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Research Article

Kronecker-Pauli Operators

Christian Rakotonirina[1,2]

1. School of Civil Engineering, Institut Supérieur de Technologie d'Antananarivo, Madagascar; 2. Laboratory of Dynamics of the Atmosphere, Climate and Oceans (DyACO), University of Antananarivo, Antananarivo, Madagascar

In quantum mechanics, classical matrix bases such as the Pauli matrices are often generalized to higher dimensions. So, it is useful to express their corresponding operators using the Dirac Bra and Ket. In this paper, to express the corresponding operators we review the Kronecker-Pauli matrices and how to construct them for an N-dimensional system, with N a prime integer, N>2. Then, we give the expression of the Kronecker-Pauli operators and show that their matrices with respect to the standard basis fulfill the conditions to form a set of Kronecker-Pauli matrices. Relationship between the Kronecker-Pauli operators and the Weyl operators has been studied.

Corresponding author: Christian Rakotonirina, rakotonirinachristianpierre@gmail.com

1. Introduction

In quantum mechanics, classical matrix bases such as the Pauli matrices are often generalized to higher dimensions. For example, the Gell-Mann matrices are generalized to any dimension without knowing the corresponding operators in the standard basis (see for example, [1]). Operators are easier to use than higher-dimensional matrices, which has led to the construction of the Gell-Mann operators by using the Dirac bra and ket (see, for example [2][3]). So, we study, in this paper, what operators whose matrices in the standard basis are the matrices in a set of Kronecker-Pauli matrices. Kronecker-Pauli matrices (KPMs) are extensions of the Pauli strings, studied in [4].

Knowing that the Kronecker product of sets of KPMs is a set of KPMs [5], we study only the case of N-dimensional, where N is a prime integer, $N \geq 3$.

By examining the sets of KPMs, we will construct the Kronecker-Pauli operators. Then, as an example, we will construct a 5×5- KPMs from the Kronecker-Pauli operators. However, we know that there are at least two sets of 5×5- KPMs. That will engage a discussion.

The paper is organized as follows. In the first section, we review the sets of KPMs, and how to construct them. In the second section, we give the expression of the Kronecker-Pauli operators. In the third section, the relation with the Weyl operators is studied. We finish the paper with a discussion and conclusion.

2. Sets of Kronecker-Pauli matrices

Definition 1

For n integer $n > 1$, let us define a set of $n \times n$- KPMs as a family $(\prod_k)_{0 \le k \le n^2-1}$ of n^2 matrices which satisfy the following properties [5] :

i. $\mathbf{S}_{n \otimes n} = \frac{1}{n} \sum_{k=0}^{n^2-1} \prod_k \otimes \prod_k$ is the $n \otimes n$ swap operator;
ii. ${\prod_k}^{\dagger} = \prod_k$, for $\left(0 \le k \le n^2 - 1\right)$, (hermiticity);
iii. ${\prod_k}^{2} = I_n$, for $\left(0 \le k \le n^2 - 1\right)$, (square root of the unit);
iv. $\mathrm{Tr}\left({\prod_k}^{\dagger} \prod_j\right) = n\delta_{kj}$, for $\left(0 \le k,\ \ j \le n^2 - 1\right)$, (orthogonality).

where δ_{kj} is the Kronecker symbol.

To construct such a family, the concept of the inverse-symmetric matrix is useful [5].

Definition 2

Let us call an inverse-symmetric matrix an invertible complex matrix $\mathbf{A} = \left(A^i_j\right)$ such that $A^j_i = \frac{1}{A^i_j}$ if $A^i_j \neq 0$.

Proposition 1

For any $n \times n$ inverse-symmetric matrix $\mathbf{A}$, with only n non-zero elements, $\mathbf{A}^2 = \mathbf{I}_n$ is the unit matrix.

Consider the case of N-dimensional matrices where N is a prime integer. According to this proposition 1, the choice of inverse-symmetric matrices ensures property iii) of definition 1. To ensure property ii) of hermiticity, consider a family of inverse-symmetric matrices whose elements are the N-th roots of the unit. The following proposition [4] ensures properties i) and iv).

In the following proposition, the matrix of an operator is its matrix in the standard basis $(\mid 0\rangle\,,\ \ \mid 1\rangle\,,\ \ \mid 2\rangle\,,\ \ \ldots\,,\ \mid N-1\rangle\ \,)$.

Proposition 2

Let $\mathbf{P}_1$, $\mathbf{P}_2$, ..., $\mathbf{P}_N$ be operators whose matrices are symmetric permutation matrices with only one unit in the diagonal.

$\prod_0 = \mathbf{P}_1$ and $\prod_1, \prod_2, \dots, \prod_{N-1}$ are operators whose matrices are obtained by replacing the "ones" in $\prod_0 = \mathbf{P}_1$ by the N-th roots of unity while keeping that they are inverse-symmetrics. We do the same to the operators $\mathbf{P}_2, \dots, \mathbf{P}_N$ in order to have the operators

$\prod_N = \mathbf{P}_2$ and $\prod_{N+1}, \prod_{N+2}, \cdots, \prod_{2N-1}$

....

$\prod_{N^2-N} = \mathbf{P}_N$ and $\prod_{N^2-N+1}, \prod_{N^2-N+2}, \cdots, \prod_{N^2-1}$

whose matrices are inverse-symmetrics.

If

1. The sum $\mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_N$ is equal to the operator whose matrices is the $N \times N$ ones matrix;
2. For any $l \in \{0, 1, 2, \ldots, N-1\}$, for any $k, \ j \in \{lN+1, lN+2, \ \ldots, \ lN+N-1\}$, for any two places in a $N \times N$-matrix, non-symmetrics with respect to the diagonal where the elements of $\prod_k$ are $e^{\frac{2i\pi p_k}{N}}$ and $e^{\frac{2i\pi r_k}{N}}$ and the elements of $\prod_j$ are $e^{\frac{2i\pi p_j}{N}}$ and $e^{\frac{2i\pi r_j}{N}}$ such that

$$e^{\frac{2i\pi(r_k+p_k)}{N}} \neq e^{\frac{2i\pi(r_j+p_j)}{N}}$$

Then

$\mathbf{S}_{N\otimes N} = \frac{1}{n}\sum_{k=0}^{n^2-1}\prod_k \otimes \prod_k$ is the $N \otimes N$ swap operator and $\mathrm{Tr}\left(\prod_k{}^{\dagger}\prod_j\right) = N\delta_{kj}$.

The following example was an example from [4], constructed following the hypotheses of Proposition 2 above. The property i) of definition 1 is checked with the help of SCILAB software.

Example 1

$$\chi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \chi_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & \eta^3 & 0 \end{pmatrix}, \chi_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & \eta & 0 \end{pmatrix},$$

$$\chi_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^3 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & \eta^4 & 0 \end{pmatrix}, \chi_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^4 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & \eta^2 & 0 \end{pmatrix},$$

$$\chi_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \chi_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & \eta^3 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \end{pmatrix}, \chi_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\chi_9 = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & \eta^4 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \end{pmatrix}, \chi_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta^3 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\chi_{11} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \chi_{12} = \begin{pmatrix} 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 1 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \end{pmatrix}, \chi_{13} = \begin{pmatrix} 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 1 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \end{pmatrix},$$

$$\chi_{14} = \begin{pmatrix} 0 & 0 & 0 & \eta^3 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 1 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \end{pmatrix}, \chi_{15} = \begin{pmatrix} 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 1 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \end{pmatrix},$$

$$\chi_{16} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \chi_{17} = \begin{pmatrix} 0 & \eta & 0 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta^3 & 0 & 0 \end{pmatrix}, \chi_{18} = \begin{pmatrix} 0 & \eta^2 & 0 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta & 0 & 0 \end{pmatrix},$$

$$\chi_{19} = \begin{pmatrix} 0 & \eta^3 & 0 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta^4 & 0 & 0 \end{pmatrix}, \chi_{20} = \begin{pmatrix} 0 & \eta^4 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \end{pmatrix}.$$

$$\chi_{21} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \chi_{22} = \begin{pmatrix} 0 & 0 & \eta & 0 & 0 \\ 0 & 0 & 0 & \eta^2 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \chi_{23} = \begin{pmatrix} 0 & 0 & \eta^2 & 0 & 0 \\ 0 & 0 & 0 & \eta^4 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\chi_{24} = \begin{pmatrix} 0 & 0 & \eta^3 & 0 & 0 \\ 0 & 0 & 0 & \eta & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \chi_{25} = \begin{pmatrix} 0 & 0 & \eta^4 & 0 & 0 \\ 0 & 0 & 0 & \eta^3 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

with $\eta = e^{\frac{2i\pi}{5}}$.

3. Kronecker-Pauli Operators

The form of a Kronecker-Pauli matrix in a set of KPMs suggests the following definition of a Kronecker-Pauli operator. For a prime integer N, our goal is to construct a system of operators whose matrices in the standard basis form a set of $N \times N$- KPMs.

Definition 2

Let us define a Kronecker-Pauli operator as the operator

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$$\prod_{\sigma,l,n,N} = \sum_{k=0}^{N-1} e^{\frac{2i\pi}{N}(k-l)n} |\sigma(\ k)\rangle \langle\ k|$$

where N is a prime integer, $n = 0, 1, 2, \ldots,\ N-1$, $l \in \{0, 1, 2, \ldots,\ N-1\}$, σ is a symmetric permutation on the set $\{0, 1, 2, \ldots,\ N-1\}$, such that $\sigma(\sigma(k)) = k$, for any $k \in \{0, 1, 2, \ldots,\ N-1\}$, $\sigma(l) = l$ and $\sigma(k) \neq k$, if $k \neq l$.

Let us take a function σ defined as

$$0 \leq \sigma(k) = (-k + 2l) modN < N$$

for any $k \in \{0, 1, 2, \ldots,\ N-1\}$.

σ is a symmetric permutation on $\{0, 1, 2, \ldots,\ N-1\}$ such that $\sigma(\sigma(k)) = k$, for any $k \in \{0, 1, 2, \ldots,\ N-1\}$, $\sigma(l) = l$ and $\sigma(k) \neq k$, if $k \neq l$. Thus, the operator

$$\prod_{l,n,N} = \sum_{k=0}^{N-1} e^{\frac{2i\pi}{N}(k-l)n} |(-k + 2l) modN\rangle \langle\ k| \quad (1)$$

is a Kronecker-Pauli operator.

However, consider the following operator

$$\prod_{r,l,n,N} = \sum_{k=0}^{N-1} e^{\frac{2i\pi}{N}(k-l)n} |[-rk + (r+1)l] modN\rangle \langle\ k|$$

for $r \in \{1, 2, \ldots,\ N-1\}$, where if $r = 1$ we will get the operator in (1). τ defined in $\{0, 1, 2, \ldots,\ N-1\}$ by $\tau(k) = (-rk + (r+1)l) modN$. $\tau(l) = l$ and $\tau(\tau(k)) = [-r\tau(k) + (r+1)l] modN = r^2k - r^2l + l$. In order that τ was a symmetric permutation $r^2k - r^2l + l = k$, $\left(r^2 - 1\right)(k - l) = 0$, for any $k \in \{0, 1, 2, \ldots,\ N-1\}$. Thus, for $k \neq l$, $r = 1$, i.e $\tau(k) = (-k + 2l) modN$.

In the rest of this paper we consider the Kroneker-Pauli operator defined in (1).

The operator can also be written as the following

$$\prod_{l,n,N} = \sum_{k=0}^{N-1} e^{\frac{2i\pi}{N}(k-l)n} |\ k\rangle \langle\sigma(\ k)| \quad = \sum_{k=0}^{N-1} e^{\frac{2i\pi}{N}(k-l)n} |\ k\rangle \langle(-k + 2l) modN|$$

In order for the matrices of the operators in this system of operators (1) to form a set of KPMs, we have to show that these matrices satisfy the hypotheses of Proposition 2 above. Let us write it as a proposition.

Proposition 3

The N^2 matrices, in the standard basis, of the system of Kronecker-Pauli operators $\left(\prod_{l,n,N}\right)_{0 \leq l,n \leq N-1}$ is a a set of $N \times N$- KPMs.

Proof

For $n = 0$, $P_{l+1} = \prod_{\sigma,l,0,N}$ is a symmetric permutation matrix with only one unit, at l-th row l-th column, on the diagonal. Let us show that $P_1 + P_2 + \ldots + P_N$ is the $N \times N$ ones matrix. To do so, let us take $j,\ m \in \{0, 1, 2, \ldots,\ N-1\}$, with $j \neq m$, and show that there is $l \in \{0, 1, 2, \ldots,\ N-1\}$, $|\ j\rangle\langle\ m|$ is a term of the operator $P_{l+1} = \prod_{\sigma,l,0,N}$.

If $j + m$ is even, there is $l \in \{0, 1, 2, \ldots,\ N-1\}$, $j + m = 2l$.

If $j + m$ is odd, for the case $0 \leq \frac{j+m+N}{2} < N$, let $l = \frac{j+m+N}{2}$, $j + m = 2l - N$. For the case $N \leq \frac{j+m+N}{2}$, $0 < j + m - N < N$, let $l = \frac{j+m-N}{2}$, $j + m = 2l + N$.

We have seen that for any case m is of the form $m = -j + 2l\ [N]$, that is $|\ j\rangle\langle\ m|$ is a term of the operator $P_{l+1} = \prod_{\sigma,l,0,N}$.

To finish show that if $|\ j\rangle\langle\ m|$ is a term of an operator $P_{l'+1} = \prod_{\sigma,l',0,N}$, then $l = l'$. If $|\ j\rangle\langle\ m|$ is both term of $P_{l+1} = \prod_{\sigma,l,0,N}$ and $P_{l'+1} = \prod_{\sigma,l',0,N}$, then $j = -m + 2l[N]$ and $j = (-m + 2l')N$. Thus, N divides $l - l'$ and that implies that $l = l'$.

Now, let us prove that for $m,\ n\ \in \{1,\ 2,\ \ldots,\ N-1\}$, $m < n$, for $j,\ k \in \{0,\ 1,\ 2,\ \ldots,\ N-1\}$, with $j \neq k$, $\sigma(j) \neq k$, $e^{\frac{2i\pi}{N}[(j-l)+(k-l)]n} \neq e^{\frac{2i\pi}{N}[(j-l)+(k-l)]m}$. To do so, let us suppose the contrary, that is suppose that there are $m,\ n\ \in \{1,\ 2,\ \ldots,\ N-1\}$, $m < n$, for $j,\ k \in \{0,\ 1,\ 2,\ \ldots,\ N-1\}$, with $j \neq k$, $\sigma(j) \neq k$, $e^{\frac{2i\pi}{N}[(j-l)+(k-l)]n} = e^{\frac{2i\pi}{N}[(j-l)+(k-l)]m}$. Then, $(j + k - 2k)(n - m) = 0 mod N$. As N is a prime integer, thus after the Euclid lemma, N divides $(j + k - 2k)$, that is $k = \sigma(j)$. It is a contradiction.

The proposition is proved.

Example 2

For $N = 3, n = 1, \sigma(0) = 2, \sigma(1) = 1, \sigma(2) = 0$

$$\prod_{1,1,3} = e^{-\frac{2i\pi}{3}}\ |\ 2\rangle\langle\ 0| + |\ 1\rangle\langle\ 1| \ + \ e^{\frac{2i\pi}{3}}\ |\ 0\rangle\langle\ 2|$$

whose matrix in the standard basis $(|\ 0\rangle,\ \ |1\ \rangle,\ \ |2\ \rangle\ )$ is

$$\prod_{1,1,3} = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}$$

with $\omega = e^{\frac{2i\pi}{3}}$.

Example 3

For $N = 5, \sigma(0) = 0, \sigma(1) = 4, \sigma(2) = 3, \sigma(3) = 2, \sigma(4) = 1$, for $n = 0,\ 1,\ 2,\ 3,\ 4$ we have respectively the following five matrices which are, in the standard basis $(|\ 0\rangle,\ \ |1\ \rangle,\ \ |2\ \rangle,\ \ |3\ \rangle,\ \ |4\ \rangle\ )$, matrices of the

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corresponding Kronecker-Pauli operators,

$$\prod_1 = \prod_{0,0,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \prod_2 = \prod_{0,1,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & \eta^3 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \end{pmatrix},$$

$$\prod_3 = \prod_{0,2,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & \eta^4 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \end{pmatrix}, \prod_4 = \prod_{0,3,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \end{pmatrix},$$

$$\prod_5 = \prod_{0,4,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & \eta^3 & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \end{pmatrix}.$$

For $\sigma(0) = 2, \sigma(1) = 1, \sigma(2) = 0, \sigma(3) = 4, \sigma(4) = 3$, for $n = 0,\ 1,\ 2,\ 3,\ 4$ we have respectively

$$\prod_6 = \prod_{1,0,5} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \prod_7 = \prod_{1,1,5} = \begin{pmatrix} 0 & 0 & \eta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & \eta^2 & 0 \end{pmatrix},$$

$$\prod_8 = \prod_{1,2,5} = \begin{pmatrix} 0 & 0 & \eta^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & \eta^4 & 0 \end{pmatrix}, \prod_9 = \prod_{1,3,5} = \begin{pmatrix} 0 & 0 & \eta^3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & \eta & 0 \end{pmatrix},$$

$$\prod_{10} = \prod_{1,4,5} = \begin{pmatrix} 0 & 0 & \eta^4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & \eta^3 & 0 \end{pmatrix}.$$

For $\sigma(0) = 4, \sigma(1) = 3, \sigma(2) = 2, \sigma(3) = 1, \sigma(4) = 0$, for $n = 0,\ 1,\ 2,\ 3,\ 4$ we have respectively

$$\prod_{11} = \prod_{2,0,5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \prod_{12} = \prod_{2,1,5} = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\prod_{13} = \prod_{2,2,5} = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \end{pmatrix}, \prod_{14} = \prod_{2,3,5} = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & \eta^3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\prod_{15} = \prod_{2,4,5} = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $\sigma(0) = 1, \sigma(1) = 0, \sigma(2) = 4, \sigma(3) = 3, \sigma(4) = 2$, for $n = 0,\ 1,\ 2,\ 3,\ 4$ we have respectively

$$\prod_{16} = \prod_{3,0,5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \prod_{17} = \prod_{3,1,5} = \begin{pmatrix} 0 & \eta^3 & 0 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta^4 & 0 & 0 \end{pmatrix},$$

$$\prod_{18} = \prod_{3,2,5} = \begin{pmatrix} 0 & \eta & 0 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta^3 & 0 & 0 \end{pmatrix}, \prod_{19} = \prod_{3,3,5} = \begin{pmatrix} 0 & \eta^4 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \end{pmatrix},$$

$$\prod_{20} = \prod_{3,4,5} = \begin{pmatrix} 0 & \eta^2 & 0 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta & 0 & 0 \end{pmatrix}.$$

For $\sigma(0) = 3, \sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 0, \sigma(4) = 4$, for $n = 0,\ 1,\ 2,\ 3,\ 4$ we have respectively

$$\prod_{21} = \prod_{4,0,5} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \prod_{22} = \prod_{4,1,5} = \begin{pmatrix} 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & \eta^3 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\prod_{23} = \prod_{4,2,5} = \begin{pmatrix} 0 & 0 & 0 & \eta^3 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \prod_{24} = \prod_{4,3,5} = \begin{pmatrix} 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & \eta^4 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\prod_{25} = \prod_{4,4,5} = \begin{pmatrix} 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. Weyl Operators

Definition 3

The following d^2 operators

$$U_{mn} = \sum_{k=0}^{d-1} e^{\frac{2i\pi}{d}kn} \mid k\rangle \left\langle \, (k+m) mod\, d \right| , \quad n, m = 0,\ 1,\ 2,\ \cdots,\ d-1$$

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are called Weyl operators.

For three-dimensional case, it has been shown that up to a phase factor a Weyl operator is product of two Kronecker-Pauli operators [4]. Now we can generalize it in playing with operators but not with matrices. We express the generalization by the following proposition.

Proposition 4

For prime integer N, for N-dimensional case the product of two Kronecker-Pauli operators is a Weyl operator up to phase factor.

Proof

For $l_1,\ l_2,\ n_1,\ n_2 \in \{0,\ 1,\ 2,\ \cdots,\ N-1\}$

$$\prod_{l_1,n_1,N}\prod_{l_2,n_2,N} = \left(\sum_{k=0}^{N-1} e^{\frac{2i\pi}{N}(k-l_1)n_1} \mid k\rangle\langle\ (-k+2l_1)\, mod\ N| \quad\right)\left(\sum_{m=0}^{N-1} e^{\frac{2i\pi}{N}(m-l_2)n_2} \mid m\rangle\langle\ (-m+2l_2)\, mod\ N| \quad\right)$$

$$= \sum_{k=0}^{N-1}\sum_{m=0}^{N-1} e^{\frac{2i\pi}{N}(k-l_1)n_1} e^{\frac{2i\pi}{N}(m-l_2)n_2} \mid k\rangle\langle\ (-k+2l_1)\, mod\ N| \quad m\rangle\langle\ (-m+2l_2)\, mod\ N|$$

As the application $k \mapsto (-k+2l_1)\, mod\ N$ is permutation on $\{0,\ 1,\ 2,\ \cdots,\ N-1\}$, then

$$\prod_{l_1,n_1,N}\prod_{l_2,n_2,N} = \sum_{k=0}^{N-1} e^{\frac{2i\pi}{N}[(k-l_1)n_1+((-k+2l_1)mod\ N-l_2)n_2]} \mid k\rangle\langle\ (-(-k+2l_1)\, mod\ N+2l_2)\, mod\ N|$$

$$= e^{\frac{2i\pi}{N}[l_1(n_2-n_1)+(l_1-l_2)n_2]} \sum_{k=0}^{N-1} e^{\frac{2i\pi}{N}[k(n_1-n_2)]} \mid k\rangle\langle\ [k+2(l_2-l_1)]\, mod\ N|$$

5. Discussion and Conclusion

Giving N a prime integer, $N > 2$, a set of $N \times N$- KPMs is well defined by N symmetric permutation matrices with on the diagonal only one entry is the unit, on other is the zero and inverse-symmetric matrices obtained in replacing the units on the N symmetric permutation matrices by N-th roots of the unit, but in respecting that the hypotheses of the Proposition 2 are satisfied. To construct the system of operators whose matrices in the standard basis form a set of $N \times N$- KPMs we have at first defined for each $l \in \{0,\ 1,\ 2,\ \ldots,\ N-1\}$ a symmetric permutation on $\{0,\ 1,\ 2,\ \ldots,\ N-1\}$. That gives only one set of $N \times N$- KPMs. However, for $N = 5$ there are at least two sets of $N \times N$- KPMs. That is due to the definition of the symmetric permutation for constructing the system of Kronecker-Pauli operators, only one set of $N \times N$- KPMs is obtained as the matrices in the standard basis. Let us call such a family of matrices a $N \times N$-Kronecker-Pauli basis.

Perhaps the other sets of $N \times N$- KPMs would be formed by the matrices of the constructed system of Kronecker-Pauli operators in other bases than the standard basis.

But the essential is obtaining a system of operators whose matrices in the standard basis fulfill the conditions to be as an $N \times N$-Kronecker-Pauli basis.

We have shown that, for N-dimensional case, with N a prime integer, N>2, the product of two Kronecker-Pauli operators is a Weyl operator, up to phase factor.

Appendix A. Kronecker Product

Definition 3

For any matrices $A = \left(A_j^i\right)_{1 \leq i \leq n, 1 \leq j \leq p} \in \mathbb{C}^{n \times p}$, $B = \left(B_j^i\right)_{1 \leq i \leq m, 1 \leq j \leq q} \in \mathbb{C}^{m \times q}$, the Kronecker product of the matrix A by the matrix B is the matrix

$$A \bigotimes B = \begin{pmatrix} A_1^1 B & A_2^1 B & \cdots & A_p^1 B \\ A_1^2 B & A_2^2 B & \cdots & A_p^2 B \\ \vdots & \vdots & \cdots & \vdots \\ A_1^n B & A_2^n B & \cdots & A_p^n B \end{pmatrix}$$

Properties

- $\bigotimes$ is associative.
- $\bigotimes$ is distributive with respect to the addition.
- For any matrices A, B, C, and D

$$(A \bigotimes B)(C \bigotimes D) = AC \bigotimes BD$$

- For any invertible matrices A and B

$$(A \bigotimes B)^{-1} = A^{-1} \bigotimes B^{-1}$$

- For any matrices A and B

$$(A \bigotimes B)^{+} = A^{+} \bigotimes B^{+}$$

Proposition 4

Let $(A_i)_{1 \leq i \leq np}$, and $(B_j)_{1 \leq j \leq mq}$ respectively be some bases of $\mathbb{C}^{n \times p}$ and $\mathbb{C}^{m \times q}$. Then, $(A_i \bigotimes B_j)_{1 \leq i \leq np, 1 \leq j \leq mq}$ is a basis of $\mathbb{C}^{nm \times pq}$.

Proposition 5

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Suppose

$$\sum_{j=1}^{m} M_j \bigotimes N_j = \sum_{i=1}^{n} A_i \bigotimes B_i$$

with the M_j's, A_i's are elements of $\mathbb{C}^{p\times q}$ and the N_j's, B_i's are elements of $\mathbb{C}^{r\times s}$.

Then

$$\sum_{j=1}^{m} M_j \bigotimes K \bigotimes N_j = \sum_{i=1}^{n} A_i \bigotimes K \bigotimes B_i$$

for any matrix K.

Proof. The proof of this proposition in [6] regards a Kronecker product of matrices as an hypermatrix.

Appendix B. Kronecker Commutation Matrices or Swap Matrices

The Kronecker product of matrices is not commutative, but there is a permutation matrix which, in multiplying to the product, commutes the product. We call such matrix Kronecker commutation matrix or swap operator.

Definition 4

The permutation matrix $K_{n\bigotimes p} \in \mathbb{C}^{np\times np}$, such that for any matrices $a \in \mathbb{C}^{n\times 1}$, $b \in \mathbb{C}^{p\times 1}$

$$K_{n\bigotimes p}(a \bigotimes b) = b \bigotimes a$$

is called $n \bigotimes p$-Kronecker commutation matrix ou swap matrix, $n \bigotimes p$-KCM.

$$K_{2\bigotimes 2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, K_{3\bigotimes 3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For constructing $n \bigotimes p$ –KCM, we can use the following rule [7].

Rule 1

Let us start in putting 1 at first row, first column, then let us pass into the second column in going down at the rate of n rows and put 1 at this place, then pass into the third column in going down at the rate of n rows and put 1, and so on until there is only for us n-1 rows for going down (then we have obtained as number of 1

: p). Then pass into the next column which is the (p + 1)-th column, put 1 at the second row of this column and repeat the process until we have only n-2 rows for going down (then we have obtained as number of 1 : 2p). After that pass into the next column which is the (2p + 2)-th column, put 1 at the third row of this column and repeat the process until we have only n-3 rows for going down (then we have obtained as number of 1 : 3p). Continuing in this way we will have that the element at n×p-th row, n×p-th column is 1.

Proposition 6

Suppose

$$K_{n\bigotimes m}=\sum_{i,j=1}^{s}A_i\bigotimes B_j$$

and

$$K_{p\bigotimes q}=\sum_{k,l=1}^{r}C_k\bigotimes D_l$$

with the A_i's are elements of $\mathbb{C}^{m\times n}$, the B_j's are elements of $\mathbb{C}^{n\times m}$, the C_k's are elements of $\mathbb{C}^{q\times p}$and the D_l's are elements of $\mathbb{C}^{p\times q}$. Then,

$$K_{np\bigotimes mq}=\sum_{i,j=1}^{s}\sum_{k,l=1}^{r}A_i\bigotimes C_k\bigotimes B_j\bigotimes D_l$$

Proof

Let (a_α), (c_β), (b_γ) and (d_δ) be, respectively, bases of $\mathbb{C}^{n\times 1}$, $\mathbb{C}^{p\times 1}$, $\mathbb{C}^{m\times 1}$ and $\mathbb{C}^{q\times 1}$. Then, $\left(a_\alpha\bigotimes c_\beta\bigotimes b_\gamma\bigotimes d_\delta\right)$ is a basis of $\mathbb{C}^{npmq\times 1}$. It is enough to prove that

$$\sum_{i,j=1}^{s}\sum_{k,l=1}^{r}A_i\bigotimes C_k\bigotimes B_j\bigotimes D_l\left(a_\alpha\bigotimes c_\beta\bigotimes b_\gamma\bigotimes d_\delta\right)=b_\gamma\bigotimes d_\delta\bigotimes a_\alpha\bigotimes c_\beta$$

We use the proposition 5. From

$$\sum_{i,j=1}^{s}A_i\bigotimes B_j\left(a_\alpha\bigotimes b_\gamma\right)=b_\gamma\bigotimes a_\alpha$$

we have

$$\sum_{i,j=1}^{s}\sum_{k,l=1}^{r}A_ia_\alpha\bigotimes C_kc_\beta\bigotimes B_jb_\gamma\bigotimes D_ld_\delta=\sum_{k,l=1}^{r}b_\gamma\bigotimes C_kc_\beta\bigotimes a_\alpha\bigotimes D_ld_\delta$$

$$=b_\gamma\bigotimes\sum_{k,l=1}^{r}C_kc_\beta\bigotimes a_\alpha\bigotimes D_ld_\delta$$

Moreover

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$$\sum_{k,l=1}^{r} C_k c_\beta \bigotimes a_\alpha \bigotimes D_l d_\delta = d_\delta \bigotimes a_\alpha \bigotimes c_\beta$$

and that ends the proof.

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References

1. ^Rakotonirina C (2007). "Expression of a tensor commutation matrix in terms of the generalized Gell-Mann matrices." Int J Math Math Sci. doi: 10.1155/2007/20672.
2. ^Kimura G (2003). "The Bloch Vector for N-Level Systems." Phys Lett A. 314:339.
3. ^Bertlmann RA, Krammer P (2008). "Bloch vectors for qudits." J Phys A Math Theor. 41(23):235303. https://doi.org/10.1088/1751-8113/41/23/235303. arXiv:0806.1174 [quant-ph].
4. a, b, cRakotonirina C (2024). "Constructing a Set of Kronecker-Pauli Matrices." Qeios. https://doi.org/10.32388/F5DGRH.2
5. a, b, cRakotonirina C, Rasamizafy S (2016). "3×3-Kronecker-Pauli Matrices." Bull Math Stat Res. 4(3):127–134.
6. ^Rakotonirina C, Rakotondramavonirina J (2014). "Tensor Commutation Matrices and Some Generalizations of the Pauli Matrices." Gen Math Notes. 23(1):1–8.
7. ^Rakotonirina C (2005). "Tensor permutation matrices in finite dimensions." arXiv:math.GM/0508053.

Declarations

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