue outcomes;
2. that R_{gen} was selected without post-selection;
3. that $\mathrm{normalClosure}(R_{gen})$ is the actual physical null subgroup;
4. that a physical dynamics realizes all, or only, the encoded free-group words;
5. that the physical admissibility domain equals $\mathrm{Adm}_{\min}$.

These are deliberately separated because otherwise a [P]-layer choice could be hidden in an [M]-layer proof.

6. The A5/2I case study

6.1. Concrete central double cover

We now specialize the abstract setup to the standard finite groups

$$G = SL(2, \mathbb{F}_5), Q = PSL(2, \mathbb{F}_5), \tag{6.1}$$

with the canonical quotient projection

$$\mathrm{proj} : SL(2, \mathbb{F}_5) \to PSL(2, \mathbb{F}_5). \tag{6.2}$$

The formal development contains a certificate identifying the quotient group as an A_5 base model and packages (6.2) as a concrete non-split central double cover. The role of this section is not to re-prove the classical abstract theory of the binary icosahedral cover, but to provide a concrete, formally checked instance of the deformation framework.

The kernel is exactly

$$K = \{I, -I\}. \tag{6.3}$$

The nontrivial element

$$z := -I \tag{6.4}$$

satisfies

$$z \neq I, z^2 = I, zg = gz \quad (g \in G). \tag{6.5}$$

The order-two central element in (6.5) reflects the same double-cover algebraic mechanism that underlies the familiar central sign in $SU(2) \to SO(3)$: a nontrivial central element squares to the identity. We use this only as an algebraic analogy. The relation $AB = zBA$ below concerns the relative lifts of two commuting downstairs elements and is not identified here with a $2\pi/4\pi$ rotation defect or with any specific continuum-defect process [6][7].

Thus,

$$\boxed{K \cong \mathbb{Z}_2.} \tag{6.6}$$

 doi.org/10.32388/8WFJ48

This is the point at which the abstract central two-torsion residue becomes genuinely binary.

Formal counterparts. `sgB1R1PSL2F5IsA5BaseModelV1`, `sgB1R1SL2ToPSL2KernelExactV1`, `sgB1R1SL2CenterMinusOneNeOneV1`, `sgB1R1SL2CenterMinusOneSquaredV1`, and `sgB1R1SL2CenterMinusOneCentralV1`.

Remark 6.1 (surjectivity gates). The abstract preservation theorems in Section 4 do not require either proj or lab to be surjective. In the present concrete cover, proj is surjective. A separate specialization gate, T1, controls whether the chosen labels generate the entire A_5-model downstairs. These conditions should not be conflated.

6.2. A concrete V4/Q8 carrier

The formal package contains a commuting pair of distinct nontrivial involutions downstairs. Together with the identity and their product, they form a four-element Klein carrier. The distinguished lifts used throughout the construction are the explicit matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \in SL(2, \mathbb{F}_5), \tag{6.7a}$$

with

$$z = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.7b}$$

All entries are understood modulo five. The formal matrix core verifies $\det A = \det B = 1$ and

$$A^2 = B^2 = z. \tag{6.7c}$$

These are `sgB1R1LiftAV1` / `sgB1R1SL2LiftAV1` and `sgB1R1LiftBV1` / `sgB1R1SL2LiftBV1` in the formal package. The cross relation $AB = zBA$ is recorded in Section 6.3; with (6.7a)–(6.7b) it can also be checked directly by multiplication in $\mathbb{F}_5$.

At the level needed here, the key formal result states that the complete inverse image of the projected four-element carrier is exactly an eight-element quaternionic carrier upstairs:

$$\mathrm{proj}^{-1}(V_4) = Q_8^{\mathrm{carrier}}. \tag{6.7}$$

This result supplies a concrete finite setting in which the binary kernel is realized nontrivially. It is stronger than merely observing that the covering kernel has two elements: it exhibits a specific relation-bearing configuration whose lifts detect the central sign.

There is a classical physical precedent for the inverse-image pattern itself. In biaxial-nematic defect theory, the rotational residual subgroup $D_2 \cong V_4 \subset SO(3)$ lifts through $SU(2) \to SO(3)$ to the quaternion group Q_8, giving the familiar Q_8 defect algebra [6][7]. Equation (6.7) should therefore not be read as a new physical classification of biaxial-nematic defects. Its role here is to formalize an abstract central-double-cover inverse-image pattern inside the finite $A_5/2I$ case. We also do not use the standard conjugacy-class classification of Q_8 defect charges; the Case A–D classification of Section 4 is instead defined by membership relative to R_p, $D(R_p)$, and Adm.

Formal counterpart. `sgB1R1ConcreteV4PreimageEqualsQ8CarrierV1`.

We emphasize that (6.7) is a finite group-theoretic statement. No topological bundle or surface obstruction is used in its proof.

6.3. Presentation-level torus relation obstruction

Let $a, b \in Q$ be the commuting projected generators and let $A, B \in G$ be the distinguished lifts. Downstairs,

$$ab = ba. \tag{6.8}$$

Upstairs, the distinguished lifts satisfy

$$\boxed{AB = zBA,} \tag{6.9}$$

where $z = -I \neq I$.

Moreover, the formal development proves that **no simultaneous choice of lifts of the two downstairs elements can preserve the commuting relation upstairs**. In other words, the relation

$$[u, v] = 1 \tag{6.10}$$

is valid for the projected pair, but no simultaneous pair of lifts above that pair realizes the same commuting relation.

Formal counterparts. `sgB1R1ConcreteTorusRelationDownstairsV1`, `sgB1R1ConcreteNoTorusRelationLiftV1`, and `sgB1R1ConcreteTorusCentralResidueV1`.

Non-Abelian defect theory provides a neighboring physical motif: when defect charges take values in a noncommutative fundamental group, their composition and crossing behavior can depend on path history, and charges are naturally tracked only up to conjugacy in the appropriate setting [6][8]. The present torus-relation witness is not identified with that continuum theory. The comparison is motivational: both settings exhibit a failure of naive path-independent composition, while the present theorem is formulated purely in terms of a declared relation subgroup and a central residue.

Firewall 6.1 (presentation level only)

The terminology "torus relation" refers here only to the group presentation with commuting generators. The formal result does **not** identify the construction with a formalized fundamental group of T^2, a CW attaching map, a principal-bundle obstruction, or a generator of $H^2(T^2; \mathbb{Z}/2)$. Such identifications would require a separate topology bridge.

6.4. Embedding the torus relation into the deformation setup

To connect the presentation-level obstruction to the deformation classification, take the two-letter alphabet

$$X_T = \{u, v\}$$

and the free group

$$F_T = F(X_T).$$

Let

doi.org/10.32388/8WFJ48

$$w_T = [u,v] = uvu^{-1}v^{-1}. \tag{6.11}$$

Freeze the singleton relation set

$$R^T_{gen} = \{w_T\}, \tag{6.12}$$

and define

$$R^T_p = \mathrm{normalClosure}(R^T_{gen}), \mathrm{Adm}^T = \mathrm{Adm}_{\min}(F_T). \tag{6.13}$$

Map the primitive letters to the commuting downstairs pair and to their distinguished upstairs lifts:

$$\lambda_T(u) = a, \lambda_T(v) = b, L_T(u) = A, L_T(v) = B. \tag{6.14}$$

The downstairs commutation relation implies

$$\mathrm{lab}_{\lambda_T}(w_T) = 1, \tag{6.15}$$

so the singleton generator passes G1. The concrete cover satisfies the centrality and exponent-two requirements, and the letter lifts cover the labels. Hence (6.12)–(6.14) instantiate the fixed mathematical deformation setup of Section 5.

This two-letter instantiation is deliberately local. The downstairs labels a, b generate the Klein four subgroup

$$\langle a, b\rangle \cong V_4,$$

so T1—generation of the full A_5 base—is **not** satisfied by the two-letter alphabet $\{u, v\}$. The torus witness at this stage is therefore an embedded V_4/Q_8-local obstruction inside the ambient $A_5/2I$ central cover. The next subsection globalizes the label set without changing the local witness word $[u, v]$.

The bridge then provides the two required domain memberships:

$$\boxed{w_T \in R^T_p,} \tag{6.16}$$

because the relation generator belongs to its normal closure, and

$$\boxed{w_T \in \mathrm{Adm}^T,} \tag{6.17}$$

because w_T is itself a commutator.

Formal counterparts. `sgB1R1TorusCommutatorMemRpV1` and `sgB1R1TorusCommutatorMemAdmV1`.

6.5. A5-globalized fixed mathematical instantiation

Extend the primitive alphabet to three letters

$$X_{T,A5} = \{u, v, c\}, \tag{6.14a}$$

while retaining the original torus relation generator $w_T = [u, v]$. Keep the lifts A, B from Section 6.2 and add

$$C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{F}_5), \tag{6.14b}$$

with c labeled by its projective image. The local u, v subsystem is unchanged, so

$$\rho([u, v]) = -I. \tag{6.14c}$$

Theorem 6.2 (full-A_5 generation with retained local residue witness). For the three-letter labeling $\lambda_{T,A5}$,

$$\boxed{\langle \lambda_{T,A5}(u), \lambda_{T,A5}(v), \lambda_{T,A5}(c) \rangle = PSL(2,5) \cong A_5.} \tag{6.14d}$$

Equivalently, the induced free-group labeling homomorphism is surjective. At the same time, the original word $[u,v]$ remains in $R_p \cap \mathrm{Adm}$, has residue $-I \neq 1$, and lies in Case B.

The formal proof avoids a 60-element global enumeration. In the generated subgroup, the projective A-letter has order 2, the added C-letter has order 5, and AC has order 3. Hence 30 divides the subgroup order, while Lagrange gives divisibility by 60. Order 30 would give an index-two normal subgroup of the simple group $PSL(2,5)$, which is impossible; hence the subgroup has order 60 and is the whole base group.

Formal counterparts. `sgB1R1TorusA5GlobalGateT1V1`, `sgB1R1TorusA5GlobalLabSurjectiveV1`, `sgB1R1TorusA5GlobalCommutatorResidueV1`, `sgB1R1TorusA5GlobalALGNondegV1`, `sgB1R1TorusA5GlobalCommutatorCaseBV1`, and `sgB1R1TorusA5GlobalWitnessIntegrityV1`.

The obstruction witness itself remains local to the V_4/Q_8 carrier; T1 says that the same fixed mathematical history alphabet generates the full ambient A_5 model.

Theorem 6.2a (exact commuting-obstruction locus).

Let $a, b \in PSL(2,5) \cong A_5$ commute. Then

$$\boxed{\neg \mathrm{CommutingLiftExists}(a,b) \Leftrightarrow \mathrm{KleinPair}(a,b).} \tag{6.14e}$$

Here $\mathrm{KleinPair}(a,b)$ means

$$a \neq 1, b \neq 1, a \neq b, a^2 = b^2 = 1, ab = ba.$$

The forward implication is the BUILD-VERIFIED locality theorem from v5.2: every commuting pair is either of Klein-four type or cyclic-related; cyclic-related pairs admit explicit simultaneous commuting lifts, so obstruction forces the Klein-four alternative.

The converse uses a different mechanism. In the concrete upstairs group $SL(2,5)$, every element X satisfying $X^2 = 1$ is either 1 or the distinguished central element $-I$. Therefore a lift A of a nontrivial base involution must satisfy $A^2 = -I$, and similarly $B^2 = -I$. If lifts of a Klein pair commuted, then

$$(AB)^2 = A^2 B^2 = (-I)(-I) = 1.$$

Hence $AB = \pm I$, so projectively $ab = 1$. Since $b^2 = 1$, this forces $a = b$, contradicting Klein-pair distinctness.

Formal counterparts. `sgB1R1SL2UniqueInvolutionNativeFactsV1`, `sgB1R1LiftOfNontrivialBaseInvolutionSquaresToMinusOneV1`, `sgB1R1ConcreteKleinPair_implies_noCommutingLiftV1`, and `sgB1R1ConcreteNoCommutingLift_iff_kleinPairV1`.

The converse is driven by the upstairs unique-involution mechanism, while the forward implication uses the A_5-specific commuting-pair dichotomy. Thus, inside the concrete A_5 base,

$$\boxed{\text{commuting-lift obstruction locus} = \text{Klein-four locus.}} \tag{6.14f}$$

This is not a claim that A_5 is the only group with a V_4/Q_8 lift pattern, nor is it a topological or physical identification.

The strengthened concrete description is

$$\boxed{\text{global } A_5 \text{ history grammar} + \text{exact } V_4/Q_8 \text{ commuting-obstruction locus.}} \tag{6.14g}$$

Theorem 6.2b (lift-independent commutator pairing and exact support).

For $a, b \in PSL(2,5) \cong A_5$, choose arbitrary lifts $A, B \in SL(2,5)$ and define

$$\kappa(a,b) := [A,B] = ABA^{-1}B^{-1}. \tag{6.14h}$$

If the lifts are changed by central signs,

$$A \mapsto zA, B \mapsto z'B, z, z' \in \ker(2I \to A_5),$$

then centrality gives

$$[zA, z'B] = [A,B].$$

Hence $\kappa(a,b)$ is independent of the lift representatives.

For a commuting downstairs pair $ab = ba$,

$$\kappa(a,b) \in \ker(2I \to A_5) = \{1, -I\}. \tag{6.14i}$$

Moreover,

$$\boxed{\kappa(a,b) = 1 \Leftrightarrow \text{CommutingLiftExists}(a,b).} \tag{6.14j}$$

Combining this with Theorem 6.2a gives

$$\boxed{\kappa(a,b) \neq 1 \Leftrightarrow \text{KleinPair}(a,b),} \tag{6.14k}$$

and therefore, because the kernel is binary,

$$\boxed{\kappa(a,b) = -I \Leftrightarrow \text{KleinPair}(a,b).} \tag{6.14l}$$

Thus, after restricting the domain to ordered commuting pairs,

$$\boxed{\text{supp}_{\text{Comm}}\kappa = \text{Klein-four locus.}} \tag{6.14m}$$

Here $\text{supp}_{\text{Comm}}\kappa$ denotes the nontrivial support of κ on the commuting-pair domain, i.e., the set of ordered commuting pairs for which $\kappa(a,b) \neq 1$.

Formal counterparts. `sgB1R1ConcreteLiftCommutatorIndependentV1`, `sgB1R1ConcreteCommutatorKappa_eq_anyLiftsV1`, `sgB1R1ConcreteCommutatorKappa_projectsToOneV1`, `sgB1R1ConcreteCommutatorKappa_binaryV1`, `sgB1R1ConcreteCommutatorKappa_eq_one_iff_commutingLiftV1`, `sgB1R1ConcreteCommutatorKappa_ne_one_iff_kleinPairV1`, `sgB1R1ConcreteCommutatorKappa_eq_minusOne_iff_kleinPairV1`, and `sgB1R1ConcreteCommutingPairKappaSupport_eq_kleinLocusV1`.

doi.org/10.32388/8WFJ48

The commutator pairing itself is standard central-extension structure. The paper-specific result is the exact determination of its nontrivial support on the commuting-pair domain in the frozen $A_5/2I$ realization.

Remark 6.2c (general mechanism versus the A_5-specific step). The converse direction of Theorem 6.2a reflects a more general mechanism: in a central double cover whose upstairs group has a unique nontrivial involution, lifts of nontrivial downstairs involutions must square to that central element, so a Klein-four pair cannot admit simultaneous commuting lifts. What is specific to the present A_5 realization is the forward classification that every commuting obstruction is forced onto the Klein-four locus.

This remark belongs to the concrete binary-cover layer. It is not promoted to Section 4, where the abstract kernel is only assumed central of exponent at most two and need not have order two.

Figure 3 summarizes the concrete double-cover mechanism, lift-choice independence of κ, and the exact identification of its restricted commuting-pair support with the Klein-four locus.

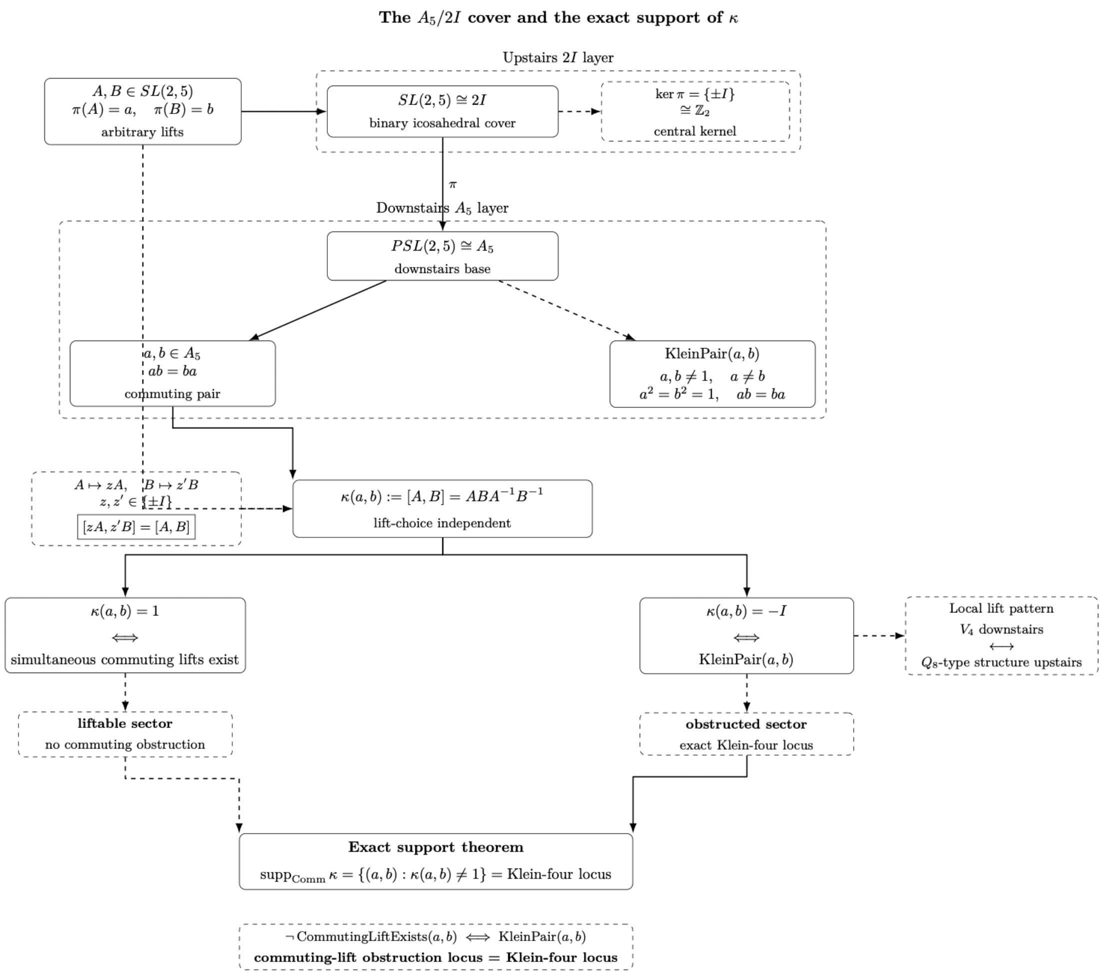

Figure 3. Exact support of the lift-independent commutator pairing for the $A_5/2I$ cover. On commuting pairs, $\kappa = -I$ exactly on the Klein-four locus, equivalently the commuting-lift obstruction locus.

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6.6. Exact residue of the torus commutator

Equation (6.9) determines the lifted commutator exactly. Indeed,

$$ABA^{-1}B^{-1} = z. \tag{6.18}$$

Consequently, the residue of the free-group torus word in the fixed mathematical deformation setup is

$$\boxed{\rho_{\mathscr{S}_T}(w_T) = z = -I.} \tag{6.19}$$

Since $z \neq 1$, this residue is nontrivial.

Formal counterparts. `sgB1R1TorusCommutatorPhiV1` and `sgB1R1TorusCommutatorResidueV1`.

Equation (6.19) is the bridge between the concrete matrix relation and the abstract deformation algebra. It is also the point at which the abstract possibility of a nontrivial central carrier becomes an explicit nontrivial algebraic history-deformation witness.

6.7. Explicit algebraic nondegeneracy

Theorem 6.3 (explicit $A_5/2I$ algebraic nondegeneracy). For the torus deformation setup $\mathscr{S}_T$, there exists an admissible null deformation with nontrivial residue:

$$\boxed{\exists w \in R_p^T \cap \mathrm{Adm}^T \quad \text{such that} \quad \rho_{\mathscr{S}_T}(w) \neq 1.} \tag{6.20}$$

Proof. Take $w = w_T$. Memberships (6.16) and (6.17) place w_T in the admissible null domain. Equation (6.19) gives

$$\rho_{\mathscr{S}_T}(w_T) = z,$$

and $z \neq 1$ by (6.5). □

Formal counterpart. `sgB1R1TorusALGNondegV1`.

Theorem 6.3 establishes ALG-NONDEG for the mathematical $A_5/2I$ case study. It rules out the possibility that the abstract preservation theory is nonempty only because the kernel was allowed to be trivial or because the chosen relation sector happened to map trivially into a nontrivial kernel.

It does **not** establish PHYS-NONDEG. The subgroup R_p^T in this theorem is the normal closure of a deliberately chosen mathematical torus relation; it is not identified with an actual physical null subgroup.

6.8. Explicit Case-B witness

The same deformation is not Case A.

Corollary 6.4 (explicit Case-B witness).

For the torus deformation setup,

$$\boxed{CaseB(w_T).} \tag{6.21}$$

Proof. We already know that $w_T \in R_p^T \cap \mathrm{Adm}^T$. If $w_T \in D(R_p^T)$, Theorem 4.2 would imply $\rho_{\mathscr{S}_T}(w_T) = 1$, contradicting (6.19) and $z \neq 1$. Hence $w_T \notin D(R_p^T)$, which is precisely Case B. □

Formal counterpart. `sgB1R1TorusCommutatorCaseBV1`.

This corollary should be read together with Firewall 4.1. The existence of an explicit Case-B word does not mean that every Case-B word flips a residue, nor does it imply that any physically allowed deformation realizes this particular word.

6.9. Status of the nondegeneracy ladder in the concrete case

The mathematical status reached by the end of this section is summarized in Table 2.

Layer	$A_5/2I$ **case-study status**	**Meaning**
STRUCT	established	the kernel is central and of exponent two
CARRIER	established	$K = \{I, -I\} \cong \mathbb{Z}_2$
ALG-NONDEG	established	$w_T \in R_p^T \cap \mathrm{Adm}^T$ has residue $-I$
explicit Case B	established	the torus commutator lies outside $D(R_p^T)$
PHYS-NONDEG	open	no actual physical history domain is identified here
physical preservation	open	no actual physical Allowed family is shown to lie in Case A
cosmological initial bit	open	no such identification is made in this paper

Table 2.

The central firewall is therefore

$$\boxed{\mathit{ALG\text{-}NONDEG} \neq \mathit{PHYS\text{-}NONDEG} \neq \text{cosmological initial bit.}} \tag{6.22}$$

The result of this section is a nontrivial algebraic binary residue and a concrete Case-B witness inside a fixed mathematical deformation setup. The witness is local to an embedded V_4/Q_8 carrier, while the globalized three-letter labeling satisfies T1 and generates the full A_5 base. Physical realization is a separate problem.

6.10. Scope of the case study

The concrete $A_5/2I$ example serves four purposes.

First, it specializes the abstract central two-torsion carrier to an exact binary carrier. Second, it demonstrates that the algebraic nondegeneracy condition is actually realized by an explicit relation rather than being a merely formal possibility. Third, it provides a finite formally checked test case for the Case A–D classification. Fourth, it places the finite example next to a classical physical motif—the V_4/Q_8 lift pattern and non-Abelian defect composition in ordered media—without identifying the two theories [6][7][8].

 doi.org/10.32388/8WFJ48

What it does not do is equally important. In particular, the existence of a D_2/Q_8 defect structure in biaxial media does not establish a physical realization of the $A_5/2I$ branch considered here. It does not derive the primitive alphabet of an actual physical dynamics; it does not derive a physical null subgroup; it does not show that the torus relation is a physically executable update; and it does not establish a persistent physical history sector. Those questions are deferred to the preregistered physical realization protocol developed later in the paper.

doi.org/10.32388/8WFJ48

Formal cross-reference for Sections 3–6

Manuscript item	Lean declaration
Definition 3.1	`SGB1R1DeformationSetupV1`
Proposition 3.2	`SGB1R1DeformationSetupV1.residue_mem_ker`
Definition 4.1	`SGB1R1DeformationSetupV1.caseASubgroup`
Theorem 4.2	`SGB1R1DeformationSetupV1.residue_caseA`
Theorem 4.3	`SGB1R1DeformationSetupV1.caseA_preserves_residue`
Theorem 4.4	`residue_mul_eq_base_iff, residue_mul_ne_base_iff`
Proposition 4.5	`caseASubgroup_normal, HistoryResidueCarrier, inducedResidueHom, historyResidueCarrier_sq_one, historyResidueCarrier_comm`
Theorem 4.6	`SGB1R1DeformationSetupV1.flip_implies_not_caseA`
Definition 4.7	`CaseA, CaseB, CaseC, CaseD`
Proposition 4.8	`SGB1R1DeformationSetupV1.cases_exhaustive and pairwise exclusion theorems`
Corollary 4.9	`SGB1R1DeformationSetupV1.flip_implies_caseB`
Case-B refinement	`CaseB0, CaseB1, flip_iff_caseB1`
Theorem 4.10	`SGB1R1DeformationSetupV1.residue_conserved_of_allowed_caseA`
Eq. (4.9) / bit-domain inclusion	`SGB1R1DeformationSetupV1.caseASubgroup_le_bitDomain`
T2 domain gate	`SGB1R1DeformationSetupV1.gateT2_iff_exists_ne_one`
T4 nondegeneracy consequences	`gateT4_witness_not_caseA, gateT4_implies_gateT2, gateT4_two_residues`
T3 Case-B/trivial-residue witness	`SGB1R1DeformationSetupV1.gateT3_witness_caseB`
Proposition 4.11	`SGB1R1DeformationSetupV1.gateT4_and_conservation_forces_avoidance`
Proposition 5.1	`sgB1R1G1FromGeneratorsV1`
Proposition 5.2	`sgB1R1CompatFromLettersV1`
Freeze-compatible setup constructor	`sgB1R1SetupOfPreregistrationV1`
Binary kernel	`sgB1R1SL2ToPSL2KernelExactV1 and support theorems`
explicit lifts A, B	`sgB1R1LiftAV1, sgB1R1LiftBV1, sgB1R1SL2LiftAV1, sgB1R1SL2LiftBV1`
V4/Q8 carrier	`sgB1R1ConcreteV4PreimageEqualsQ8CarrierV1`

Manuscript item	Lean declaration
Downstairs torus relation	`sgB1R1ConcreteTorusRelationDownstairsV1`
No commuting lift	`sgB1R1ConcreteNoTorusRelationLiftV1`
Central residue relation	`sgB1R1ConcreteTorusCentralResidueV1`
Torus word in R_p	`sgB1R1TorusCommutatorMemRpV1`
Torus word in Adm	`sgB1R1TorusCommutatorMemAdmV1`
Exact torus residue	`sgB1R1TorusCommutatorResidueV1`
Theorem 6.3	`sgB1R1TorusALGNondegV1`
Corollary 6.4	`sgB1R1TorusCommutatorCaseBV1`
Theorem 6.2 / global T1	`sgB1R1TorusA5GlobalGateT1V1`
Global lab surjectivity	`sgB1R1TorusA5GlobalLabSurjectiveV1`
Global witness integrity	`sgB1R1TorusA5GlobalWitnessIntegrityV1`

Table 3.

7. Encoding and lift-choice auxiliary results

The preservation theorems of Sections 3–4 are carrier-level statements formulated in terms of a deformation setup $\mathscr{S}$. They do not require a choice of set-theoretic section of a central extension, nor do they require a positive-word encoding. The present section addresses a different question: **when a concrete history word is evaluated by choosing representatives upstairs, which changes of that auxiliary choice leave the resulting lift unchanged?**

This distinction is important. Choice-independence is not automatic, and it is not equivalent to the Case-A preservation theorem. The formal package contains both positive and negative results that delimit the exact scope currently verified.

7.1. *Group-element-indexed section choice*

Let

$$p : G \to Q$$

be a group homomorphism and let

$$s : Q \to G$$

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be a set-theoretic section,

$$p(s(q)) = q \qquad (q \in Q). \tag{7.1}$$

For a positive word

$$w = (q_1, \dots, q_n) \in List(Q),$$

define its section lift by the ordered product

$$L_s(w) := s(q_1) \cdots s(q_n). \tag{7.2}$$

This is the object implemented by `sgB1R1WordLiftV1`. The indexing set here is **the downstairs group Q itself**. Thus the formal result in this subsection must not be conflated with an arbitrary primitive alphabet X, nor with the letterwise lift $L : X \to G$ used in the freeze-compatible interface of Section 5.

Suppose now that Z is a commutative group embedded centrally in G, that its image lies in $\ker p$, and that

$$z^2 = 1 \qquad (z \in Z). \tag{7.3}$$

A function

$$b : Q \to Z$$

changes the section to

$$s_b(q) := \iota(b(q))s(q). \tag{7.4}$$

The formal change-of-section theorem gives

$$\boxed{L_{s_b}(w) = \iota\left(\prod_{q_i \in w} b(q_i)\right) L_s(w).} \tag{7.5}$$

More explicitly, the central correction is the ordered-list product of the values of b along the word. This statement is `sgB1R1TwistedWordLiftV1`.

Equation (7.5) supplies the basic warning for the rest of this section: a word lift is generally **choice dependent**. A central extension by itself does not make a history-word residue canonical.

Firewall 7.1 — section choice is not graph switching

The freedom in (7.4) is a choice of representative for each element of Q. It is not the vertex switching operation of gain-graph theory, where a function on graph vertices modifies edge gains by a gauge transformation. The two operations have different domains and should not be identified.

Likewise, (7.4) is not the same object as a primitive-letter lift

$$L : X \to G$$

in Section 5. The formal package contains results for both styles of encoding, but they occur in different modules and under different hypotheses.

doi.org/10.32388/8WFJ48

7.2. Positive-word parity invariance

The correction factor in (7.5) disappears when every downstairs group element occurs an even number of times in the positive word.

Proposition 7.2 (positive-word count-parity invariance). Assume (7.3). Let $w \in \mathrm{List}(Q)$ satisfy

$$\forall q \in Q, \mathrm{count}_w(q) \equiv 0 \pmod 2. \tag{7.6}$$

Then for every section twist $b : Q \to Z$,

$$\boxed{L_{s_b}(w) = L_s(w).} \tag{7.7}$$

The formal counterpart is `sgB1R1CountParityWordResidueInvarianceV1`.

The earlier doubled-word theorem is a special case. If w is a permutation of

$$w' + +w',$$

then every multiplicity is even, and hence (7.7) follows. This relation between the older doubled-word hypothesis and the stronger count-parity formulation is formalized by `sgB1R1PermDoubledViaCountParityV1`.

For positive words, condition (7.6) is naturally read as triviality in the mod-two abelianization of the free group on symbols indexed by Q. The present paper uses this observation only as interpretation. The verified theorem is the concrete count-parity statement above; no Hopf-formula formalization is required for it.

Scope of Proposition 7.2

The carrier type is `List Q`, hence **positive words only**. Formal inverses are not represented as signed letters. Consequently, Proposition 7.2 must not be silently promoted to a theorem about arbitrary reduced words in a free group. In particular, a commutator

$$x_a x_b x_a^{-1} x_b^{-1}$$

belongs to the mod-two abelianization kernel, but its four displayed signed letters are not a doubled positive list. A signed-word extension is a separate formalization problem.

7.3. Closure alone is not sufficient

The parity hypothesis is not a cosmetic strengthening of downstairs closure. The package contains a concrete negative witness.

In the $SL(2, \mathbb{F}_5) \to PSL(2, \mathbb{F}_5)$ model, let A, B be the concrete lifts used in Section 6 and set $C = AB$. Consider the positive three-letter word represented downstairs by

$$([A], [B], [C^{-1}]). \tag{7.8}$$

It is closed downstairs. Under one section choice its lift product is $+I$; after changing the sign of the representative of $[A]$ alone, the same downstairs word lifts to $-I$. Therefore

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$$\boxed{\text{downstairs closure does not imply section-independent residue.}} \tag{7.9}$$

This is the content of `sgB1R1ClosedWordResidueNotSignInvariantV1`.

The same module gives a positive companion: the even-multiplicity word

$$([A],[B],[A],[B]) \tag{7.10}$$

has the same nontrivial residue $-I$ for every tested sign choice, formalized by `sgB1R1EvenWordResidueConstantV1`, with nontriviality recorded by `sgB1R1EvenWordResidueNontrivialV1`.

The paired positive and negative results delimit the current conclusion sharply:

$$\boxed{\text{choice-independence is purchased by a parity condition, not by closure alone.}} \tag{7.11}$$

This is an encoding-level statement. It is auxiliary to, and logically distinct from, the Case-A deformation theorem of Section 4.

Section 6 supplies a complementary positive result. Although closure of a general word does not remove section dependence, the commutator of a commuting downstairs pair is invariant under every central change of lift:

$$[zA, z'B] = [A, B]. \tag{7.11a}$$

Thus the paper contains both sides of the boundary:

$$\boxed{\text{general closed-word residue can be section-dependent}}$$

but

$$\boxed{\text{commuting-pair commutator residue is lift-choice independent.}} \tag{7.11b}$$

The difference is structural: central factors cancel inside the group commutator. This positive statement is formalized by `sgB1R1ConcreteLiftCommutatorIndependentV1`.

7.4. Refinement invariance for admissible positive histories

A second encoding issue arises when a primitive move is subdivided into finer moves. Let X and X' be alphabets and let

$$rf : X \to \mathrm{List}(X')$$

replace each source letter by a word in the refined alphabet. The induced refinement of a positive history $h \in \mathrm{List}(X)$ is obtained by concatenation.

The formal package first proves that even-count admissibility is preserved by arbitrary such letterwise refinement:

$$[\forall x \in X, \mathrm{count}_h(x) \equiv 0 \pmod 2] \implies \left[\forall y \in X', \mathrm{count}_{rf(h)}(y) \equiv 0 \pmod 2\right]. \tag{7.12}$$

This is `sgB1R1RefinePreservesEvenCountV1`.

Now let

$$L : X \to G, L' : X' \to G$$

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be letterwise lifts. Assume that after refining a source letter, the refined lift agrees with the source lift up to a central exponent-two discrepancy:

$$L'(rf(x)) = \iota(d(x))L(x), d(x) \in Z. \tag{7.13}$$

Here $L'(rf(x))$ denotes the ordered product of the refined letter lifts.

Proposition 7.3 (refinement invariance).

If every source letter occurs an even number of times in h, then

$$\boxed{L'(rf(h)) = L(h).} \tag{7.14}$$

This is `sgB1R1RefinementResidueInvarianceV1`.

The mechanism is the same as in Proposition 7.2. Every local refinement discrepancy is central and of exponent two; even global multiplicity cancels the discrepancies pairwise.

This result supports a limited encoding-robustness statement: **within the positive-word, even-count domain and under a label-compatible refinement satisfying (7.13), subdividing primitive moves does not change the computed lift.** It does not prove that every physically meaningful change of primitive alphabet is such a refinement.

7.5. Auxiliary-status firewall

The results of this section are useful but deliberately excluded from the main theorem spine for four reasons.

First, the section-choice theorems are stated for group-element-indexed positive words, while the preregistered physical interface starts from a primitive alphabet X. Second, signed-word generalization remains outside the verified scope. Third, section twisting, primitive-letter lift choice, and graph switching are different operations. Fourth, the physical status of a primitive alphabet is a modeling decision rather than a theorem of the auxiliary algebra.

The build-provenance status of these auxiliary modules is tracked separately from that of the main theorem spine. Historical header annotations or unresolved release-engineering provenance in an auxiliary module do not alter the formal dependency of Theorems 1–4 or of the torus-deformation bridge.

Accordingly, we use the following implication structure only:

$$\boxed{\text{specified parity/refinement hypotheses} \Longrightarrow \text{specified encoding invariance.}} \tag{7.15}$$

We do **not** claim the converse in full generality, nor do we claim that actual physical histories satisfy the auxiliary hypotheses.

8. Physical realization protocol: an open bridge

Sections 3–7 establish an algebraic preservation framework and a concrete nontrivial $A_5/2I$ algebraic witness. They do not identify that witness with a physically realized history sector. This section records the preregistered bridge that would be required for such an identification.

doi.org/10.32388/8WFJ48

The purpose of including the protocol in the paper is methodological: the physical eligibility criteria and the preservation test are to be fixed **before** observing whether the desired binary residue survives. The section therefore reports a test architecture, not a positive physical result.

8.1. Physical data that remain to be frozen

An actual physical application requires a local dynamics

$$\Phi = (\mathrm{State}, X, \mathrm{LegalMove}, \mathrm{next}), \tag{8.1}$$

where X is a typed primitive move alphabet and legality may depend on the current state. The present paper does not select or freeze a final physical move catalog; any such choice belongs to the downstream physical preregistration rather than to the algebraic result.

In addition, a physical run must specify, independently of the residue outcome,

$$R_{phys}, \mathrm{Adm}_{phys}, \mathrm{Allowed}_{phys}. \tag{8.2}$$

These objects play different roles:

- R_{phys} declares which history differences are physically null/relation-preserving;
- Adm_{phys} declares the physical domain on which a central residue is considered an admissible sector observable;
- $\mathrm{Allowed}_{phys}$ specifies actual dynamical deformations.

For a genuinely local dynamics, the last item need not be state-independent. One may instead use $\mathrm{Allowed}_H \subseteq F$ or a relation $\mathrm{Allowed}(H, w)$. Theorem 4.4 applies pointwise,

$$\rho(Hw) = \rho(H) \Leftrightarrow \rho(w) = 1,$$

so state dependence belongs to the physical bridge rather than to the algebraic definition of the preservation kernel.

The canonical mathematical choice

$$\mathrm{Adm}_{\min} = \langle \text{squares and commutators} \rangle \tag{8.3}$$

used in the algebraic case study is not automatically identified with Adm_{phys}.

Firewall 8.1

The explicit torus relation of Section 6 proves algebraic nondegeneracy for a mathematical relation choice. It does not prove that the same relation belongs to the actual physical null set.

8.2. Bit-blind physical domain and relative evaluation fiber

The preregistration protocol separates the physical domain from the residue-evaluation domain.

First define a bit-blind predicate

$$\mathrm{PhysicalEligible}(H), \tag{8.4}$$

without reference to the central sign, the desired NONDEG outcome, or a later preservation result. The physical history domain is

$$\mathscr{H}_{phys} = \{H : \mathrm{PhysicalEligible}(H)\}. \tag{8.5}$$

The first-run protocol then uses a frozen reference history $H_0 \in \mathscr{H}_{phys}$ and a base descriptor

$$\beta(H) := (p(H), \nu(H)), \tag{8.6}$$

where $p(H)$ is the coarse A_5 image and $\nu(H)$ is the parity vector associated with the frozen primitive alphabet. The evaluation fiber is

$$\mathscr{H}_{H_0} = \{H \in \mathscr{H}_{phys} : \beta(H) = \beta(H_0)\}. \tag{8.7}$$

The relative observable is then defined, when the algebraic carrier connection is available, by

$$\varepsilon_{rel}(H) := \varepsilon(HH_0^{-1}). \tag{8.8}$$

This construction avoids shrinking the entire physical ontology to a single residue-compatible fiber. The fiber is an evaluation domain inside the independently frozen physical domain.

Firewall 8.2

$$\boxed{\mathscr{H}_{phys} \neq \mathscr{H}_{H_0} \quad \text{in general.}} \tag{8.9}$$

The residue definition is not allowed to determine `PhysicalEligible` retrospectively.

8.3. Required physical gates

The first-run physical test is ordered as

$$\boxed{FREEZE \to NAT \to PHYS\text{-}CLOSURE \to FIBER\text{-}CLOSURE \to PHYS\text{-}NONDEG \to PRES.} \tag{8.10}$$

The order matters: a later positive residue result may not be used to revise an earlier ontology choice.

NAT — result-blind naturality audit

NAT is a procedural audit that checks that the history type, physical eligibility rule, allowed-step rule, reference history, scope, and primitive ontology were frozen without forbidden reference to the residue result. It includes pre-label digesting and a forbidden-term/rationale audit. NAT is therefore not merely an unrestricted proposition supplied after the fact.

PHYS-CLOSURE — closure of the physical domain

Let AllowedStep(H,H′) be a bit-blind transition predicate. The physical domain must satisfy

$$\mathrm{PhysicalEligible}(H) \wedge \mathrm{AllowedStep}(H, H') \implies \mathrm{PhysicalEligible}(H'). \tag{8.11}$$

Failure here is a mismatch between the declared physical scope and the proposed dynamics; it is not evidence for or against a history bit.

FIBER-CLOSURE — closure of the evaluation fiber

For the single-reference first run, one additionally asks

$$H \in \mathscr{H}_{H_0} \wedge \mathrm{AllowedStep}(H, H') \Longrightarrow H' \in \mathscr{H}_{H_0}. \tag{8.12}$$

If (8.12) fails, it does not by itself constitute preservation failure. It means that the dynamics crosses the chosen β-fiber, so the claim must be restricted to within-fiber evolution or a separately preregistered multi-fiber extension must be used.

PHYS-NONDEG — physical realization of both residue sectors

The physical nondegeneracy test is

$$\boxed{\exists H_1, H_2 \in \mathscr{H}_{H_0}, \varepsilon_{rel}(H_1) \neq \varepsilon_{rel}(H_2).} \tag{8.13}$$

This is stronger than the algebraic nondegeneracy result of Section 6 because the witnesses must be actual physically eligible histories in the frozen evaluation fiber.

PRES — preservation under actual allowed evolution

Define a fiber step by requiring both endpoints to lie in $\mathscr{H}_{H_0}$. PRES asks

$$\boxed{\mathrm{FiberStep}_{H_0}(H, H') \Longrightarrow \varepsilon_{rel}(H) = \varepsilon_{rel}(H').} \tag{8.14}$$

For release, local edge preservation is intended to be accompanied by the corresponding reachable-path statement: histories with different relative residues are not connected by an allowed path inside the evaluation fiber. This is the physical counterpart of Section 4: nondegeneracy supplies multiple sectors, while preservation constrains which transitions can connect them.

8.4. Algebraic PASS does not imply physical PASS

The status ladder is therefore

$$\boxed{STRUCT \to CARRIER \to ALG\text{-}NONDEG \to PHYS\text{-}NONDEG \to PRES.} \tag{8.15}$$

The $A_5/2I$ formal case study establishes the first three levels in the mathematical model: the kernel is central of exponent two, it is a nontrivial binary carrier, and the torus deformation gives an explicit algebraically nontrivial witness. It does **not** establish either of the final two physical levels.

Accordingly,

$$\boxed{ALG\text{-}NONDEG \neq PHYS\text{-}NONDEG \neq \text{cosmological initial bit.}} \tag{8.16}$$

Even a successful PHYS-NONDEG and PRES test would initially justify only the wording **preserved nondegenerate relative history-sector bit candidate** within the frozen physical model. An identification with a cosmological initial choice would require additional assumptions concerning initial accessibility, inheritance, persistence, and physical generation.

doi.org/10.32388/8WFJ48

8.5. The finite-state issue is logically separate

The abstract results of Sections 3–4 make no global finiteness assumption on the state space. A history may be a finite resolved trajectory even when the ambient state space is infinite.

The physical program originally isolated a cardinality question because the candidate update grammar can change the number of local defects or sites. The present design therefore audits several distinct finiteness levels rather than assuming a global finite universe from the outset:

$$\boxed{H < X < L < G,} \tag{8.17}$$

where, schematically,

- H: individual histories are finite;
- X: the primitive move alphabet is finite;
- L: the transition structure is locally finite;
- G: the entire state space is globally finite.

The open logical question is which downstream statements require which level. In particular, global Fintype State may be needed for an exhaustive executable census without being logically necessary for the residue-preservation theorem itself.

Thus no value of a global $N_{\max}$ is assumed in the verified core of this paper. If a finite audit window is later introduced for computation, its computational role must be distinguished from a physical theorem that the universe-level state space is globally finite.

8.6. Post-realization quotient and compression tests

The preregistered program contains later quotient tests, but they are downstream of an actual Φ freeze and are not part of the present verified result.

For a fixed nontrivial quotient candidate q, one may ask whether the coarse dynamics descends,

$$q \circ T = \overline{T} \circ q, \tag{8.18}$$

by explicitly constructing $\overline{T}$. Such descent alone is not evidence that the residue is informationally rich: a constant quotient may satisfy (8.18) trivially. The quotient and its domain therefore have to be preregistered and nontrivial on that domain.

A subsequent sufficient-compression audit may ask whether the relative residue is constant on a proposed quotient kernel. These tests belong to a later quotient/compression phase. They do not enter the proof of Theorems 1–5 and are not used to select the physical ontology.

8.7. Release criterion for the physical branch

The protocol permits a positive physical-history-sector statement only if the result-blind ontology audit and both closure gates pass, both relative residue sectors are physically realized, and the residue is preserved under the frozen allowed

 doi.org/10.32388/8WFJ48

dynamics.

In schematic form,

$$\boxed{NAT \wedge PHYS\text{-}CLOSURE \wedge FIBER\text{-}CLOSURE \wedge PHYS\text{-}NONDEG \wedge PRES} \tag{8.19}$$

is the first-run release condition for a preserved nondegenerate relative history-sector bit candidate.

No such physical PASS is claimed in the present paper. The purpose of Section 8 is to make the missing bridge explicit and falsifiable while keeping the algebraic results of Sections 3–7 independent of its outcome.

9. Relation to existing frameworks and claim boundary

This section positions the present construction relative to several established literatures. The purpose is primarily negative and delimiting: the paper does not introduce central extensions, gain graphs, partition refinement, predictive state minimization, quantum histories, or blind analysis. Its narrower contribution is to combine a parametric admissible-relation deformation setup with a central exponent-two residue, to isolate exact preservation and commuting-obstruction criteria, and to exhibit a formally checked nontrivial $A_5/2I$ instance. The physical realization problem remains open.

9.1. Central extensions, Schur-type constructions, and the binary icosahedral case

Central extensions and their relation to second homology are classical; standard references include Hopf's original second-homology formula and Brown's treatment of group cohomology [9][10]. Modern treatments of universal and relative central extensions continue to formulate the kernel of the universal construction homologically [11]. Nothing in the present paper is intended to replace that theory.

Likewise, the group-theoretic identification used in the concrete case is classical. Standard treatments of finite simple groups include the isomorphism $PSL(2,5) \cong A_5$, while the binary icosahedral cover is realized by $SL(2,5)$ with central subgroup $\{\pm I\}$ of order two [12][13]. Thus the specialization

$$1 \longrightarrow \{\pm 1\} \longrightarrow 2I \longrightarrow A_5 \longrightarrow 1$$

is background structure, not a novelty claim of this work.

The paper also deliberately avoids claiming a new Hopf-formula theorem or a formalized homology–cohomology naturality theorem. The verified main spine uses only the elementary consequences of two properties of the kernel $K = \ker(\mathrm{proj})$: centrality and exponent at most two. The homological language explains why the denominator-like subgroup is natural, but it is not required for the formal preservation results proved here.

A particularly close literature match to the mod-two numerator/denominator pattern is now available. Cruz, Ortega, and Segovia define $S(F) = \langle x^2 : x \in F \rangle$ and obtain the Hopf-type formula

$$\mathscr{M}(G;\mathbb{Z}_2) \cong \frac{S(F) \cap R}{[F,R]R^2} \tag{9.1}$$

doi.org/10.32388/8WFJ48

for their unoriented Schur multiplier [14]. Because the subgroup generated by squares contains the commutator subgroup, this has the same numerator/denominator pattern as $R \cap F^2[F,F]$ over $R^2[F,R]$ in the notation used in the present paper. This correspondence is important bibliographically: the denominator structure is not new. At the same time, Cruz–Ortega–Segovia identify their multiplier with the cohomology group $H^2(G;\mathbb{Z}_2)$. The present paper does **not** identify that cohomological object with a homological $H_2(G;\mathbb{F}_2)$ object, does not formalize (9.1) in Lean, and does not use such an identification in Theorems 4.2–4.10 or Proposition 4.11.

The point of the present construction is therefore not the existence of a central extension. It is the use of a separately declared normal relation subgroup R_p, constrained by $R_p \leq \ker(lab)$, as the deformation domain on which a central residue is tested. Changing R_p changes the source-side quotient and the collection of relation classes under consideration even though the target extension is held fixed.

9.2. Gain and voltage graphs

Gain and voltage graphs provide a natural neighboring language for transition systems whose oriented edges are labeled by group elements [15][16]. In the signed-graph special case, Harary's balance notion is an early canonical reference [17]. In gain-graph language, reversing an oriented edge inverts its gain, and a closed circle is balanced when the ordered product of its gains is the identity [16]. This is directly relevant when an actual dynamics Φ is represented by a transition graph Γ_Φ together with a group-valued edge labeling.

The present algebraic core, however, is not identified with gain-graph theory. Its basic data are the group homomorphisms and subgroups in the deformation setup

$$\mathscr{S} = (F, G, Q, \phi, \mathrm{proj}, \mathrm{lab}, R_p, \mathrm{Adm}),$$

and the main theorems do not require that F be a graph fundamental group or even a free group. A graph becomes relevant only in a downstream realization where a concrete transition system supplies histories and legal moves.

A second distinction concerns changes of representation. Gain-graph switching is a vertex-based gauge transformation. By contrast, the auxiliary Lean results discussed in Section 7 concern choices of representatives or lifts of symbols in a central extension. These are different operations and are not identified in this paper.

Accordingly, we do not claim a new theory of gain graphs, voltage graphs, balance, switching, deletion, contraction, or graph quotients. Gain-graph language is used only where it accurately describes a possible transition-graph realization of the algebraic data.

9.3. State aggregation, lumpability, bisimulation, and partition refinement

The general problem of reducing a state space while preserving prescribed behavior is well established. Burke and Rosenblatt studied conditions under which a function of a Markov chain remains Markovian [18], while Kemeny and Snell give the classical finite-Markov-chain treatment of lumpability [19]. In process semantics, bisimulation belongs to the established concurrency literature, with Milner's CCS providing a standard early reference [20]. In algorithmic settings,

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Paige and Tarjan developed efficient partition-refinement procedures for relational coarsest-partition problems [21]. These traditions make clear that "find the coarsest quotient preserving a property" is not, by itself, a new problem schema.

The compression questions deferred to later stages of the present program should therefore be viewed as instances or relatives of established aggregation and refinement problems. The special feature here is the observable to be preserved: a central-extension residue attached to admissible relation histories, rather than a transition probability law or a generic behavioral equivalence.

The current paper does **not** prove that its residue-preserving quotient is the coarsest such quotient, does not provide a new partition-refinement algorithm, and does not claim that the Case A–D classification is a replacement for bisimulation or lumpability. The formal contribution is upstream of that optimization problem: it identifies a sufficient deformation class on which the residue is forced to be constant and a necessary condition for a residue change.

9.4. Causal states and sufficient compression

Computational mechanics supplies another important comparison. Crutchfield and Young introduced statistical-complexity constructions that became foundational to the later causal-state program [22]. Shalizi and Crutchfield subsequently formulate causal states by grouping histories with the same predictive consequences and establish minimality and uniqueness properties for the resulting predictive representation [23]. This provides an established example in which histories are quotiented by a task-specific equivalence relation to obtain a minimal sufficient representation.

Our later phrase "minimal sufficient compression" should be read in that broad methodological sense, not as a claim to have generalized causal-state theory. The sufficient object in computational mechanics is predictive: it preserves the conditional distribution of future behavior. The object considered here is algebraic: it preserves a central residue on an admissible relation sector. These are different target observables and different ambient structures.

For that reason, the present paper stops before a minimality theorem. It provides the residue and deformation predicates that a future minimization procedure would have to preserve. Existing partition-refinement or sufficient-representation machinery may then be reusable, but that algorithmic reduction is not part of the verified claim set here.

9.5. Histories, discrete covariance, topological sectors, and quantum cosmology

The use of histories as primary objects has several established precedents in quantum foundations and quantum cosmology. Griffiths' consistent-histories formalism and the Gell-Mann–Hartle decoherent-histories program organize quantum alternatives as histories rather than as a single sequence of externally measured states [1][24]. Hartle's spacetime formulation makes the connection to quantum cosmology especially explicit: a generalized quantum theory is specified by fine-grained histories, allowed coarse grainings of those histories, and a decoherence functional, with general relativity treated through four-dimensional histories of geometry and matter [2]. Sorkin's quantum-measure formulation likewise places the generalized measure on a history space [3].

The present usage is substantially narrower. A finite resolved history here is an operationally specified finite trajectory or word in a transition/deformation system. No decoherence functional, Hilbert-space projector algebra, quantum-probability consistency condition, or quantum-measure sum rule is assumed. The algebraic residue is therefore not a consistent-histories probability and is not proposed as a replacement for histories-based quantum mechanics. The relevant inheritance from this literature is only the methodological possibility that a physical question may naturally be posed on classes of histories rather than on instantaneous endpoints alone.

9.5.1. Causal-set sequential growth: discrete histories and covariance

A closer structural precedent is supplied by causal-set dynamics. Rideout and Sorkin formulate classical sequential growth as a stochastic growth process of finite causal sets and impose a discrete form of general covariance together with causality conditions [4]. In that setting, different natural labelings or birth orders can describe the same unlabeled causal set, so the dynamics must not assign physical significance to purely label-dependent differences. The central problem is therefore already deformation-like: which distinctions introduced by a discrete growth history are physical and which are artifacts of representation?

The comparison with the present paper is useful but not an identification. Causal-set discrete general covariance removes label-dependent distinctions associated with different growth descriptions of the same causet. Our setup instead begins with an independently declared relation subgroup R_p, an admissibility domain, and a central extension, and asks whether a deformation changes a residue in $K = \ker(\mathrm{proj})$. Thus

$$\boxed{\text{causal-set covariance: discard unphysical growth-label distinctions}}$$

and

$$\boxed{\text{present problem: determine which declared history deformations preserve a central residue}}$$

are neighboring questions with different observables and different algebraic data. No causal-set dynamics is constructed or derived in this paper.

9.5.2. Topological geons and mapping-class-group sectors

Topological-geon quantum gravity provides a second, conceptually deep precedent. For a fixed spatial topology, nontrivial large diffeomorphisms lead to a mapping class group whose inequivalent unitary representations define distinct quantum sectors; Sorkin and Surya analyze the resulting geon statistics and sector structure [5]. Related work on spin and statistics emphasizes that nontrivial spatial topology can support half-odd-integral spin and nonstandard statistics even though the underlying gravitational field is bosonic [25].

This literature makes precise a general lesson relevant here: agreement of a coarse geometric configuration need not exhaust all information carried by the route through configuration space. In the geon setting that extra structure is topological and representation-theoretic, arising from the mapping class group. In the present setting the extra structure is deliberately more elementary and finite: it is a residue of a declared deformation in a central extension.

Accordingly, we do **not** claim that the residue of Sections 3–6 is a discretization of a geon θ-sector, a representation of a spatial mapping class group, or a homotopy invariant of a gravitational configuration space. Establishing such an equivalence would require a separate topology-to-deformation bridge that is absent here. The geon literature is therefore a conceptual precedent for endpoint-insufficient sector information, not a proof of the present $A_5/2I$ mechanism.

9.5.3. Spin foams, triangulation changes, and local move invariance

Spin-foam and state-sum approaches supply a third neighboring question: how should amplitudes behave when a discrete spacetime history is changed by local moves? In the three-dimensional Ponzano–Regge model, a properly regularized state sum can be triangulation independent, and local Pachner moves provide the natural comparison between different triangulations [26]. More generally, the behavior of spin-foam amplitudes under Pachner moves is used to probe discretization dependence and the restoration or failure of topological invariance.

The analogy to the present Case-A problem is limited but structurally useful. Pachner-type moves ask whether a physical amplitude is unchanged under a specified local change of discrete history or triangulation. The present theorem asks whether a central residue is unchanged under a specified algebraic deformation. In schematic form,

$$\text{local discrete move} \longrightarrow \text{invariance test}$$

is common to both. The invariant itself, the ambient category, and the required consistency conditions are different. We do not identify Case A with Pachner equivalence, nor do we construct a spin-foam amplitude.

9.5.4. Binary icosahedral cosmic topology as a group-specific precedent

There is also a group-specific cosmological precedent for $2I$. The Poincaré dodecahedral space is the spherical space form associated with the binary icosahedral group; it has been studied as a possible finite cosmic topology, including in connection with large-angle cosmic-microwave-background correlations [27][28]. In that literature the binary icosahedral group acts as a deck/holonomy group of the spatial 3-manifold.

This precedent is relevant because it shows that $2I$ is not foreign to foundational cosmology. It does **not**, however, support the specific history-residue mechanism of this paper. The two uses of $2I$ are logically distinct:

$$\boxed{\text{Poincar cosmic topology: } 2I \text{ characterizes a spatial quotient/topology}}$$

whereas

$$\boxed{\text{present case study: } 1 \to \mathbb{Z}_2 \to 2I \to A_5 \to 1 \text{ supplies a central residue carrier.}}$$

No claim is made that physical space is $S^3/2I$, that the torus commutator of Section 6 is a loop in the Poincaré dodecahedral space, or that observational work on dodecahedral cosmology constitutes evidence for the physical NONDEG/PRES gates of Section 8.

9.5.5. Synthesis and claim boundary

Taken together, these literatures establish substantial prior art for four ingredients that motivate the present problem:

doi.org/10.32388/8WFJ48

histories as primary alternatives	(histories-based quantum cosmology),
discrete growth with representation-independence	(causal sets),
residual sectors beyond a coarse configuration label	(topological geons),
invariance under declared local changes	(spin foams/state sums).

The contribution claimed here is not any one of these ideas in isolation. In its present form, it has four mathematical pillars:

1. declared separation of R_p, Adm, and $\mathrm{Allowed}$;

an exact setup-specific preservation kernel together with the setup-independent sharpness identity

$$D(R_p) = \bigcap_{\mathscr{S}' \in \mathfrak{S}(F, R_p)} \ker \phi_{\mathscr{S}'};$$

1. the setup-independent carrier $R_p/D(R_p)$, its setup-induced residue map, and the exact B_0/B_1 refinement;
2. a full-A_5 generating history labeling together with a lift-independent binary commutator pairing whose nontrivial support is exactly the Klein-four commuting-obstruction locus in the $SL(2,5) \to PSL(2,5)$ cover.

The strict ALG-NONDEG/PHYS-NONDEG/PRES firewall is retained. The exact relation of this finite algebraic interface to any specific quantum-cosmological dynamics remains an open bridge.

9.6. Blind analysis and pre-result freezing

Blind analysis is a well-established strategy for reducing experimenter bias in data analysis. Roodman's particle-physics account and the later review by Klein and Roodman describe procedures in which analysts deliberately prevent themselves from knowing the answer while selection and analysis choices are being fixed [29][30]. MacCoun and Perlmutter argue for broader use of the same bias-control logic beyond particle physics [31]. The methodological analogy to the present protocol is intentional.

Here, the object being blinded is not an experimental measurement. Instead, the primitive alphabet, relation/nullity rule, admissibility convention, carrier choice, and pass/fail criteria are to be frozen before the target residue or physical verdict is inspected. This is best understood as an adaptation of the logic of blind analysis to a theory-construction workflow.

We make no claim that preregistration of theoretical algebraic model construction is itself new, nor do we claim that the protocol eliminates all researcher degrees of freedom. Its narrower role is auditable: it records which definitions were fixed before NONDEG/PRES or compression outcomes were used to evaluate the branch.

9.7. Non-Abelian defect topology in ordered media

The topology of defects in ordered media supplies a direct neighboring physical literature for the concrete V_4/Q_8 material used in Section 6. Toulouse identified the biaxial-nematic order-parameter structure associated with the residual D_2 rotational symmetry, and Mermin's general review places such classifications in the homotopy theory of defects [6][7]. In this setting, the lift from the rotational $D_2 \cong V_4$ subgroup to the double cover is quaternionic, giving the standard Q_8 defect algebra. Poenaru and Toulouse further analyzed the crossing of defects in noncommutative settings, where path

doi.org/10.32388/8WFJ48

history and conjugacy enter the composition of defect charges [8]. Modern reviews of nematic defect topology provide broader context for these homotopy-based classifications [32].

This literature changes the interpretation of Section 6 in an important but limited way. The V_4/Q_8 inverse-image pattern is not itself a new physical structure introduced by the present paper; it is a classical motif in defect topology. What is specific to the present construction is the deformation architecture built around a separately declared normal relation subgroup R_p, an admissibility subgroup Adm, the denominator $D(R_p) = R_p^2[F, R_p]$, the Case A–D classification, and the preregistered separation between algebraic and physical nondegeneracy. Those ingredients are not identified here with the conjugacy-class classification of non-Abelian defect charges.

The comparison also has a strict scope boundary. Ordered-media defect theory begins from an order-parameter manifold and its homotopy groups; the present algebraic core begins from group homomorphisms and a declared relation subgroup and explicitly refrains from constructing a continuum order-parameter space. Accordingly, the existence of physical D_2/Q_8 defect phenomena does not establish a physical realization of the $A_5/2I$ history-residue branch. The analogy supplies motivation and prior art, not a PHYS-NONDEG result.

9.8. What is and is not claimed as the contribution

The literature above motivates a deliberately narrow claim boundary.

The following are **not** claimed as new:

1. central extensions, Schur multipliers, or the binary icosahedral cover of A_5;
2. group-labeled or gain-labeled graphs and their balance language;
3. state aggregation, lumpability, bisimulation, or coarsest partition refinement as general methodologies;
4. causal-state or minimal-sufficient predictive representations;
5. consistent/decoherent or sum-over-histories quantum cosmology as a framework;
6. causal-set sequential growth and discrete general covariance;
7. topological-geon sectors and mapping-class-group representations in quantum gravity;
8. Pachner-move or triangulation-invariance questions in spin-foam/state-sum models;
9. Poincaré dodecahedral cosmic topology or the use of $2I$ as a spatial deck/holonomy group;
10. blind analysis as a bias-control methodology;
11. the topological classification of non-Abelian defects in biaxial nematics or other ordered media.

The contribution claimed here is instead the following four-part combination:

1. declared separation of R_p, Adm, and $\mathrm{Allowed}$;
2. exact setup-specific preservation together with universal residue-blind sharpness of $D(R_p)$;
3. the setup-independent carrier $R_p/D(R_p)$, its setup-induced residue map, and the exact B_0/B_1 classification;
4. full-A_5 generation together with the exact Klein-four commuting-obstruction locus in the binary icosahedral cover.

doi.org/10.32388/8WFJ48

The underlying proofs are elementary; the intended value is the reduction, the explicit claim boundary, and the formally checked integration of these ingredients. In the limited literature survey conducted for this draft, which was not systematic or exhaustive, we did not identify this exact combination as a previously named framework. This observation is not an exhaustive novelty theorem and should not be presented as one.

10. Formal verification and provenance

The algebraic development underlying this paper is implemented in Lean. This section separates three different claims that are easy to conflate: (i) what statements are represented in source code, (ii) what static source hygiene has been audited, and (iii) what has been reported to compile in the accepted project workspace.

10.1. Formal source set

The final direct SGB1R1/history-residue source set contains eighteen Lean files. They comprise 5,193 source lines and 209 declarations introduced with the theorem keyword. Static audit of the accepted direct sources gives sorry=0, no explicit axiom declarations, and no admit tactic; the only literal admit match in the obstruction-locality source is ordinary English in a comment.

These counts are provenance data, not mathematical results. The seventeen direct files are distinct from the complete buildable project snapshot, which also contains the required local dependency, the aggregator, and Lake environment metadata.

The formal material is divided into a main spine and auxiliary support.

Main spine

The main paper claims are traced primarily to:

- `SGB1R1CaseClassificationV1.lean`;
- `SGB1R1ExactFlipCarrierV1.lean`;
- `SGB1R1UniversalSafeSectorV1.lean`;
- `SGB1R1PreregistrationInterfaceV1.lean`;
- `SGB1R1SL2F5ConcreteBridgeV1.lean`;
- `SGB1R1TorusRelationObstructionV1.lean`;
- `SGB1R1TorusDeformationBridgeV1.lean`;
- `SGB1R1TorusA5GlobalWitnessV1.lean`;
- `SGB1R1A5CommutingObstructionLocalityV1.lean`;
- `SGB1R1A5CommutingObstructionExactLocusV1.lean`;
- `SGB1R1A5CommutatorPairingSupportV1.lean`.

doi.org/10.32388/8WFJ48

Auxiliary verification

The remaining modules support section/representative-choice parity, closed-word counterexamples, count-parity bridges, refinement invariance, the explicit V_4/Q_8 matrix core, and project adapters. Some auxiliary modules have narrower domains than the main deformation theorem, for example positive-word restrictions in the section-choice analysis.

10.2. Statement-to-Lean traceability

The paper maintains a one-to-one traceability table between its formal statements and Lean theorem identifiers. In particular, the verified claim set is restricted to:

1. Case-A denominator annihilation;
2. residue preservation under Case-A deformations;
3. the necessary condition that any residue-changing deformation lies outside Case A;
4. the one-way Case-B localization under the additional $R_p \cap \mathrm{Adm}$ domain condition;
5. preservation under an allowed family contained in the Case-A denominator;
6. the exact flip criterion and the B_0/B_1 refinement of Case B;
7. the normal quotient carrier $R_p/D(R_p)$, its exponent-two/commutative structure, and the induced residue homomorphism;
8. the gate-level compatibility results that T4 realizes algebraic nondegeneracy, that a T3 witness can be Case B with trivial residue, and that conservation on `Allowed` forces every allowed difference to have trivial residue;
9. the concrete $A_5/2I$ torus witness together with BUILD-VERIFIED full-A_5 T1 generation.

The directionality is part of the formal claim. In particular, Case B is **not** asserted to imply residue change. The GateT1T4 results sharpen rather than weaken this firewall: they formally separate Case-B membership, nontrivial-residue realization, and conservation of a physically declared allowed family.

10.3. What is not formally claimed

Several mathematically natural extensions are intentionally excluded from the verified claim set.

First, the paper does not claim a Lean formalization of the Hopf formula. The notation $R_p^2[F, R_p]$ is used because it is structurally suggestive, but the preservation theorem is proved directly from centrality and exponent two.

Second, no explicit homological pushforward

$$H_2(G_R; \mathbb{F}_2) \to H_2(A_5; \mathbb{F}_2)$$

or homology–cohomology pairing naturality theorem is claimed as formalized here.

Third, the torus relation is not identified in Lean with a generator of $H^2(T^2; \mathbb{Z}_2)$, a principal-bundle obstruction, or a theorem about the topology of a torus CW complex. The existing formal files explicitly maintain this firewall.

doi.org/10.32388/8WFJ48

Fourth, the formal package does not establish PHYS-NONDEG, PRES for an actual physical dynamics, or a cosmological initial bit. Those are downstream tests described in Section 8.

10.4. Build provenance

The final self-contained v5.5 repair snapshot is

`A5_History_v55_SUPPORT_FIREWALL_REPAIR_CANDIDATE_2026-08-15.zip`

with SHA256

`18070a8a84442a195acb08a89693eb180fcbf3a3f560e22533165ff2a8525fab`.

The author reported a successful full lake build of this exact integrated snapshot on August 15, 2026. It is therefore treated here as **BUILD VERIFIED by author report**. The assistant did not independently execute Lean in the author's environment, and no raw full-build log is embedded in the present manuscript package unless separately archived.

The integrated aggregator A5.lean has SHA256

`52e54c389828997323d88bdff9473fa2383331fe7c868540b940679c35644c18`.

The project root directly contains `A5/`, `A5.lean`, `lakefile.lean`, `lean-toolchain`, and `lake-manifest.json`. The toolchain and pinned package environment are unchanged from the preceding build-verified snapshot.

The final direct paper-specific accounting is 18 SGB1R1 source files, 5,193 source lines, and 209 declarations introduced with the theorem keyword.

The v5.5 repair adds the literal restricted-domain support theorem `sgB1R1ConcreteCommutingPairKappaSupport_eq_kleinLocusV1` and repairs source comments so that Hopf/H_2/p_* language is clearly marked as external interpretation rather than Lean-established content.

Static audit of the accepted direct source set gives `sorry=0`, no `admit` tactic, and no explicit axiom declarations.

10.5. Reproducibility role of the formalization

The Lean development is used here primarily as a claim-control device. It prevents the paper from silently strengthening a one-way implication, replacing an arbitrary admissible subgroup by a convenient special case, or identifying an algebraic relation obstruction with a topological statement that has not been formalized.

The intended reproducibility object is therefore not merely a collection of proofs. It is the tuple

$$(\text{paper statement}, \text{Lean identifier}, \text{source digest}, \text{build provenance}),$$

which allows later revisions to distinguish a change of exposition from a change of mathematical content.

11. Discussion

11.1. What the preservation theorem accomplishes

The main theorem is deliberately simple. Once a relation element maps into a central kernel of exponent at most two, its square is annihilated and its commutator with an arbitrary element is annihilated. The subgroup generated by those operations is therefore residue-trivial. The preservation theorem is essentially the multiplicative consequence of that observation.

The usefulness of the result lies less in proof depth than in the shape of the reduction. A potentially broad question—whether detailed histories carry a conserved relation-sensitive distinction—is decomposed into auditable predicates:

$$R_p \lhd F, R_p \leq \ker(\mathrm{lab}), K \leq Z(G), K^2 = 1,$$

followed by membership in

$$D(R_p) = \langle r^2, [f, r] \rangle.$$

This makes the conservation statement robust under later changes of physical interpretation. A future physical model may fail to justify the proposed R_p, may place allowed moves outside Adm, or may fail NONDEG/PRES; none of those failures changes the abstract implication proved in Section 4.

11.2. Why the concrete $A_5/2I$ witness matters

The abstract setup permits vacuous cases. If $K = 1$, every residue statement is trivial. Even if $K \neq 1$, it is possible that the relevant relation sector never reaches a nonidentity kernel element. The concrete torus bridge prevents the $A_5/2I$ case study from remaining at that purely structural level.

In the $SL(2,5)/PSL(2,5)$ model, the kernel is genuinely binary, and the formal bridge constructs a commutator that lies in the selected relation and admissibility sectors while carrying the nontrivial central element. Thus the specialization reaches

$$CARRIER + ALG\text{-}NONDEG,$$

not merely a statement that a two-element kernel exists.

This still does not establish a physical sector. The witness is algebraic and presentation-level. Its role is to show that the abstract preservation problem has a nontrivial concrete instance before any physical dynamics is asserted.

A useful algebraic contrast is that A_5 is non-Abelian simple and therefore has trivial abelianization,

$$A_5^{ab} = 1,$$

whereas its binary central cover has

$$\ker(2I \to A_5) \cong \mathbb{Z}_2.$$

doi.org/10.32388/8WFJ48

Thus a binary distinction may live in lift/relation data without appearing as an Abelian quotient label of the A_5 base itself. This is only a group-theoretic interpretation; it does not produce a physical or cosmological bit.

Theorems 6.2a–6.2b sharpen this statement from localization to exact support. Inside the concrete A_5 base,

$$\boxed{\text{the Klein-four locus is exactly the commuting-lift obstruction locus,}}$$

and the lift-independent central commutator pairing satisfies

$$\boxed{\operatorname{supp}_{\mathrm{Comm}}\kappa = \text{Klein-four locus.}}$$

Thus the local V_4/Q_8 subsystem is not merely a convenient witness carrier: it is the exact support of the binary commuting obstruction in the frozen $A_5/2I$ realization. This remains an A_5-internal classification statement and does not assert uniqueness of this mechanism among all groups.

11.3. Relation to topology and histories-based quantum cosmology

The torus language is suggestive because a commuting pair downstairs and a noncommuting pair of lifts are familiar ingredients in lifting-obstruction problems. A related physical precedent occurs in non-Abelian defect topology: biaxial-nematic defects exhibit the classical D_2/Q_8 lift pattern, and defect composition can carry noncommutative path-history information [6][7][8]. However, the current formal development intentionally stops before identifying the presentation-level commutator with a topological class of a torus, a principal bundle, a defect charge in an order-parameter manifold, or a specific element of $H^2(T^2;\mathbb{Z}_2)$.

The wider quantum-cosmological literature clarifies why this restraint is compatible with a nontrivial foundational motivation. Hartle's sum-over-histories quantum cosmology, Sorkin's quantum-measure program, causal-set sequential growth, topological-geon sectors, and spin-foam move invariance all provide established settings in which the physically relevant object is richer than an instantaneous endpoint or in which equivalence under changes of history must be specified rather than assumed [2][3][4][5][26]. The present paper should therefore be read as a finite algebraic preservation model in that broad problem family, not as a replacement for any of those theories.

The use of $2I$ also has a separate cosmological precedent in Poincaré dodecahedral-space models [27][28]. That fact motivates group-specific comparison but does not identify the present central-extension residue with cosmic topology. In particular,

$$2I \text{ as a deck group of } S^3/2I$$

and

$$2I \to A_5 \text{ as a central extension carrying a binary residue}$$

are distinct mathematical roles.

This restraint is important for two reasons. First, a free fundamental group has no nontrivial degree-two lifting obstruction of the relevant sort, so a bare transition graph must not be conflated with a relation-bearing two-dimensional presentation. Second, the current theorem needs no topological or quantum-cosmological identification: the exact flip criterion and quotient-carrier construction follow directly from the algebraic homomorphism, subgroup, centrality, and exponent-two conditions, while full-A_5 generation is a separate finite-group statement. A later topology or dynamics bridge could enrich the interpretation, but it is not needed to support the claims made here.

11.4. Encoding dependence and the role of preregistration

Section 7 shows that representative choice is not automatically harmless. In the currently formalized group-element-indexed positive-word setting, even-count conditions guarantee invariance under section changes, while closure alone does not. This is a concrete warning against choosing a primitive encoding only after inspecting the target residue.

For that reason, the physical branch freezes the primitive alphabet, relation generators, nullity rule, admissibility convention, and carrier choice before NONDEG/PRES are evaluated. The methodological point is modest: preregistration does not make a physical model correct, but it makes one class of post-selection visible and auditable.

The same logic also explains why the paper distinguishes the mathematical normal closure of a frozen relation set from the physical statement that those relations are genuinely null. The former is an algebraic consequence; the latter is a modeling choice that must be justified independently.

11.5. Compression is downstream, not part of the present theorem

The present paper does not solve the minimal-compression problem. Once a physical dynamics and a physical residue domain have been frozen, one may ask whether a coarse quotient q supports a descended dynamics

$$q \circ T = \overline{T} \circ q$$

while failing to retain enough information to reconstruct the residue. That question naturally connects to established work on state aggregation, bisimulation, partition refinement, lumpability, and task-specific sufficient representations.

The current contribution is upstream: it specifies the observable and identifies a sufficient residue-safe deformation class. This is the data that any later quotient/minimization procedure must preserve. In particular, a constant quotient is not evidence of meaningful dynamical compression, and no claim of a coarsest or irreducible quotient is made without a completeness theorem for the candidate compression family.

11.6. Physical realization remains the decisive open bridge

The physical branch requires more than algebraic nondegeneracy. At minimum, an actual local dynamics Φ must determine the primitive moves and physically allowed histories in a result-blind way, preserve the declared physical domain, support the chosen relative evaluation fiber or force a preregistered revision, realize both residue sectors, and preserve the residue under the relevant evolution.

The release ladder is therefore

$$ALG\text{-}NONDEG \not\Rightarrow PHYS\text{-}NONDEG \not\Rightarrow PRES \not\Rightarrow \text{cosmological initial bit.}$$

 doi.org/10.32388/8WFJ48

Even a physical NONDEG+PRES result would establish at most a preserved physical history-sector candidate. Additional claims about cosmological initial selection, inheritance, or long-time transmission would require separate arguments.

A stronger downstream question, deliberately not part of the present result, is whether physically realized residue sectors have asymmetric reachability or survival times—for example, whether one sector lies in a recurrent or closed component while another is transient or dynamically erased. Such an asymmetry would be a falsifiable dynamical statement beyond PHYS-NONDEG and PRES.

11.7. Formalization as claim control

The formal development is useful here not because the underlying algebra is difficult, but because the research program contains several nearby statements whose directions differ subtly. Examples include

$$\text{residue change} \Rightarrow \neg\text{CaseA},$$

versus the false strengthening

$$\text{CaseB} \Rightarrow \text{residue change}.$$

Similarly, the formal source distinguishes group-element-indexed section changes from primitive-letter lift choices and from graph switching, and it explicitly avoids claiming a topological obstruction theorem. In this setting, formalization functions as a firewall against conceptual promotion: a suggestive interpretation is not silently upgraded into a theorem.

11.8. Limitations

The main limitations are correspondingly clear.

1. The individual arguments are elementary and the paper does not introduce a new homological or cohomological invariant beyond standard central-extension machinery. Its contribution instead lies in the exact preservation/sharpness architecture, the complete determination of the commuting-obstruction locus, and the exact support theorem for the concrete lift-independent commutator pairing in the $A_5/2I$ case.
2. The current paper does not formalize the Hopf formula or a homological pushforward/naturality statement.
3. The auxiliary section-choice formalization is narrower than the abstract deformation theorem and currently includes positive-word restrictions.
4. The concrete torus witness is algebraic/presentation-level, not a formal topology theorem.
5. The actual physical dynamics, physical nullity rule, and physical admissibility domain remain open.
6. No minimal compression or quotient-irreducibility theorem is proved.
7. The final self-contained v5.4 snapshot is BUILD VERIFIED by author report. The remaining provenance refinement is archival rather than mathematical: preserving the raw full-build log alongside the frozen source snapshot would strengthen the public reproducibility record.

These limitations define the next work rather than being hidden assumptions of the present result.

 doi.org/10.32388/8WFJ48

12. Conclusion

We have separated a history-sensitive binary-sector question into an abstract deformation theorem, a concrete $A_5/2I$ algebraic instance, and an explicitly open physical realization problem. For a parametric deformation setup with central kernel of exponent at most two, the subgroup

$$D(R_p) = \langle r^2, [f, r] \rangle$$

is residue-trivial. Consequently, Case-A deformations preserve the residue, and any residue change must lie outside Case A; within the additional null-and-admissible domain it localizes one-way to Case B.

The $SL(2,5)/PSL(2,5)$ specialization makes the target kernel genuinely binary. The original torus commutator is an embedded V_4/Q_8-local witness with nontrivial residue and Case-B membership, while the three-letter globalized labeling generates the full A_5 base. The exact-locus theorem shows, moreover, that for commuting pairs in the concrete A_5 base, failure of simultaneous commuting liftability is equivalent to Klein-four type. The lift-independent central commutator pairing gives a positive binary invariant on this commuting locus, with nontrivial support exactly equal to the Klein-four locus. Together with the exact flip criterion, the exact family-level conservation kernel, the quotient carrier $R_p/D(R_p) \to K$, and the universal sharpness of $D(R_p)$, this makes the mathematical architecture nonvacuous and explicit while preserving the distinction between algebraic and physical nondegeneracy.

The main practical outcome is therefore an auditable interface:

$$\boxed{\begin{array}{c}\text{history deformation} \longrightarrow \text{relation/admissibility classification} \\ \longrightarrow \text{central residue preservation test}\end{array}}$$

that can be frozen before a downstream physical verdict is inspected. The next decisive question is not whether the algebra can produce a binary residue—it can in the concrete case—but whether an independently specified physical dynamics realizes and preserves that residue without post-selection. Until those gates are passed, no physical history bit or cosmological initial bit is claimed.

Appendix A. Preregistration and freeze record for the physical bridge

A.1. Separation of physical eligibility and residue evaluation

The v3 protocol fixes the distinction

$$\mathscr{H}_{phys} = \{H : \mathrm{PhysicalEligible}(H)\}$$

versus

$$\mathscr{H}_{H_0} = \{H \in \mathscr{H}_{phys} : \beta(H) = \beta(H_0)\}.$$

The first is a bit-blind physical domain; the second is the relative-residue evaluation fiber. The design decision Q-15=C defines

doi.org/10.32388/8WFJ48

$$\beta(H) = (p(H), \nu(H)), \varepsilon_{rel}(H) = \varepsilon(HH_0^{-1}).$$

The reference history H_0, the physical domain, and the primitive ontology are to be frozen before NONDEG or PRES is evaluated.

A.2. Ordered first-run gates

The preregistered order is

$$FREEZE \to NAT \to PHYS\text{-}CLOSURE \to FIBER\text{-}CLOSURE \to NONDEG \to PRES.$$

The corresponding meanings are:

- **NAT:** result-blind provenance and rationale audit;
- **PHYS-CLOSURE:** actual allowed dynamics preserves the declared physical domain;
- **FIBER-CLOSURE:** the first-run relative evaluation fiber is preserved, or else the claim is restricted/repreregistered;
- **NONDEG:** both relative residue values occur among physically eligible histories in the frozen fiber;
- **PRES:** the relative residue is preserved under allowed evolution, with a reachable-path sector-separation theorem reserved for release.

The protocol permits a positive first-run history-sector statement only when all of these gates pass.

A.3. Later quotient and compression stages

MSI-8 is preregistered before the result but evaluated only after an actual dynamics Φ has been frozen. A fixed nontrivial quotient q is tested for an explicit descent

$$q \circ T = \overline{T} \circ q.$$

A constant quotient is excluded as evidence for a dynamics/readout mismatch.

MG3 is separated into a preregistration gate for candidate compression families and a later sufficient-compression test. The intended question is whether the relative residue is constant on a proposed compression kernel. Absolute irreducibility is not claimed without a completeness theorem for the compression grammar.

A.4. D0 and state-space finiteness

The historical D0 v1 preregistration recorded that the candidate $U0/U1/U2/U3$ grammar may change defect number and left a global finite upper bound $N_{\max}$ open. Subsequent design analysis separated this issue into a logical-necessity audit and a computational-finiteness audit. The present manuscript therefore does not treat a global $N_{\max}$ as an assumption of the algebraic core.

The paper uses the hierarchy

$$H < X < L < G$$

to distinguish finite histories, a finite primitive alphabet, local finiteness, and global finiteness. Which levels are required by an eventual physical implementation remains an open protocol question. Exhaustive computation may require a finite audit window even when the mathematical preservation theorem does not require a globally finite state space.

A.5. Release-language firewall

The preregistration record enforces the ladder

$$ALG\text{-}NONDEG \neq PHYS\text{-}NONDEG \neq \text{cosmological initial bit}.$$

The present paper reaches the algebraic level in the concrete $A_5/2I$ case. It does not report a completed physical freeze and does not claim a physical or cosmological binary sector.

Statements and Declarations

Funding

This research received no external funding.

Potential Competing Interests

The author declares no competing interests.

Data Availability

This paper contains no experimental data. The finite $A_5/2I$ history-deformation definitions, relation and admissibility structures, residue constructions, finite group certificates, theorem statements, provenance records, and claim-control tables used in the argument are stated in the manuscript or contained in the accompanying formal source package and are reproducible from the definitions and frozen source snapshot supplied with the paper.

Code Availability

The Lean 4 audit files are available at https://github.com/nuu/A5_History under the Apache License Version 2.0.

Author Contribution

M. Numagaki is the sole author and is responsible for the conception of the theoretical framework, the mathematical analysis, the history-deformation and central-extension architecture, the formalization strategy and build verification, the claim-control design, and the writing and revision of the manuscript.

Use of Generative AI

During preparation of this manuscript, AI-assisted tools were used for copyediting, formatting, English grammar and style refinement, improvements to the clarity and organization of the mathematical exposition, and assistance in drafting and revising formal Lean code. All mathematical claims, proof acceptance decisions, build-verification decisions, scientific interpretations, and conclusions were reviewed and approved by the author. The author takes full responsibility for the content of the manuscript and the accompanying formal source package.

Acknowledgments

The formal audit style and claim-dependency discipline used in this paper are informed by the Lean 4 theorem prover and the Mathlib mathematical library. The author thanks the developer communities of both projects.

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Supplementary data: available at https://doi.org/10.32388/8WFJ48

Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.