Review of: "A Convergence Not Metrizable"

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Title: A convergence not metrizable.

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The author gives an example of a space of functions between two metric spaces where the pointwise convergence is not metrizable. The proof is not quite convincing - see the remarks below. The hypotheses must be carefully specified. The metric spaces with the mentioned property deserve a more detailed examination. For instance, give examples of metric spaces not "strongly second numerable".

Engelking, Ryszard; Sieklucki, Karol

Topology. A geometric approach. Transl. from the Polish by Adam Ostaszewski.

Sigma Series in Pure Mathematics. 4. Berlin: Heldermann. viii, 429 p. (1992).

Kelley, John L.

General topology. 2nd ed.

Graduate Texts in Mathematics. 27. New York - Heidelberg - Berlin: Springer-Verlag. XIV, 298 p.,

Remarks

page 1

- "For the purpose of only to fix terminology" delete the text in red
- "strongly second numerable" I consider this term inappropriate

A topological space containing a dense countable set is called separable, so you can call a metric space satisfying the required condition as "a double separable space" or a "strongly separable space"

also replace "numerable subset" by "countable subset"

- in Prop. 0.2

"there does not exist"

"non-unitary path-component" is not quite usual, better say "containing a non-trivial path"

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- "It follows that $\phi : M \to N$ is discontinuous at every point in M." add the text in blue
- "approximated by a sequence"
- in the definition of the sets F^1_m and G^1_m

- replace the Portuguese "y" by "and"

- the same two lines below and replace $f^1_{u,m}$ with $f^1_{U,m}$ (check this in the whole paper)
- Warning. The following situation can occur. It is possible to exist a point $x_m \in M$ such that $\frac{1}{m+1} < d(x_1, x_m) < \frac{1}{m}$.

Then $x_m \notin B[x, \frac{1}{m+1}] \cup (M \setminus B(x,$

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m+1)), being possible that f

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U,m (x m) ∈ (0,1)

and that (f 1

m (x)) m ∈ N

does not converge to $\phi(x) \in \{a,b\} = \{\psi(0),\psi(1)\}$.

• in the second item of the proof you write balls as B(x;r) (with semicolon :) while

in the first item you have used a dot (,). Use a unitary style.

• ”such that

$B(x_i,\delta) \cap B(x_j,\delta) = \varnothing$,

for all i,j ∈ {1,...,n} with i 6= j.” write it in this way

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References

I think that it must be ”General” not ”Geral”

This is a book on topology inaccessible to the majority of the readers. I consider it more

appropriate to quote a standard text in topology.

I suggest the following proof. There is no need to make a distinction between the cases

$n = 1$ and $n > 1$.

Proof. Write the set D as $D = \{x_k : k \in \mathbb{N}\}$, where $x_i \neq x_j$ for $i \neq j$. Let $a \neq b$ be

two points in N for which there exists a path $\psi : [0,1] \to N$ with $\psi(0) = a$ and $\psi(1) = b$.

Consider the function $\phi : M \to N$ given by

$$\phi(x) = \begin{cases} a & \text{if } x \in D \\ b & \text{if } x \in M \setminus D. \end{cases}$$

Then ϕ is discontinuous at every point $x \in M$.

For $n \in \mathbb{N}$ let $D_n = \{x_1,\dots,x_n\}$ and let $\phi_n : M \to N$ be given by

$$\phi_n(x) = \begin{cases} a & \text{if } x \in D_n \\ b & \text{if } x \in M \setminus D_n. \end{cases}$$

Then the sequence (ϕ_n) is pointwise convergent to ϕ.

Indeed, let $x \in M$. If $x \in D$, say $x = x_k$ for some $k \in \mathbb{N}$, then $\phi_n(x) = a = \phi(x)$ for all

$n \geq k$.

If $x \in M \setminus D$, then $x \notin D_n$ for all $n \in \mathbb{N}$, so that $\phi_n(x) = b = \phi(x)$, for all $n \in \mathbb{N}$.

For $m,n \in \mathbb{N}$ put

$$A^n_m = \bigcup_{i=1}^{n} B\left(x_i, \frac{1}{m+1}\right) \quad \text{and} \quad B^n_m = M \setminus \bigcup_{i=1}^{n} B\left(x_i, \frac{1}{m}\right).$$

Then A^n_m and B^n_m are closed subsets of M and $A^n_m \cap B^n_m = \varnothing$. Indeed, if $x \in A^n_m \cap B^n_m$

m ,

then

$d(x\ j\ ,x) \leq$

1

$m + 1$

for some $j \in \{1,...,n\}$,

and

$d(x\ i\ ,x) \geq$

1

m

for all $i \in \{1,...,n\}$.

This leads to the contradiction

1

m

$\leq d(x\ j\ ,x) \leq$

1

$m + 1$.

By Urysohn's Lemma there exists a continuous function f n

U,m

: $M \rightarrow [0,1]$ such that

f n

U,m (A

n

m) $\subset \{1\}$ and f

n

U,m (B

n

m) $\subset \{0\}$. Let f

n

m

$= \psi \circ f$ 1

U,m . We shall show that the sequence

(f n

m) $m \in N$

is pointwise convergent to ϕ n for every $n \in N$.

Let $n \in N$ and $x \in M$. If $x \in D\ n \subset A\ n$

m , then f

n

U,m (x) = 1 and f

n

m (x) = b = ϕ n (x), for

all m ∈ N.

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If x / ∈ D n , then δ n := inf{d(x i ,x) : 1 ≤ i ≤ n} > 0, and, for all m ≥ 1/δ n ,

d(x i ,x) ≥ δ n ≥

1

m ,

so that x / ∈ B(x i ,1/m) for all 1 ≤ i ≤ n, that is, x ∈ B n

m . Hence, f

n

U,m (x) = 0 and

f n

m (x) = a = ϕ n (x), for all m ≥ 1/δ n .

From here, the proof can be finished as in your manuscript.