Research Article

First Integrals Without Integrating Factors or Symmetries

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A new method for finding first integrals of ordinary differential equations (ODEs) is presented. The approach is based on the complete integrability of the Pfaffian system associated with the ODE, defined on a suitable jet space. Remarkably, the method does not require the use of symmetries or integrating factors. Examples of second-, third-, and fourth-order ODEs are provided to illustrate the method, including cases where classical approaches fail. This work extends the range of tools available for the analysis and solution of ODEs.

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1. Introduction

First integrals are a key concept in the study of ordinary differential equations (ODEs), which represent conserved quantities when the ODE models a physical system. The knowledge of first integrals is an important feature that not only enables the reduction of the order of the ODE, but also facilitates the determination of the general solution, or the validation of numerical methods.

Over the years, numerous techniques have been developed to determine first integrals of ODEs. The most common approaches are based on Lie symmetries. They are extremely successful, but the determination of symmetries can be infeasible for certain classes of ODEs. This limitation has led to the enrichment of the approach through the introduction of more general classes of vector fields, such as dynamical symmetries, nonlocal symmetries, $\mathcal{C}^\infty$-symmetries, etc.[1][2][3][4][5][6][7][8][9][10][11].

A different point of view to find first integrals is based on the utilization of symbolic computation techniques to determine integrating factors in ODEs. The corresponding integrating factor determining equations, a system of partial differential equations (PDEs), must be solved, and then the first integral can be computed, effectively reducing the order of the ODE. Due to the difficulty of solving these determining equations, various techniques have been developed to aid in the search for integrating factors[5][12][13][14][15][16][17][18].

While both approaches have significantly advanced the study of first integrals, they come with inherent difficulties, which motivate the research of alternative strategies for finding first integrals. In this paper a novel approach is presented, that does not require the use of either symmetries or integrating factors. It is based on the complete integrability of the Pfaffian system associated with the ODE, defined on the corresponding jet bundle, which is the natural framework for the geometric study of ODEs[19][20].

To provide motivation, consider a first-order ODE of the form

$$u_1 = \phi(x, u), \tag{1}$$

where $u_1 = \frac{du}{dx}$ denotes the derivative of the dependent variable u with respect to the independent variable x, and ϕ is a smooth function defined on an open set $U \subseteq \mathbb{R}^2$. Equation (1) is usually written in differential form as

$$\theta_\phi := -\phi dx + du \equiv 0.$$

We have, by dimensionality, that $d\theta_\phi \wedge \theta_\phi = 0$, and therefore, by Frobenius theorem there exists, locally, a smooth function $F = F(x, u)$ such that $dF = \mu\theta_\phi$, with μ a non-vanishing smooth function. The function F can be found by solving the PDE:

$$dF \wedge \theta_\phi = 0.$$

Once a particular solution is found, the general solution to (1) could be expressed in implicit form as

$$F(x, u) = C,$$

where C is an arbitrary constant.

In this work, we generalize this idea to arbitrary order, presenting a novel method for finding first integrals. While our approach may be more computationally intensive than some traditional techniques, its fundamental advantage lies in its ability to address problems where standard methods, particularly those reliant on Lie symmetries, fail. After introducing the necessary notation and basic definitions in Section 2, we prove the main theoretical result in Section 3. We then demonstrate the practical power of our method. Section 4 particularizes the technique for second-order ODEs, while Section 5 applies it to challenging third- and fourth-order ODEs, including examples that are intractable with standard symmetry analysis.

2. Preliminaries

Consider an mth-order ODE

$$u_m = \phi(x, u, \dots, u_{m-1}), \quad m \geq 2, \tag{2}$$

where (x, u) represent the independent and dependent variables, respectively, and u_i denotes the ith-order derivative of u with respect to x, for $1 \leq i \leq m$. We consider $(x, u, u_1, \dots, u_{m-1})$ as the standard coordinates of the corresponding $(m-1)$th-order jet bundle, denoted by $J^{m-1}(\mathbb{R}, \mathbb{R})$[19] for details); and ϕ is a smooth function defined on an open subset $U \subseteq J^{m-1}(\mathbb{R}, \mathbb{R})$.

Throughout the text, when we say that a fact holds *locally*, we mean that it is valid in some open subset of U. For simplicity, we will continue to use the same notation U to denote such open subsets. Also, we will implicitly use the notation u_0 to denote u, for the sake of uniformity.

Recall that an important feature of the jet bundle is the contact ideal, which encodes the geometrical structure of this space. The contact ideal is generated by the contact forms, which are 1-forms that vanish on any section of the jet bundle that corresponds to the prolongation of a smooth function $u = f(x)$. The contact forms in the jet bundle $J^N(\mathbb{R}, \mathbb{R})$, $N \geq 1$, are given by:

$$\begin{aligned} \theta^0 &= u_1 dx - du, \\ \theta^1 &= u_2 dx - du_1, \\ &\vdots \\ \theta^{N-1} &= u_N dx - du_{N-1}. \end{aligned}$$

Suppose that there exist, locally, two functions $H = H(x, u, \dots, u_{m-1})$ and $\delta = \delta(x, u, \dots, u_{m-1})$, with δ is non-vanishing, satisfying

$$D_x H = \delta(u_m - \phi(x, u, \dots, u_{m-1})),$$

where D_x denotes the total derivative operator[1][2]

$$D_x = \partial_x + \sum_{i=0}^{m-1} u_{i+1} \partial_{u_i}.$$

The function H is called a first integral of equation (2), and δ an integrating factor.

The knowledge of a first integral allows us to reduce the order of the ODE (2), since equation (2) is, locally, equivalent to the family of $(m-1)$th-order ODEs

$$H(x, u, \ldots, u_{m-1}) = C, \quad C \in \mathbb{R}. \tag{3}$$

On the other hand, it is well known[20] that the ODE (2) is encoded by its associated Pfaffian system, i.e., by the differential ideal $\mathcal{I}_\phi$ of the algebra $\Omega^*(U)$ generated by the contact forms

$$\theta_0, \theta_1, \ldots, \theta_{m-2},$$

together with the 1-form

$$\theta_\phi := -\phi dx + du_{m-1}. \tag{4}$$

This Pfaffian system $\mathcal{I}_\phi$ is in correspondence with the involutive rank-1 distribution generated by the vector field associated to equation (2):

$$A_\phi = \partial_x + u_1 \partial_u + \cdots + u_{m-1} \partial_{u_{m-2}} + \phi \partial_{u_{m-1}}.$$

Therefore, $\mathcal{I}_\phi$ is completely integrable, in the sense that Frobenius theorem applies, i.e., there exist, locally, $m-1$ smooth functions $F_1, \ldots, F_{m-1}$ such that $\mathcal{I}_\phi$ is generated by the 1-forms $dF_1, \ldots, dF_{m-1}$ [20].

Finally, recall that in the case of a single 1-form ω, which corresponds to a corank-1 distribution, Frobenius theorem establishes that the requirement

$$d\omega \wedge \omega = 0$$

is equivalent to the (local) existence of a smooth function F such that $dF = \mu\omega$ for a certain non-vanishing function μ. The 1-form ω is said to be Frobenius integrable.

3. Main result

Consider an mth-order ODE as in (2). Unlike in the first-order case, the 1-form θ_ϕ defined in (4) is not, in general, Frobenius integrable. Nevertheless, since the Pfaffian system $\mathcal{I}_\phi$ is completely integrable there must exist, locally, a function $F = F(x, u, \ldots, u_{m-1})$ such that $dF \in \mathcal{I}_\phi$, that is,

$$dF = \mu\theta_\phi + \sum_{i=0}^{m-2} \alpha_i \theta_i,$$

for certain smooth functions $\mu, \alpha_0, \ldots, \alpha_{m-2}$ defined on U.

If necessary, we shrink the open set U so that the function μ is non-vanishing. We then define

$$\omega_{(\gamma_0, \ldots, \gamma_{m-2})} := \theta_\phi + \sum_{i=0}^{m-2} \gamma_i \theta_i, \tag{5}$$

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where $\gamma_i := \frac{\alpha_i}{\mu}$. With this definition, we have

$$dF = \mu\omega_{(\gamma_0,\dots,\gamma_{m-2})},$$

which implies that $\omega_{(\gamma_0,\dots,\gamma_{m-2})}$ is Frobenius integrable. Thus, it must satisfy

$$d\omega_{(\gamma_0,\dots,\gamma_{m-2})} \wedge \omega_{(\gamma_0,\dots,\gamma_{m-2})} = 0, \tag{6}$$

which is a first-order PDE system for the functions γ_i, $0 \le i \le m-2$.

On the other hand, observe that the smooth function F must satisfy the PDE system

$$dF \wedge \omega_{(\gamma_0,\dots,\gamma_{m-2})} = 0. \tag{7}$$

Now, we are in a position to state and prove the main result of this paper:

Theorem 3.1. Consider an mth-order ODE given by (2). A first integral F can be determined by first obtaining a particular solution to the PDE system (6) for the functions γ_i, $0 \le i \le m-2$, and subsequently solving the PDE system (7) for F.

Proof. Suppose that the PDE system (6) admits the particular solution

$$\gamma_i = \gamma_i(x, u, \dots, u_{m-1}), \text{ for } 0 \le i \le m-2.$$

Then, we use γ_i to define the 1-form $\omega_{(\gamma_0,\dots,\gamma_{m-2})}$ according to equation (5), which is therefore Frobenius integrable.

Consider now a smooth function $F = F(x, u, \dots, u_{m-1})$ satisfying the PDE system (7). Then, there exists, locally, a certain non-vanishing function μ such that

$$dF = \mu\omega_{(\gamma_0,\dots,\gamma_{m-2})}.$$

Expanding both sides of this expression, we have

$$\begin{aligned} F_x dx + F_u du + \cdots + F_{u_{m-1}} du_{m-1} &= \mu\left(\theta_\phi + \sum_{i=0}^{m-2}\gamma_i\theta_i\right) \\ &= \mu\left(-\phi dx + du_{m-1} + \sum_{i=0}^{m-2}\gamma_i(-u_{i+1}dx + du_i)\right), \end{aligned}$$

and by comparing the coefficients of the 1-forms $dx, du, du_1, \dots, du_{m-1}$, we obtain

$$\begin{aligned} F_x &= -\mu\phi - \sum_{i=0}^{m-2}\mu\gamma_i u_{i+1}, \\ F_{u_i} &= \mu\gamma_i, \text{ for } 0 \le i \le m-2, \\ F_{u_{m-1}} &= \mu. \end{aligned} \tag{8}$$

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So, finally, to check that F is a first integral for the ODE (2) we apply the total differential operator to F, and substitute the expressions (8):

$$\begin{aligned} D_x(F) &= F_x + \sum_{i=0}^{m-1} u_{i+1} F_{u_i} \\ &= -\mu\phi - \sum_{i=0}^{m-2} \mu\gamma_i u_{i+1} + \sum_{i=0}^{m-2} \mu\gamma_i u_{i+1} + \mu u_m \\ &= \mu(u_m - \phi). \end{aligned}$$

□

Remark 3.1. The PDE system (7) is a homogeneous linear system of first-order PDEs for the function F. In contrast, the PDE system (6) is a system of first-order PDEs for the functions γ_i, which is not generally linear. Moreover, for an mth-order ODE, the PDE system (6) consists of $\binom{m+1}{3}$ equations with $m-1$ unknown functions $\gamma_0, \ldots, \gamma_{m-2}$. As a result, solving this system is typically challenging. A practical strategy to address this complexity is to assume some standard ansatz for the functions γ_i, such as a dependence on fewer variables or linearity in some variable. As we will see in the examples, this kind of assumption not only reduces the complexity of the involved PDEs, but also allows us to write some equations as polynomials, in such a way that they can be split into simpler equations.

Remark 3.2. In certain cases, multiple particular solutions to the PDE system(6) can be identified. Each of these solutions may lead to the construction of distinct, independent first integrals, so increasing the number of conserved quantities or even providing the general solution of the ODE. For an illustration, see Example 4.1.

4. Second-order ODEs

In this section, we will explore the application of our results to the particular case of second-order ODEs. Given the ODE

$$u_2 = \phi(x, u, u_1), \tag{9}$$

we define the 1-form given by (5)

$$\begin{aligned} \omega_{(\gamma_0)} &:= -\phi dx + du_1 + \gamma_0 \theta_0 \\ &= (-\phi + \gamma_0 u_1) dx - \gamma_0 du + du_1, \end{aligned} \tag{10}$$

where $\gamma_0 = \gamma_0(x, u, u_1)$ is a smooth function to be determined.

In this case, condition (6), $d\omega_{(\gamma_0)} \wedge \omega_{(\gamma_0)} = 0$, reduces to the single PDE for γ_0:

$$\gamma_{0x} + \gamma_{0u} u_1 + \gamma_{0u_1} \phi + \gamma_0^2 - \phi_u - \gamma_0 \phi_{u_1} = 0. \tag{11}$$

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Once a particular solution $\gamma_0 = \gamma_0(x, u, u_1)$ is found, a first integral $F = F(x, u, u_1)$ can be obtained by solving the PDE system (7):

$$\gamma_0 F_{u_1} + F_u = 0, \tag{12a}$$
$$\gamma_0 F_x + (\gamma_0 u_1 - \phi) F_u = 0, \tag{12b}$$
$$F_x + (\phi - \gamma_0 u_1) F_{u_1} = 0. \tag{12c}$$

The following illustrative example showcases how to use the results above to find first integrals of a second-order ODE.

Example 4.1. Consider the second-order ODE given by:

$$u_2 = \frac{(3xu_1 + u)(xu_1 - u)}{2x^2 u}. \tag{13}$$

To solve the determining equation (11) for γ_0 we assume the ansatz $\gamma_0 = g(x, u)u_1 + h(x, u)$, in such a way that (11) simplifies to the polynomial:

$$D_2 u_1^2 + D_1 u_1 u + D_0 = 0,$$

where

$$D_0 := -2x^2u^2h^2 - 2x^2u^2h_x + u^3 g - 2xu^2 h - u^2, \tag{14a}$$
$$D_1 := -4x^2u^2gh - 2x^2u^2h_u - 2x^2u^2g_x + 6x^2uh, \tag{14b}$$
$$D_2 := -2x^2u^2g^2 - 2x^2u^2g_u + 3x^2ug - 3x^2. \tag{14c}$$

Setting the coefficients $D_0 = D_1 = D_2 = 0$ we obtain a system of three PDEs for the functions g and h. With the aid of a computer algebra system, we find the particular solution

$$g = \frac{1}{u}, \quad h = 0,$$

and thus we take $\gamma_0 = \frac{u_1}{u}$.

By substituting this expression into (12) and clearing the denominators, we obtain the following system of equations:

$$u_1 F_{u_1} + u F_u = 0, \tag{15a}$$
$$u(x^2u_1^2 - 2xuu_1 - u^2)F_u - 2x^2uu_1F_x = 0, \tag{15b}$$
$$(x^2u_1^2 - 2xuu_1 - u^2)F_{u_1} + 2x^2uF_x = 0. \tag{15c}$$

The reader can verify that a particular solution is given by

$$F(x, u, u_1) = \ln(x) + 2\,\text{arctanh}\Big(\frac{xu_1}{u}\Big), \tag{16}$$

which is therefore a first integral of the ODE (13).

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Notably, another particular solution to (14) can be checked to be

$$g = \frac{3}{2u}, \quad h = -\frac{1}{2x},$$

and, consequently, the choice $\gamma_0 = \frac{3xu_1 - u}{2xu}$ allows us to find another first integral of (13), provided the corresponding PDE system is solved:

$$\begin{aligned} 2x^2 F_x - (xu_1 + u)F_{u_1} &= 0, && (17a)\\ 2xuF_u + (3xu_1 - u)F_{u_1} &= 0, && (17b)\\ (xu - 3x^2u_1)F_x - (xuu_1 + u^2)F_u &= 0. && (17c) \end{aligned}$$

A particular solution for (17) is given by

$$(x, u, u_1) = \frac{xu_1 - u}{\sqrt{xu^3}}, \tag{18}$$

which is, therefore, another first integral of (13).

The first integrals given by (16) and (18) describe the general solutions of the ODE (13) in implicit form.

In the following example, we use our approach to find a first integral and a 1-parameter family of solutions to a second-order ODE that does not admit Lie point symmetries.

Example 4.2. Consider the following second-order ODE

$$u_2 = 1 + xu_1 - \frac{2xu}{u_1}. \tag{19}$$

The reader can check that it does not admit Lie point symmetries, so standard procedures cannot be applied.

To find a first integral we first tackle the PDE (11), which is, in this case,

$$u_1^2\gamma_{0x} + u_1^3\gamma_{0u} + \left(u_1^2 + xu_1^3 - 2xuu_1\right)\gamma_{0u_1} + u_1^2\gamma_0^2 - \left(2xu + xu_1^2\right)\gamma_0 + 2xu_1 = 0.$$

By using the natural ansatz $\gamma_0 = \gamma_0(x, u_1)$ this equation can be regarded as a polynomial in the u variable:

$$D_1 u + D_0 = 0,$$

where

$$\begin{aligned} D_1 &= 2xu_1\gamma_{0u_1} + 2x\gamma_0, && (20a)\\ D_0 &= -u_1^2\gamma_{0x} - \left(u_1^2 + xu_1^3\right)\gamma_{0u_1} - u_1^2\gamma_0^2 + xu_1^2\gamma_0 - 2xu_1. && (20b) \end{aligned}$$

The particular solution $\gamma_0 = \frac{1}{u_1}$ for the system given by $D_0 = D_1 = 0$ is easy to find. Now, we write system (12):

$$
\begin{aligned}
F_{u_1} + u_1 F_u &= 0, && (21a)\\
x(u_1^2 - 2u)F_u - F_x &= 0, && (21b)\\
x(u_1^2 - 2u)F_{u_1} + u_1 F_x &= 0. && (21c)
\end{aligned}
$$

The general solution to equation (21a) is

$$
F(x, u, u_1) = g(x, u_1^2 - 2u),
$$

where $g = g(x, y)$ is an arbitrary smooth function. From equations (21b) and (21c) it follows that g must satisfy

$$
2xyg_y + g_x = 0. \tag{22}
$$

This equation admits the particular solution $g(x, y) = ye^{-x^2}$, therefore a particular solution to the system (21) is

$$
F(x, u, u_1) = (u_1^2 - 2u)e^{-x^2},
$$

which is a first integral of the ODE (19).

Remarkably, even if for the reduced ODE

$$
(u_1^2 - 2u)e^{-x^2} = C, \quad C \in \mathbb{R},
$$

no straightforward analytical method yields a general solution, in the particular case $C = 0$ the solutions are given by the family

$$
u(x) = \frac{1}{2}(x + K)^2, \quad K \in \mathbb{R},
$$

which in turn is a 1-parameter family of solutions for the ODE (19). Thus, our method has facilitated a partial integration of an ODE for which classical approaches appear to be ineffective.

The preceding examples illustrate the effectiveness of the method in obtaining first integrals. In the next subsection, we will see that, remarkably, for autonomous second-order ODEs, equation (???) admits a prescribed solution. This ensures that the method introduced in this work can always be applied in this important class of problems.

4.1. Autonomous second-order ODEs

In the case of an autonomous second-order ODE

$$
u_2 = \phi(u, u_1), \tag{23}
$$

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the smooth function $\gamma_0 = \frac{\phi}{u_1}$ is always a particular solution to equation (11), as can easily be checked. By substituting in (12) the PDE system for the first integral $F = F(x, u, u_1)$ reduces to:

$$\begin{aligned} \phi F_{u_1} + u_1 F_u &= 0, \\ F_x &= 0. \end{aligned} \tag{24}$$

Incidentally, this first integral corresponds to the *foliation energy* introduced in [?].

Observe that the solutions are of the form $F = F(u, u_1)$, so the reduced equation

$$F(u, u_1) = C, \quad C \in \mathbb{R}, \tag{25}$$

is also autonomous. If u_1 can be explicitly isolated in (25) as a function of u (and possibly C), the equation can be solved by quadrature, leading to a 2-parameter family of solutions for equation (23).

Example 4.3. Consider the autonomous second-order ODE given by

$$u_2 = \frac{u_1^2 - u^2 u_1 - 2u_1}{u^2 + u}, \tag{26}$$

which only have the trivial Lie point symmetry ∂_x. In this case the PDE system (24),

$$\begin{aligned} F_x &= 0, \\ \frac{u_1^2 - u^2 u_1 - 2u_1}{u^2 + u} F_{u_1} + u_1 F_u &= 0, \end{aligned}$$

admits the particular solution

$$F(u, u_1) = \frac{u^2 + u u_1 + u_1 - 2}{u}.$$

Therefore, equation (26) can be reduced to the first-order family of ODEs:

$$\frac{u^2 + u u_1 + u_1 - 2}{u} = C, \tag{27}$$

with $C \in \mathbb{R}$.

This family of ODEs can be solved by quadrature, leading to the general solution of (26), which is implicitly expressed as:

$$x + \frac{1}{2} \ln\left(Cu - u^2 + 2\right) + \frac{C+2}{\sqrt{C^2+8}} \operatorname{arctanh}\left(\frac{C - 2u}{\sqrt{C^2+8}}\right) = K,$$

where $C, K \in \mathbb{R}$.

5. Higher-order ODEs

In this section we will show examples of how our approach can be successfully applied to third- and fourth-order ODEs. The general procedure is the same as for second-order ODEs, but the complexity of the PDE systems increases with the order of the ODE.

Example 5.1. Consider the third-order ODE

$$u_3 = \frac{u + xu_1 - xu_1u_2^2 - uu_2^2}{2u_2}. \tag{28}$$

According to Theorem 3.1, we need to solve the PDE system (6),

$$d\omega_{(\gamma_0,\gamma_1)} \wedge \omega_{(\gamma_0,\gamma_1)} = 0,$$

for the functions γ_0, γ_1. By using the ansatz $\gamma_0 = \gamma_0(x, u_2), \gamma_1 = \gamma_1(x, u_2)$, the resulting PDE system becomes:

$$\left(\gamma_{1x} + \gamma_0 + \frac{x(u_2^2-1)}{2u_2}\right)\gamma_0 - \left(\gamma_{0x} + \frac{(u_2^2-1)}{2u_2}\right)\gamma_1 = 0, \tag{29a}$$

$$-\gamma_{1u_2}\gamma_0 + \gamma_{0u_2}\gamma_1 = 0, \tag{29b}$$

$$-\gamma_{0x} - \frac{(u_2^2-1)}{2u_2} + \gamma_{0u_2}\left(\gamma_1 u_2 + \gamma_0 u_1 + \frac{xu_1u_2^2 + uu_2^2 - xu_1 - u}{2u_2}\right) - \left(\gamma_{1u_2}u_2 + \gamma_1 + \gamma_{0u_2}u_1 + xu_1 + u - \frac{xu_1u_2^2 + uu_2^2 - xu_1 - u}{2u_2^2}\right)\gamma_0 = 0, \tag{29c}$$

$$-\gamma_{1x} - \gamma_0 - \frac{x(u_2^2-1)}{2u_2} + \gamma_{1u_2}\left(\gamma_1 u_2 + \gamma_0 u_1 + \frac{xu_1u_2^2 + uu_2^2 - xu_1 - u}{2u_2}\right) - \left(\gamma_{1u_2}u_2 + \gamma_1 + \gamma_{0u_2}u_1 + xu_1 + u - \frac{xu_1u_2^2 + uu_2^2 - xu_1 - u}{2u_2^2}\right)\gamma_1 = 0. \tag{29d}$$

A particular solution can be found by using a computer algebra system, and it is given by

$$\gamma_0 = \frac{x(1-u_2^2)}{2u_2},$$
$$\gamma_1 = 0.$$

With these values for γ_0 and γ_1, the PDE system (7) for the first integral $F = F(x, u, u_1, u_2)$ becomes

$$xF_x - uF_u = 0, \tag{30a}$$
$$-xu_2^2F_{u_2} + xF_{u_2} + 2u_2F_u = 0, \tag{30b}$$
$$-uu_2^2F_{u_2} + 2u_2F_x + uF_{u_2} = 0, \tag{30c}$$
$$F_{u_1} = 0, \tag{30d}$$

and a particular solution can be checked to be

$$F = e^{xu}(u_2^2 - 1). \tag{31}$$

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The reduced ODE

$$e^{xu}(u_2^2 - 1) = C, \quad C \in \mathbb{R},$$

does not appear to admit an explicit closed-form solution in the general case. However, the particular choice $C = 0$ leads to a notable simplification, yielding the 2-parameter family of solutions for (28):

$$u(x) = \pm\frac{1}{2}x^2 + K_1 x + K_2, \quad K_1, K_2 \in \mathbb{R}.$$

Observe that the ODE (28) does not possess Lie point symmetries, as the reader may check. Nevertheless, we successfully derived a first integral (31), enabling partial integration of the ODE, without relying on Lie point symmetries or integrating factors.

Example 5.2. Consider the fourth-order ODE

$$u_4 = \frac{e^{x+u}(xu_1 - u_1 u_3 + x - u_3 + 2) - 2u_3}{x - 3u_3 + 2e^{x+u}}. \tag{32}$$

In this case, we have

$$\theta_\phi = -\frac{e^{x+u}(xu_1 - u_1 u_3 + x - u_3 + 2) - 2u_3}{x - 3u_3 + 2e^{x+u}}\,dx + du_3,$$

and then the 1-form

$$\omega_{(\gamma_0,\gamma_1,\gamma_2)} = \theta_\phi + \gamma_0\theta_0 + \gamma_1\theta_1 + \gamma_2\theta_2,$$

must satisfy condition (6):

$$d\omega_{(\gamma_0,\gamma_1,\gamma_2)} \wedge \omega_{(\gamma_0,\gamma_1,\gamma_2)} = 0.$$

This system of 10 PDEs for γ_0, γ_1 and γ_2 is too involved to be included in the text. However, by using the ansatz

$$\gamma_0 = \gamma_0(x, u, u_3), \quad \gamma_1 = \gamma_1(x, u, u_3), \quad \gamma_2 = \gamma_2(x, u, u_3),$$

we obtain the particular solution

$$\begin{aligned} \gamma_0 &= \frac{e^{x+u}(x - u_3)}{2e^{x+u} + x - 3u_3}, \\ \gamma_1 &= 0, \\ \gamma_2 &= 0, \end{aligned} \tag{33}$$

using a computer algebra system.

Upon substituting these expressions into the PDE system (7), we obtain the following system of equations for the first integral F:

$$e^{x+u}(x-u_3)F_{u_3}+(2e^{x+u}+x-3u_3)F_u=0, \tag{34a}$$
$$(e^{x+u}(x-u_3+2)-2u_3)F_u+e^{x+u}(u_3-x)F_x=0, \tag{34b}$$
$$(e^{x+u}(u_3-x-2)+2u_3)F_{u_3}+(2e^{x+u}+x-3u_3)F_x=0, \tag{34c}$$
$$F_{u_1}=0, \tag{34d}$$
$$F_{u_2}=0. \tag{34e}$$

The reader can verify that a particular solution is given by

$$F=(u_3-e^{x+u})(u_3-x)^2.$$

Again, we want to point out that our approach has allowed us to find the first integral (35) for the ODE (32) without making use of Lie point symmetries (which are not available for this equation) or integrating factors. Moreover, even if the reduced ODE

$$(u_3-e^{x+u})(u_3-x)^2=C, \quad C\in\mathbb{R},$$

is not easily solved for arbitrary $C\in\mathbb{R}$, the particular case $C=0$ provides the 3-parameter family of solutions for (32):

$$u(x)=\frac{1}{24}x^4+\frac{K_1}{2}x^2+K_2x+K_3,$$

where $K_1,K_2,K_3\in\mathbb{R}$.

6. Comparison with other methods

In this section we will show the relationship between the method presented in this paper and similar approaches such as the λ-symmetries method, the S-function method, and the extended Prelle–Singer method for higher-order equations.

First, the method presented in this work is connected to the theory of canonical λ-symmetries for second-order ODEs. Specifically, equation (11) serves as the determining equation for a canonical λ-symmetry of the ODE (9), as established in equation (5) in reference[21]. Therefore, identifying the function $\lambda=\lambda(x,u,u_1)$ that makes ∂_u a λ-symmetry is equivalent to finding a solution $\gamma_0=\gamma_0(x,u,u_1)$ in the first step of Theorem 3.1.

Also, our approach is related to the framework of the S-function method, introduced in [22][23] for second-order ODEs and generalized in [?,?,?], as well as to the extended Prelle–Singer method for higher-order equations developed in [14]. Both are extensions of the Prelle–Singer procedure [?], originally formulated for first-order ODEs, and both aim at finding first integrals and solutions in terms of elementary functions.

These extended Prelle–Singer methods share with the approach of this paper the goal of constructing first integrals for higher-order ordinary differential equations by identifying auxiliary functions. In the extended Prelle–Singer framework one introduces the so-called null forms S_i (alongside an integrating factor R) so that the modified differential 1-form becomes exact and hence integrable to a first integral I. Analogously, our geometric approach seeks coefficients γ_j for the contact 1-forms to find a Frobenius-integrable 1-form on the relevant jet bundle. In fact, one may view each S_i in the Prelle–Singer method as playing the same role as $-\gamma_{i-1}$ in our construction. Thus, the underlying idea of introducing "extra" functions whose compatibility yields a first integral is common to both approaches.

In contrast, the two approaches differ fundamentally in their nature. Whereas the Prelle–Singer method is essentially algebraic, constructing and manipulating modified differential forms until they become exact, our approach is intrinsically geometric, exploiting the structure of the contact distribution of the jet bundle and the integrability of differential ideals to obtain first integrals.

Moreover, there is an operational difference residing in how these auxiliary functions are used and obtained. The Prelle–Singer method pairs the functions S_i with an explicit integrating factor R that must be solved for from an overdetermined system of algebraic PDEs; once the functions S_i and R are known, one performs an integration to recover the first integral. In contrast, our geometric strategy bypasses any integrating factor by embedding the problem in the language of jet bundles and Pfaffian ideals: we directly enforce the Frobenius integrability condition $d\omega \wedge \omega = 0$ to solve for the γ_j, and then integrate a homogeneous linear PDE system for the first integral itself. This difference is illustrated in the following example.

Example 6.1. In Example 4.1 in [14] it is considered as second-order ODE, which adapted to our notation becomes

$$u_2 = \frac{2u-1}{1+u^2}\,u_1^2. \tag{36}$$

The authors obtain the following equations:

$$S_x + u_1 S_u + \left(\frac{2u-1}{1+u^2}u_1^2\right) S_{u_1} = 2\frac{u(u+1)-1}{(1+u^2)^2}u_1^2 + \frac{2(2u-1)u_1}{1+u^2}S + S^2, \tag{37}$$

$$R_x + u_1 R_u + \left(\frac{2u-1}{1+u^2}u_1^2\right) R_{u_1} = -R\left(S + \frac{2(2u-1)u_1}{1+u^2}\right), \tag{38}$$

$$R_u = R_{u_1} S + R S_{u_1}, \tag{39}$$

and the following particular solution for them:

$$S = \frac{(2u-1)}{(1+u^2)}u_1, \quad R = -\frac{e^{\arctan(u)}}{(1+u^2)}. \tag{40}$$

The corresponding first integral is given in equation (4.15) in{m/14/}, which written in our notation becomes:

$$I = \frac{u_1 e^{\arctan(u)}}{(1+u^2)}. \tag{41}$$

On the other hand, following the procedure of the present work we write down equation (11):

$$\gamma_{0x} + \gamma_{0u} u_1 + \left(\frac{(2u-1)}{(1+u^2)}u_1^2\right)\gamma_{0u_1} + \gamma_0^2 + 2\frac{u(u+1)-1}{(1+u^2)^2}u_1^2 - \gamma_0\frac{2(2u-1)u_1}{1+u^2} = 0, \tag{42}$$

which coincides with equation (37) by substituting S by $-\gamma_0$. We can consider the solution

$$\gamma_0 = -\frac{(2u-1)}{(1+u^2)}u_1,$$

which corresponds to the one obtained in equation (40) for S, and we write the PDE system (12):

$$F_x - \frac{u_1}{x}F_{u_1} = 0, \tag{43a}$$

$$F_u + \frac{2xu\,u_1 - xu_1 + u^2 + 1}{x(u^2+1)}F_{u_1} = 0, \tag{43b}$$

$$\frac{2xu\,u_1 - xu_1 + u^2 + 1}{x(u^2+1)}F_x + \frac{u_1}{x}F_u = 0. \tag{43c}$$

A solution to this system is the first integral

$$F = \frac{(xu_1 - u^2 - 1)e^{\arctan(u)}}{1+u^2}, \tag{44}$$

and we can derive from it, as a byproduct, the integrating factor

$$\mu = \frac{x e^{\arctan(u)}}{1+u^2}. \tag{45}$$

The reader can observe that neither the first integral (44) obtained with our method coincides with expression (41), nor does the integrating factor (45) coincide with the one provided by the extended Prelle–Singer method in equation (40).

7. Conclusions

In this work, it has been presented an approach to find first integrals of mth-order ODEs that does not require the computation of integrating factors or the knowledge of Lie point symmetries. The approach, which is based on the complete integrability of the Pfaffian system associated to the ODE, requires the solution of PDE systems, usually through the assumption of ansätze for the unknown functions and the use of computer algebra systems. The effectiveness of the method has been illustrated with examples of second-, third-, and fourth-order ODEs, showing that it can be applied to a wide range of ODEs, including those that do not admit Lie point symmetries.

We acknowledge that the method may appear complex, compared to some traditional techniques. However, this perceived complexity, involving the formulation of an ansatz and reliance on symbolic computation, is a direct cost for greater scope. The primary benefit of our approach is precisely its ability to succeed where traditional methods fail, particularly for ODEs that lack the required Lie point symmetries for standard integration techniques to work. This more intricate structure of the method is not a weakness but rather the very source of its strength, providing a systematic pathway to solve problems that would otherwise be intractable.

On the other hand, the implementation of the method in computer algebra systems is straightforward, and it could be particularly useful. Such an implementation could be combined with a systematic exploration of different ansätze, for example, by progressively selecting an increasing number of variables for the functions γ_i to depend on. This combination could enhance existing algorithms, increasing the likelihood of finding first integrals of ODEs. In this way, our approach offers a complementary tool to the methods currently available, potentially extending the range of solvable problems.

Statements and Declarations

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Use of Generative-AI Tools Declaration

During the preparation of this work, the author used ChatGPT exclusively for grammar and language refinement. After using this service, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.

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