Research Article

A New Metric for Quantifying the Tail Heaviness of Probability Distributions

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Tail heaviness is an important characteristic of probability distributions. Effectively quantifying tail heaviness is crucial for constructing statistical models and comparing different distributions. Traditionally, kurtosis is employed as a metric for measuring tail heaviness; however, it suffers from several well-known limitations or drawbacks. Although several alternative measures have been proposed in the literature, how to appropriately quantify tail heaviness remains an unresolved issue. This paper proposes a novel method for measuring tail weight and introduces a new metric for quantifying tail heaviness, termed the "Tail Heaviness Index." Our analysis of tail weight focuses primarily on the downside tail (i.e., the right tail) of unimodal distributions. First, we define the "Tail Probability Function" (TPF) as the normalized exceedance probability function. Subsequently, we define the tail weight as the product of the area under the TPF curve and the average density of the tail region. Furthermore, we define the tail heaviness index as the ratio of the tail weight of a given distribution to the tail weight of the benchmark exponential distribution. This paper presents closed-form expressions for the tail heaviness index for seven well-known distributions and compares them with the tail weight coefficient (TWC) proposed by Ortega[1].

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1. Introduction

The tail of a probability distribution refers to extreme values that are far away from its central tendency (e.g., mean or mode). Consequently, tail heaviness refers to how much weight (i.e., the probability mass) that the distribution assigns to these extreme values. An effective method for quantifying tail heaviness is important for numerous practical applications involving statistical modeling, distribution comparison, and model selection.

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Kurtosis is traditionally used as a metric for measuring tail heaviness[2][3][1]. For a probability distribution of the random variable Y with finite moments, kurtosis is the normalized fourth moment, defined as

$$\kappa = \frac{\mathbb{E}\left[(Y-\mu)^4\right]}{\left\{\mathbb{E}\left[(Y-\mu)^2\right]\right\}^2} = \frac{\mu_4}{\sigma^4} \tag{1}$$

where $\mathbb{E}[\bullet]$ represents the mathematical expectation, $\mu_4 = E\left[(Y-\mu)^4\right]$ is the fourth central moment and σ is the standard deviation.

However, kurtosis suffers from several well-known limitations or drawbacks. First, and most importantly, there is no consensus in the statistics community on what kurtosis actually measures[2]. In fact, for a long time, kurtosis has been mistakenly interpreted as a measure of the peakedness of distributions. Westfall[3] studied this misconception and concluded that "kurtosis should never be defined in terms of peakedness," and that "the relationship of peakedness with kurtosis is now officially over." Second, some well-known heavy-tailed distributions, such as the Pareto distribution and the Cauchy distribution, do not have a defined kurtosis because these distributions lack finite moments. Third, kurtosis is highly sensitive to outliers in the observed data[2].

A number of studies have proposed alternative measures to quantify the tail heaviness of probability distributions. Schuster[4] introduced the right-tail exponent measure to classify distributions into three classes: short- medium-, and long-tailed. Brys et al.[2] proposed several tail weight measures based on robust measures of skewness. They considered separate assessments for left and right tail weight measures to accommodate asymmetric distributions. Jordanova and Petkova[5] proposed a measure for the tail heaviness and introduced a classification of distributions with respect to tail heaviness. Ortega[1] proposed a tail weight coefficient (TWC) based on the tail set that is constructed using a fixed cut-off value. The TWC for a distribution F is defined as

$$C_{tw}(F) = m_{tw}\left(F, \hat{c}\right). \tag{2}$$

For continuous distributions, $m_{tw}\left(F, \hat{c}\right)$ is given by $E\left[Y^{'}|Y^{'} > \hat{c}\right]$, $Y^{'} = (Y-\mu)/\sigma$, and $\hat{c} \approx 1.5718$ is used as the cut-off value. With this fixed cut-off value, the TWC is very close to 2 for the normal distribution and is 2.57 for the exponential distribution[1].

However, existing literature indicates considerable uncertainty, controversy, and error in connection with the distinctions between tail weights[6]. Therefore, how to appropriately measure tail weights and quantify tail heaviness remains an unresolved issue.

In this paper, we propose a novel method for measuring tail weight and introduces a new metric for quantifying tail heaviness, termed the “tail heaviness index”. In the following sections, Section 2 defines the tail probability function, tail area, and tail weight. Section 3 describes the proposed tail heaviness index. Section 4 presents closed-form expressions for the tail heaviness index for seven well-known distributions and compares them with the tail weight coefficient (TWC) proposed by Ortega[1]. Section 5 and Section 6 present discussion and conclusions, respectively.

2. Definitions: tail probability function, tail area, and tail weight

We consider a unimodal distribution of a continuous random variable Y defined on the support $(-\infty, \infty)$, with probability density function (PDF) $p(y)$. We partition this distribution at its mode into two parts: the left portion $(-\infty, y_{mode}]$ and the right portion $[y_{mode}, \infty)$. Our analysis of tail weight focuses primarily on the right portion (i.e., the downside tail, called the right tail or simply “tail” hereafter). This focus is particularly relevant in finance, where the downside tail of asset return distributions is a fundamental and essential factor for the analysis of financial market risks[7]. It is worth noting that that there is currently no rigorous or universally accepted definition for the "tail"[8][9]. We argue that analyzing the right portion $[y_{mode}, \infty)$ of the distribution provides a reasonable and practical approach to studying tails.

Definition 1. Let $T(y)$ denote the tail probability function (TPF). It is defined as the exceedance probability function $Pr(Y > y)$ normalized at the mode y_{mode}. That is,

$$T(y) = \frac{\Pr(Y > y)}{\Pr(Y > y_{mode})} = \frac{1}{P_T}\int_y^\infty p(t)dt = 1 - \int_{y_{mode}}^{y} \frac{p(t)}{P_T}dt, \tag{3}$$

where P_T is the probability mass of the tail, given by

$$P_T = Pr(Y > y_{mode}) = \int_{y_{mode}}^\infty p(y)dy, \tag{4}$$

which ensures that the integral $\int_{y_{mode}}^\infty \frac{p(y)}{P_T}dy = 1$ holds true. This normalization ensures that the total probability mass in the tails of any distribution remains consistent, thereby rendering comparisons between different distributions meaningful.

The TPF $T(y)$ is a dimensionless probability measure and a function of y (distance). It begins at a peak value of 1 at the mode, decays as y increases, and approaches 0 as $y \to \infty$. Therefore, $T(y)$ characterizes

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the tail behavior—specifically, how fast the *probability mass* dissipates beyond the mode.

Notably, the exceedance probability function $Pr(Y > y)$ is precisely the complimentary cumulative distribution function. Moreover, when "y" represents time, this function is referred to as the "survival function." Consequently, the TPF $T(y)$ is equivalent to the survival function normalized at the mode.

Definition 2. Let ω denote the tail area. It is defined as the area under the TPF $T(y)$ curve,

$$\omega = \int_{y_{mode}}^{\infty} T(y)dy. \tag{5}$$

Substituting $T(y) = \frac{1}{P_T}\int_y^{\infty} p(t)dt$ into Eq. (5) yields

$$\omega = \frac{1}{P_T}\int_{y_{mode}}^{\infty}\left(\int_y^{\infty} p(t)dt\right)dy. \tag{6}$$

Now interchange integration order in the region: $y_{\text{mode}} \le y \le t < \infty$. The inner integral: $\int_{y_{\text{mode}}}^{t} dy = t - y_{\text{mode}}$. Thus,

$$\omega = \frac{1}{P_T}\int_{y_{\text{mode}}}^{\infty}(t - y_{\text{mode}})p(t)dt = \frac{1}{P_T}\int_{y_{\text{mode}}}^{\infty} tp(t)dt - y_{\text{mode}}. \tag{7}$$

This is a general expression for the tail area that works for any unimodal distribution. The tail area ω quantifies the geometric properties of the tail. Since the TPF $T(y)$ is dimensionless, the tail area ω possesses the dimension $[L]$ (i.e., [length]) and depends on the scale of the distribution.

Definition 3. Let Ω denote the tail weight. It is defined as the tail area ω times the tail's average density quantified by the tail informity β. That is,

$$\Omega = (\text{tail area}) \times (tail's \text{ average density}) = \omega\beta. \tag{8}$$

The concept of informity (including discrete informity for discrete distributions and continuous informity for continuous distributions) and informity theory are proposed by Huang[10]. The continuous informity β for the right tail of a unimodal distribution is given by

$$\beta = \int_{y_{mode}}^{\infty}\left[\frac{p(y)}{P_T}\right]^2 dy = \frac{1}{P_T^2}\int_{y_{mode}}^{\infty} p(y)^2 dy. \tag{9}$$

In informity theory, the reciprocal of continuous informity is defined as the uncertainty length (δ)[10]. Thus, the tail weight Ω can also be defined as

$$\Omega = \frac{\omega}{\delta}. \tag{10}$$

That is, the proposed tail weight Ω is equivalent to the tail area ω normalized by the tail uncertainty length δ. Since both ω and δ possess the dimension $[L]$ (i.e., $[length]$), Ω is dimensionless; it also exhibits scale invariance. These properties constitute precisely the appropriate characteristics required for measuring tail weight.

3. The proposed tail heaviness index

The notion of whether a distribution is considered "heavy-tailed" is inherently relative[11]. To establish a reference point, we use the exponential distribution as a benchmark for comparing tail weights. In this framework, a distribution is considered heavy-tailed if its tail weight is heavier than the tail weight of the exponential distribution. In other words, a "heavy-tailed" distribution decays more slowly than the exponential distribution, so extreme values are more likely. This approach aligns with Nair et al.[11] for defining the class of heavy-tailed distributions. Bryson[12] also considered a heavy-tailed distribution to be one whose tail is heavier than the tail of the exponential distribution.

The PDF of the exponential distribution is

$$p(y) = \lambda e^{-\lambda y}, \tag{11}$$

where λ is the rate (or inverse scale) parameter.

The exponential distribution has the mode at y=0. Therefore, the entire distribution is considered as the "tail" and $P_T = 1$. Its TPF $T(y)$ is

$$T(y) = 1 - \int_0^y \lambda e^{-\lambda t} dt = 1 - \left(1 - e^{-\lambda y}\right) = e^{-\lambda y}. \tag{12}$$

Its tail area is

$$\omega_{\text{exp}} = \int_0^\infty T(y) dy = \int_0^\infty e^{-\lambda y} dy = \frac{1}{\lambda}. \tag{13}$$

The informity of the exponential distribution is[10]

$$\beta_{\text{exp}} = \int_0^\infty \lambda^2 e^{-2\lambda y} dy = \frac{\lambda}{2}. \tag{14}$$

Thus, the tail weight of the exponential distribution is

$$\Omega_{\text{exp}} = \omega_{\text{exp}} \beta_{\text{exp}} = \frac{1}{2}. \tag{15}$$

We use Ω_{exp} as a benchmark tail weight to define the tail heaviness index.

Definition 4. Let θ denote the tail heaviness index for a given distribution. It is defined as the ratio of its tail weight Ω to the benchmark tail weight Ω_{exp}. That is,

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$$\theta = \frac{\Omega}{\Omega_{\exp}} = 2\Omega = 2\omega\beta. \tag{16}$$

The proposed tail heaviness index θ is dimensionless and possesses scale invariance and translation invariance.

4. Tail heaviness indices for seven well-known distributions

This section presents the tail heaviness indices for seven well-known distributions: the normal, Laplace, Student's *t*, Pareto, Gamma, lognormal, and Chi-squared distributions. We compare these indices with the normalized tail weight coefficient (TWC). The TWC values are drawn from the study by Ortega[1]. We adopt the TWC value for the exponential distribution (2.57) as the benchmark for normalization. The purpose of this normalization is to render the comparison between these two metrics more practically meaningful.

4.1. The normal distribution

The PDF of the normal distribution $N(\mu, \sigma)$ is

$$p(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right], \tag{17}$$

where μ is the mean and σ is the standard deviation.

Since the normal distribution is symmetric, its mode is μ, its tail probability mass $P_T = 0.5$, and its TPF $T(y)$ is

$$T(y) = 1 - 2\int_{\mu}^{y} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right] dt = 2\left[1 - \Phi\left(\frac{y-\mu}{\sigma}\right)\right], \tag{18}$$

where $\Phi(.)$ is the cumulative distribution function (CDF) of the normal distribution.

Its tail area is

$$\omega_{normal} = \sigma\sqrt{\frac{2}{\pi}}. \tag{19}$$

Its tail informity is

$$\beta_{normal} = \frac{1}{\sigma\sqrt{\pi}}. \tag{20}$$

Then, the tail heaviness index for the normal distribution is

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$$\theta_{normal} = 2\omega_{normal}\beta_{normal} = \frac{2\sqrt{2}}{\pi} = 0.9003 < 1. \tag{21}$$

Since $\theta_{normal} = 0.9003 < 1$, the normal distribution is a light-tailed distribution, which aligns with our intuition.

According to Ortega[1], the TWC value for the normal distribution is 2. Therefore, its normalized TWC is 2/2.57=0.7782 < 1.

4.2. The Laplace distribution

The PDF of the Laplace distribution is

$$p(y) = \frac{1}{2b}\exp\left(-\frac{|y-\mu|}{b}\right), \tag{22}$$

where μ is the location and b is the scale.

Since the Laplace distribution is symmetric, its mode is μ, and its tail probability mass $P_T = 0.5$. Its tail area is

$$\omega_{Laplace} = b. \tag{23}$$

Its tail informity is

$$\beta_{Laplace} = \frac{1}{2b}. \tag{24}$$

Then, the tail heaviness index for the Laplace distribution is

$$\theta_{Laplace} = 2\omega_{Laplace}\beta_{Laplace} = 1, \tag{25}$$

which aligns with the benchmark value for the exponential distribution. This result is expected, as both the Laplace and exponential distributions exhibit identical exponential decay characteristics. However, the TWC value for the Laplace distribution is 2.27[1]. Consequently, its normalized TWC is 2.27/2.57 = 0.8833 < 1. Thus, it is evident that TWC fails to accurately measure the tail heaviness of the Laplace distribution.

4.3. The t-distribution

The PDF of the t-distribution is

$$p(y) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{y^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \tag{26}$$

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where ν is the degrees of freedom.

Since the t-distribution is symmetric, its mode is 0, and its tail probability mass $P_T = 0.5$. Its tail area is

$$\omega_t = \frac{2\sqrt{\nu}}{\nu - 1}\frac{\Gamma\ \left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\Gamma\ \left(\frac{\nu}{2}\right)}, \nu > 1. \tag{27}$$

Its tail informity is

$$\beta_t = \frac{2\Gamma(\frac{\nu+1}{2})^2}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})^2}\frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu + 1)}. \tag{28}$$

Then, the tail heaviness index for the t-distribution is

$$\theta_t = 2\omega_t\beta_t = \frac{8\Gamma(\frac{\nu+1}{2})^3\Gamma\left(\nu + \frac{1}{2}\right)}{\pi(\nu - 1)\Gamma(\frac{\nu}{2})^3\Gamma(\nu + 1)}, \nu > 1. \tag{29}$$

Figure 1 displays the plot of θ_t as a function of the degrees of freedom ν, and compares it with the normalized TWC. As can be seen from Figure 1, the proposed tail heaviness index exhibits the same trend as the normalized TWC: as the degrees of freedom increases, both indices demonstrate a downward trend. xm=1. As $\nu \rightarrow \infty$ α→∞,, each index converges to its value under the normal distribution: specifically, the tail heaviness index approaches 0.9003, while the normalized TWC approaches 0.7782. Note that when ν = 1 (i.e., the Cauchy distribution), neither of these indices is defined.

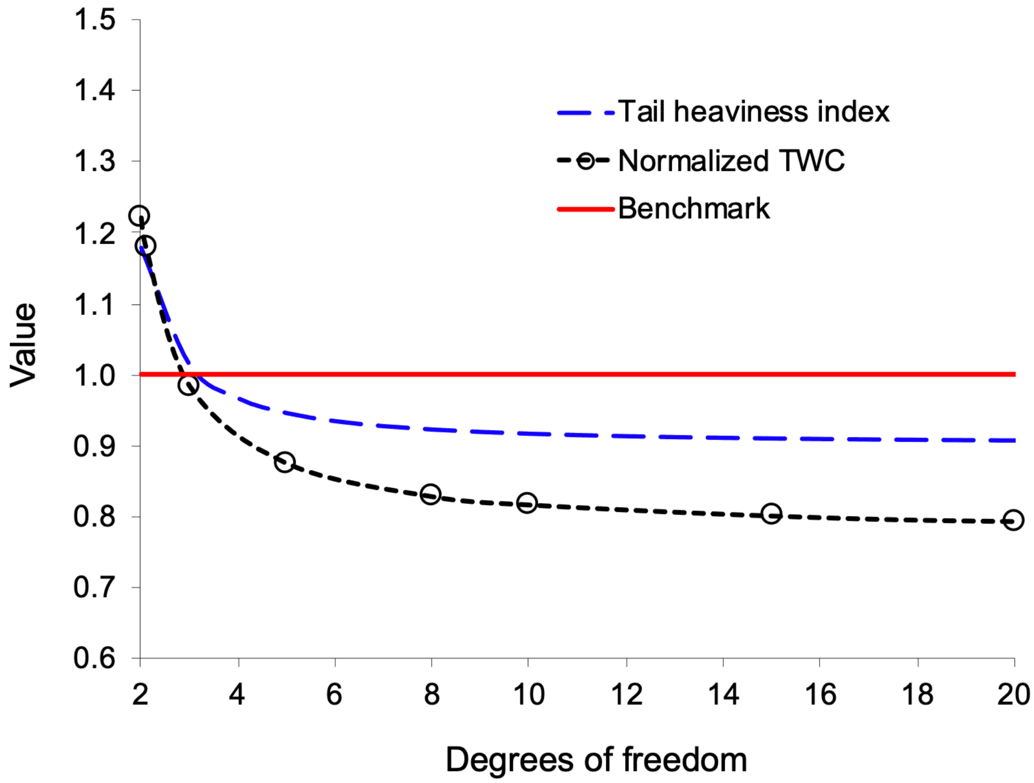

Figure 1. Tail heaviness index for the t-distribution as a function of the degrees of freedom, compared with the normalized tail weight coefficient (TWC)

4.4. The Pareto distribution

The PDF of the Pareto distribution is

$$p(y) = \begin{cases} \frac{\alpha y_m^\alpha}{y^{\alpha+1}}, & y \geq y_m, \\ 0, & y < y_m, \end{cases} \tag{30}$$

where α is the shape parameter and y_m is the minimum value of the distribution and serves as the scale parameter.

The mode of the Pareto distribution is at its minimum value y_m. Therefore, the entire distribution is considered as the "tail" and $P_T = 1$. Its TPF $T(y)$ is

$$T(y) = 1 - \int_{y_m}^{y} \frac{\alpha t_m^\alpha}{t^{\alpha+1}} dt = \left(\frac{y_m}{y}\right)^\alpha. \tag{31}$$

Its tail area is

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$$\omega_{Pareto} = \int_{y_m}^{\infty} \left(\frac{y_m}{y}\right)^{\alpha} dy = \frac{y_m}{\alpha - 1}, \alpha > 1. \tag{32}$$

The informity of the Pareto distribution is[10]

$$\beta_{Pareto} = \int_{y_m}^{\infty} \left[\frac{\alpha\, y_m^{\alpha}}{y^{\alpha+1}}\right]^2 dy = \frac{\alpha^2}{(2\alpha+1)}\frac{1}{y_m}. \tag{33}$$

Then, the tail heaviness index for the Pareto distribution is

$$\theta_{Pareto} = 2\omega_{Pareto}\beta_{Pareto} = \frac{2\alpha^2}{(\alpha-1)(2\alpha+1)}. \tag{34}$$

Figure 2 displays the plot of θ_{Pareto} as a function of α, and compares it with the normalized TWC. As shown in Figure 2, the proposed tail heaviness index remains consistent with the normalized TWC, with the exception of an anomalous drop in the normalized TWC value at $\alpha = 2$. Both indices are always greater than 1, indicating that the Pareto distribution is a heavy-tailed distribution regardless of its shape parameter value.

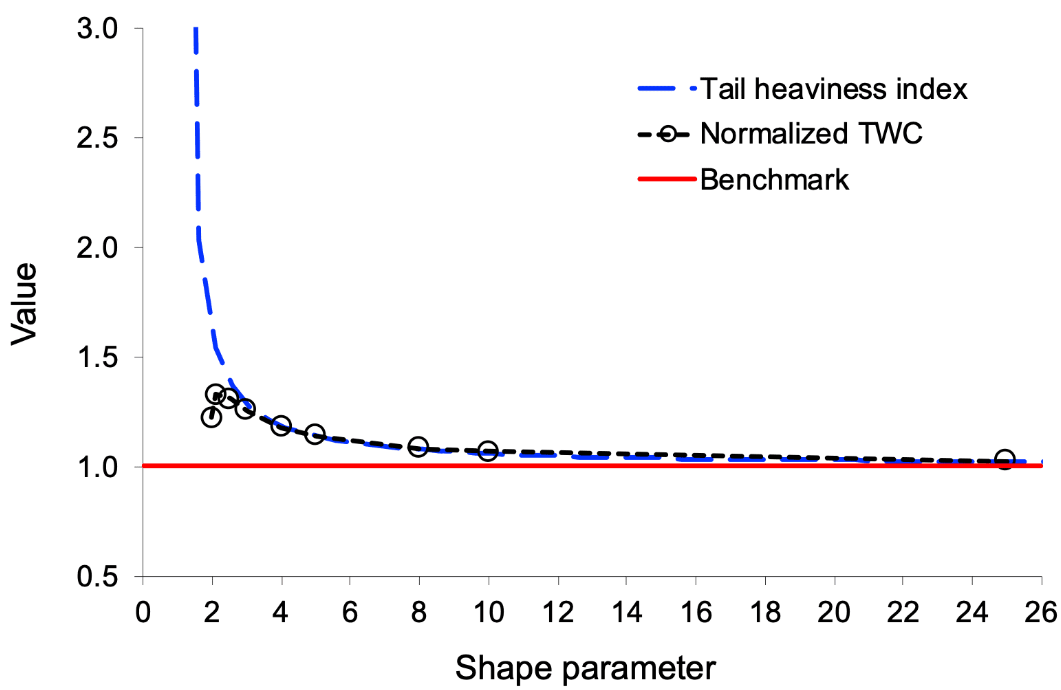

Figure 2. Tail heaviness index for the Pareto distribution as a function of the shape parameter, compared with the normalized tail weight coefficient (TWC)

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4.5. The Gamma distribution

The PDF of the Gamma distribution is

$$p(y) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y}, \tag{35}$$

where λ is the rate parameter and α is the shape parameter.

The mode of the Gamma distribution depends on the shape parameter,

$$y_{mode} = \begin{cases} 0, & 0 < \alpha < 1, \\ \frac{\alpha-1}{\lambda}, & \alpha \geq 1\ . \end{cases} \tag{36}$$

Its tail probability mass P_T is

$$P_T = \begin{cases} 1, & 0 < \alpha < 1\ , \\ \frac{\Gamma(\alpha,\alpha-1)}{\Gamma(\alpha)}, & \alpha \geq 1. \end{cases} \tag{37}$$

Its tail area is

$$\omega_{Gamma} = \begin{cases} \frac{\alpha}{\lambda}, & 0 < \alpha < 1, \\ \frac{1}{\lambda}\left(1 + \frac{\left(\alpha-1\right)^{\alpha} e^{-(\alpha-1)}}{\Gamma(\alpha,\alpha-1)}\right), & \alpha \geq 1. \end{cases} \tag{38}$$

Its tail informity is

$$\beta_{Gamma} = \begin{cases} \frac{\lambda}{2^{2\alpha-1}\Gamma(\alpha)^2 P_R^2}\Gamma(2\alpha-1), & 0 < \alpha < 1, \\ \frac{\lambda}{2^{2\alpha-1}\Gamma(\alpha)^2 P_R^2}\Gamma\left(2\alpha-1,\ 2(\alpha-1)\right), & \alpha \geq 1. \end{cases} \tag{39}$$

Then, the tail heaviness index for the Gamma distribution is

$$\theta_{Gamma} = \begin{cases} \frac{1}{2^{2\alpha}\Gamma(\alpha)^2}\Gamma(2\alpha-1)\left(1 + \frac{\left(\alpha-1\right)^{\alpha} e^{-(\alpha-1)}}{\Gamma(\alpha,\alpha-1)}\right), & 0 < \alpha < 1, \\ \frac{1}{2^{2\alpha}\Gamma(\alpha,\alpha-1)^2}\Gamma\left(2\alpha-1,\ 2(\alpha-1)\right)\left(1 + \frac{\left(\alpha-1\right)^{\alpha} e^{-(\alpha-1)}}{\Gamma(\alpha,\alpha-1)}\right), & \alpha \geq 1. \end{cases} \tag{40}$$

Figure 3 displays the plot of θ_{Gamma} as a function of α, and compare it with the normalized TWC. As can be seen from Figure 3, overall, the trend of the proposed tail heaviness index is consistent with the trend of the normalized TWC. Both indices decrease as the shape parameter α increases. xm=1. When $\alpha \to \infty$ α→∞,, each index converges to its value under the normal distribution: specifically, the tail heaviness index approaches 0.9003, while the normalized TWC approaches 0.7782.

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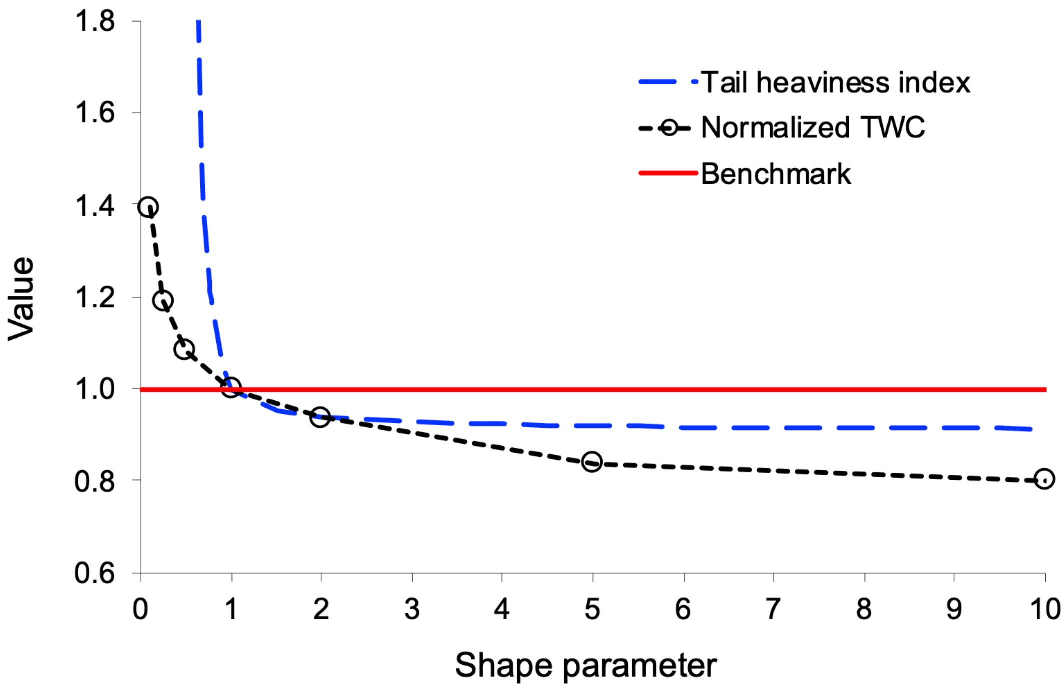

Figure 3. Tail heaviness index for the Gamma distribution as a function of the shape parameter, compared with the normalized tail weight coefficient (TWC)

4.6. The lognormal distribution

The PDF of the lognormal distribution is

$$p(y) = \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}. \tag{41}$$

The mode the lognormal distribution is at $y_{mode} = e^{\mu-\sigma^2}$. Its tail probability mass is

$$P_T = \Phi(\sigma). \tag{42}$$

Its tail area is

$$\omega_{lognormal} = e^{\mu+\sigma^2/2} \frac{\Phi(2\sigma)}{\Phi(\sigma)} - e^{\mu-\sigma^2}. \tag{43}$$

Its tail informity is

$$\beta_{lognormal} = \frac{1}{2\sigma\sqrt{\pi}} e^{-\mu+\frac{\sigma^2}{4}} \frac{\Phi(\frac{\sigma}{\sqrt{2}})}{\Phi(\sigma)^2}. \tag{44}$$

Then, the tail heaviness index for the lognormal distribution is

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$$\theta_{lognormal} = [e^{\sigma^2/2}\frac{\Phi(2\sigma)}{\Phi(\sigma)} - e^{-\sigma^2}]\frac{1}{\sigma\sqrt{\pi}}e^{\frac{\sigma^2}{4}}\frac{\Phi(\frac{\sigma}{\sqrt{2}})}{\Phi(\sigma)^2}. \tag{45}$$

Figure 4 displays the plot of $\theta_{lognormal}$ as a function of σ (sigma), and compares it with the normalized TWC. As can be seen from Figure 4, the proposed tail heaviness index and the normalized TWC exhibit the same trend: both indices increase with increasing σ. xm=1. However, the former shows a nonlinear increase, while the latter shows an approximately linear increase.

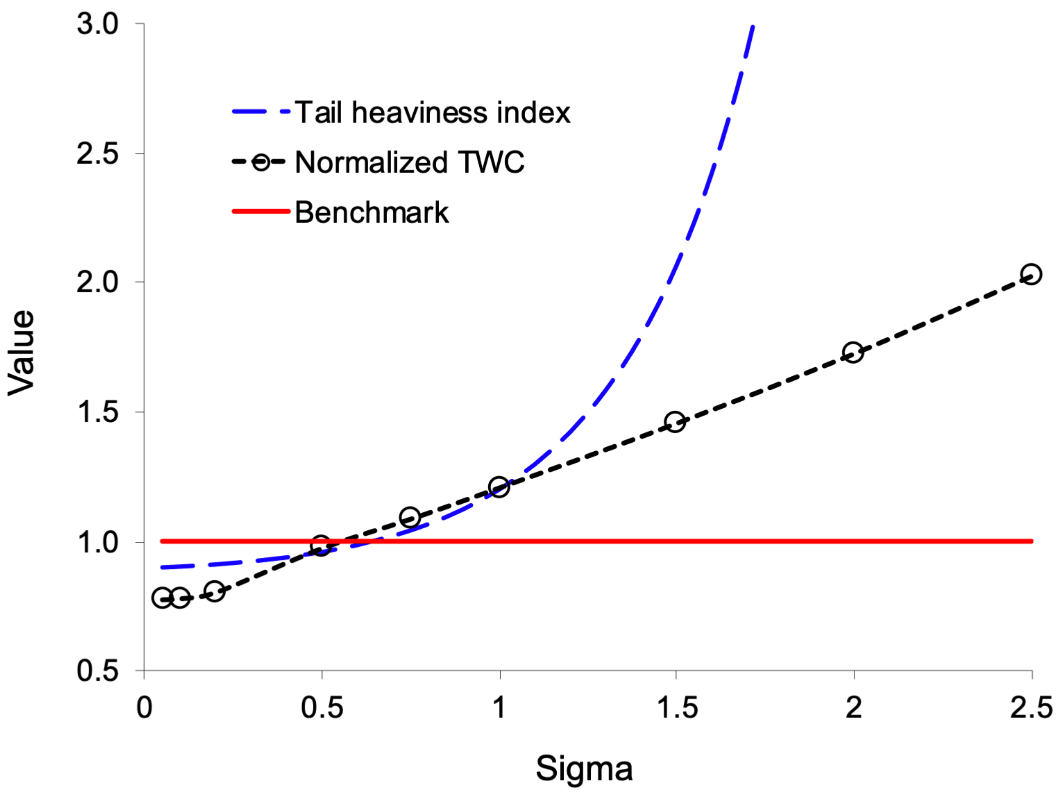

Figure 4. Tail heaviness index for the lognormal distribution as a function of σ (sigma), compared with the normalized tail weight coefficient (TWC)

4.7. The Chi-squared distribution

The PDF of the Chi-squared distribution is

$$p(y) = \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})}y^{\left(\frac{k}{2}\right)-1}e^{-\frac{y}{2}}. \tag{46}$$

The mode of the Chi-squared distribution is at $y_{mode} = k - 2$ for $k \geq 2$. Its tail probability mass is

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$$P_T = \frac{\Gamma\left(\frac{k}{2}, \frac{k-2}{2}\right)}{\Gamma(\frac{k}{2})}. \tag{47}$$

Its tail area is

$$\omega_{Chi-squared} = 2\,\frac{\Gamma\left(\frac{k}{2}+1,\ \ \frac{k-2}{2}\right)}{\Gamma\left(\frac{k}{2}, \frac{k-2}{2}\right)} - (k-2). \tag{48}$$

Its tail informity is

$$\beta_{Chi-squared} = \frac{\Gamma(k-1,\ \ k-2)}{2^k\Gamma\left(\frac{k}{2}, \frac{k-2}{2}\right)^2}. \tag{49}$$

Then, the tail heaviness index for the Chi-squared distribution is

$$\theta_{Chi-squared} = \left[2\,\frac{\Gamma\left(\frac{k}{2}+1, \frac{k-2}{2}\right)}{\Gamma\left(\frac{k}{2}, \frac{k-2}{2}\right)} - (k-2)\right] \frac{\Gamma(k-1, k-2)}{2^{k-1}\Gamma\left(\frac{k}{2}, \frac{k-2}{2}\right)^2}. \tag{50}$$

Figure 5 displays the plot of $\theta_{Chi-squared}$ as a function of the degrees of freedom k, and compares it with the normalized TWC. As can be seen from Figure 5, the proposed tail heaviness index exhibits the same trend as the normalized TWC: as the degrees of freedom increases, both indices demonstrate a downward trend. xm=1. As $k \to \infty$ α→∞,, each index converges to its value under the normal distribution: the tail heaviness index approaches 0.9003, while the normalized TWC approaches 0.7782.

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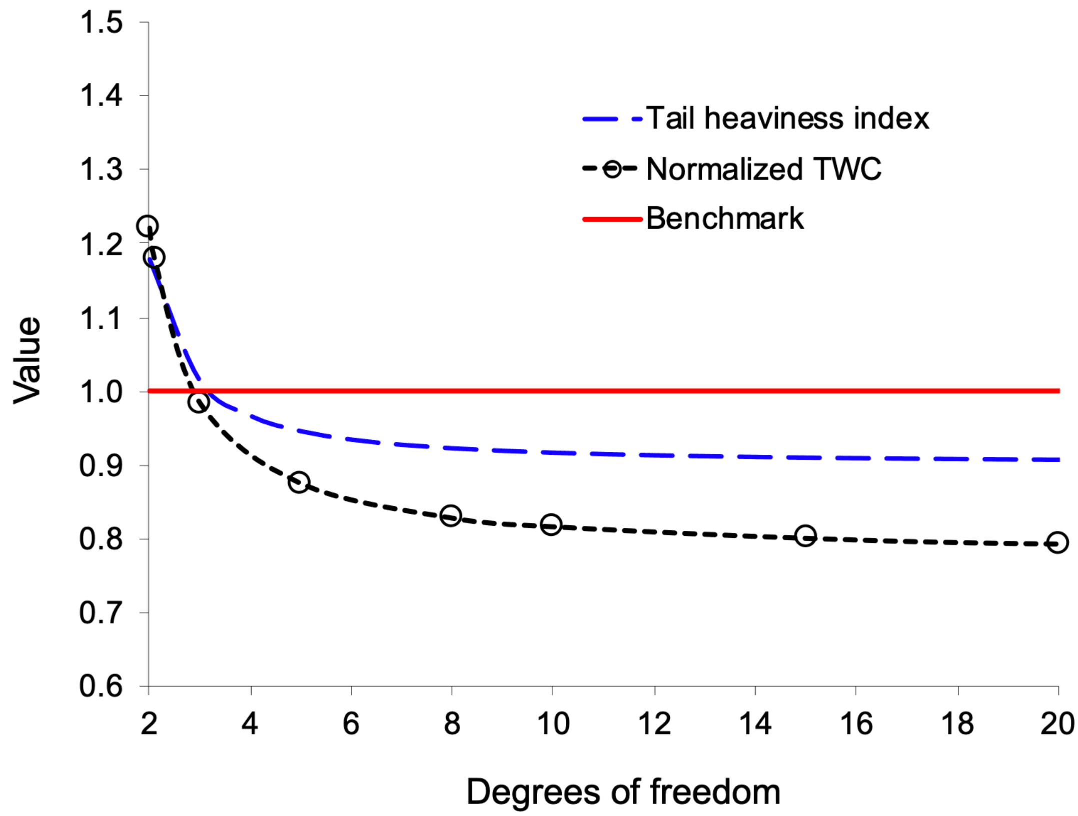

Figure 5. Tail heaviness index for the Chi-squared distribution as a function of the degrees of freedom, compared with the normalized tail weight coefficient (TWC)

5. Discussion

It is important to note that our analysis of tail weight is based on the TPF $T(y)$, rather than the PDF $p(y)$. This is because the TPF $T(y)$ effectively characterizes tail behavior, as it describes how fast the *probability mass* dissipates beyond the mode. Conversely, the PDF $p(y)$ is not a suitable object for studying tail weight[13].

Table 1 presents the expressions for the TPF $T(y)$ of the normal, exponential, and Cauchy distributions. These three distributions exhibit three distinct decay patterns:

- Exponential distribution → exponential decay: serves as a benchmark for measuring the tail heaviness of different distributions
- Normal distribution → super-exponential decay: a typical light-tailed distribution
- Cauchy distribution→ power-law decay: a typical heavy-tailed distribution

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Distribution	$T(y)$	$T(y)$ **for large** ***y***
Normal: $N(\mu, \sigma)$	$2[1 - \Phi(\frac{y-\mu}{\sigma})]$	$\sim \frac{\sigma}{(y-\mu)\sqrt{\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$
Exponential: $e(0, \lambda)$	$e^{-\lambda y}$	$e^{-\lambda y}$
Cauchy: $C(y_0, \gamma)$	$1 - \frac{2}{\pi}\arctan\left(\frac{y-y_0}{\gamma}\right)$	$\sim \frac{2\gamma}{\pi(y-y_0)}$

Table 1. Expressions for the TPF $T(y)$ of the normal, exponential, and Cauchy distributions

Figure 6 presents a comparison of the TPF curves of the normal distribution $N(0,1)$, the exponential distribution $e(0,1)$, and the Cauchy distribution $C(0,1)$. As can be observed from Figure 6, all three TPF curves originate at a peak value of 1 at *y*=0 (i.e., the mode), and exhibit a decaying trend as *y* increases. As expected, the normal distribution decays faster than the exponential distribution, while the Cauchy distribution decays significantly more slowly than the exponential distribution. The Cauchy distribution is considered an extremely heavy-tailed distribution; its tail area is infinite, and consequently, its tail heaviness index is also infinite.

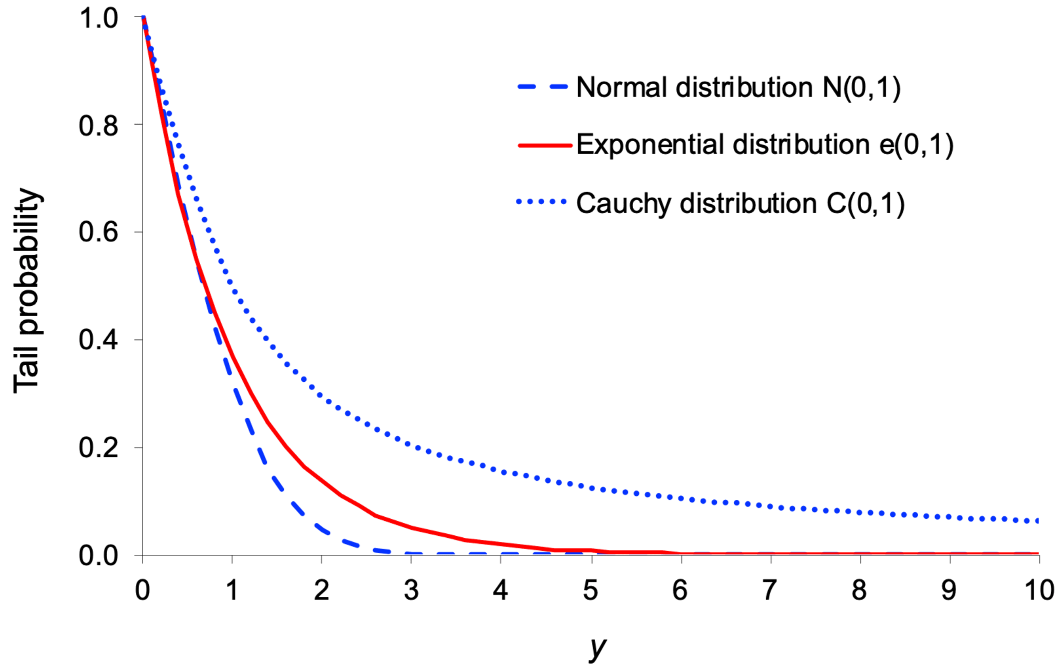

Figure 6. Comparison of the tail probability function (TPF) curves of the normal distribution $N(0,1)$, the exponential distribution $e(0,1)$, and the Cauchy distribution $C(0,1)$

It is also worth noting that the proposed tail weight and the tail heaviness index do not rely on moments. This constitutes a significant advantage compared to traditional measures of tail weight, such as kurtosis. Recall that the proposed tail weight is the product of the tail area ω and the tail informity β. For almost all practical distributions, the tail informity β is well-defined. On the other hand, for most distributions, the tail area ω is also well-defined, with the exception of a few extremely heavy-tailed distributions (such as the Cauchy distribution). Therefore, the proposed tail weight and the tail heaviness index are robust metrics.

6. Conclusions

The proposed tail heaviness index θ is simple to interpret, as it is constructed using the exponential distribution as the benchmark for comparing tail weights. If $\theta < 1$, the distribution is classified as light-tailed; conversely, if $\theta > 1$, the distribution is considered heavy-tailed. For most distributions, the value of θ is finite; only for extremely heavy-tailed distributions, such as the Cauchy distribution, θ tends

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toward infinity. For the seven well-known distributions discussed in Section 4, their corresponding θ values (or expressions) align with our intuitive understanding of their respective distributional shapes.

The proposed tail heaviness index θ possesses scale invariance and translation invariance. That is, it is independent of the scale parameter of the distribution and remains unchanged even if the distribution is shifted (i.e., a constant is added to the random variable). However, as expected, the proposed tail heaviness index θ depends on the shape parameter. This is similar to other tail weight measures, such as kurtosis and the tail weight coefficient (TWC).

The proposed tail heaviness index θ is applicable to any unimodal distribution, whether it is symmetric or asymmetric. Although this paper focuses on the right-tail of a distribution, the proposed method can also be applied to the left-tail of the distribution.

This paper presents the tail heaviness indices for seven well-known distributions. Further work is needed to derive the tail heaviness indices for more distributions, thereby expanding the applicability of the proposed method.

Statements and Declarations

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Potential Competing Interests

No potential competing interests to declare.

Data Availability

Data are contained within this article.

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