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Research Article

# New Computational Methods Using Seventh-Derivative Type for the Solution of First-Order Initial Value Problems

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**This study uses finite power series as the basis function and interpolation and collocation techniques to study a class of implicit block methods of a seventh-derivative type. Discrete schemes are implicit two-point block methods that are obtained by selecting collocation points carefully and unevenly in order to improve the stability of the methods through testing. Nevertheless, in contrast to other current numerical equations, these methods require seventh-derivative functions. The novel techniques are identified, examined, and shown to be A-stable and convergent. Newton Raphson's approach is used to accomplish method implementation. Trials demonstrated the effectiveness and precision of the derived equations in terms of computational time and absolute errors on a variety of first-order initial value issues, such as first-order, second-order linear differential systems, and the SIR model. When compared to similar methods that are currently in the literature, the suggested methods produce better numerical results.**

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## 1. Introduction

Over the years, stiff differential equations have been explored in an effort to create suitable and reliable numerical methods. It is important to remember that [1] was the first to study the most effective numerical strategy for solving stiff ODEs. Diverse academics define this fascinating field of study in different ways. As such, it can be described as ill-conditioned equations. First-order initial value problems of the form of Equation (1) should be examined in order to reveal the nature of the ill-conditioning's stiffness and to highlight the necessity of developing effective numerical techniques for stiff differential equations:

$$y' = f(x,y), \quad a \leq x \leq b, \quad y(0) = y_0, \tag{1}$$

where the step size is denoted by $h$ and $x_n = x_0 + nh$. According to [2], the initial value problems with stiff ordinary differential equations occur in many fields of engineering science, particularly in the studies of electrical circuits, vibrations, chemical reactions, and so on. Stiff differential equations are ubiquitous in astrochemical kinetics and many non-industrial areas like weather prediction and biology. A set of differential equations is 'stiff' when an excessively small step is needed to obtain correct integration.

Furthermore, a stiff system of equations is one for which $|\lambda_{max}|$ (where $\lambda$ is the eigenvalues) is enormous, meaning that only unreasonable restrictions on $h$ (that is, an excessively small $h$ that necessitates an excessive number of steps to solve the initial value problem) can guarantee stability, the error bound, or both. In this context, enormous refers to a scale of $\frac{1}{b}$. Thus, an equation with $|\lambda_{max}|$, may also be viewed as stiff if we must solve it for great values of time, where $f : [x_n, x_N] \times R^m \to R^m$ in (1) is continuous and differentiable; so that, $f$ is assumed to satisfy the existence and uniqueness theorem within the interval of $[a, b]$; while stability is clearly necessary, it is not sufficient to acquire precise solutions to stiff ordinary differential equation systems. One frequently noticed occurrence is that many implicit methods appear to fall short of the predicted accuracy order when used in stiff situations. We refer to this process as order reduction. Runge-Kutta techniques undoubtedly result in order reduction, while backward differentiation formula techniques do not. Furthermore, because the step size is limited to preserve the methods' potential accuracy, explicit methods are unable to solve stiff ODEs. Using proper implicit methods solves this problem (see [3]). Nonetheless, a few well-known numerical techniques are the Runge-Kutta methods in [4], the Euler method by [5], and linear multistep approaches in [6]. Furthermore, many research and engineering domains face challenging problems of a rigid character that are outside the scope of the aforementioned approaches. Thus, there is a need to create more practical approximation techniques. Furthermore, for stiff IVPs, [7] developed a diagonally implicit block backward differentiation formula. Implicit linear block multistep algorithms for first-order stiff and non-stiff IVPs have been devised and implemented, respectively, in [8][9][10][11][12][13][14]. Remarkably, [15][16] also created and applied an implicit four-point hybrid block integrator on stiff models connected to specific real-world scenarios, using a technique that was almost as good as other approaches already in use. In [17][18], an additional implicit block technique has been explored for utilizing the Chebyshev polynomial to solve stiff IVPs. Nevertheless, their techniques rely on the approximation of perturbed collocation.

Among other places, [19][20][21][22] have proposed applications of multi-derivatives block approaches to first-order stiff initial-value problems. Higher derivative approaches, however, generally have the drawback of requiring the provision and evaluation of derivative functions, leading to a greater number of function evaluations. Therefore, if numerical methods are not sufficiently stable, that is, if the numerical errors are not checked by the zero-stability and consistency properties—this shortcoming can lead to round-off errors in the global iterations.

As a result, using collocation and interpolation techniques, [23] developed and used the fourth derivative $k-$point block formula on first-order stiff IVPs. Similarly, for solving (1), [24] suggested a third derivative trigonometrically fitted block technique of a low order 2. Equation (1) was solved by [25] using second-derivative methods. Furthermore, [26] considered a seventh-order second-derivative block technique for solving (1) directly, with numerical results better than those in [27].

The work in [28], developed a continuous implicit seventh-eight approach of uniform order 8 for the direct solution of (1) by using a power series basis function using collocation and interpolation techniques. But the previously mentioned also took into account relevant issues such as the Prothero-Robinson oscillatory problem, the growth, and SIR models. In [16], an optimal family of block techniques is applied to solve models of infectious diseases using fixed and adaptive strategies. The approaches did not only take into account numerical accuracy but also the precision factor, among others, which is the negative logarithm of the absolute errors of the methods.

Summarily, in contrast to conventional approaches, the idea behind the inclusion of the seventh derivative is to investigate the effect of the non-uniform distribution of collocation points on the stability of numerical methods with higher-order accuracy for the direct solution of (1). The proposed methods are a group of discrete schemes of functions of first order with type seventh-derivative, which makes them significant. In contrast to other existing approaches that have a constant $k^{th}-$points of collocation, they also have a strategic non-uniform distribution and positioning of collocation points with a higher order of accuracy. Although providing the previously indicated total derivative functions in the proposed approaches is a burden imposed by these techniques, the significance of the derived methods is demonstrated by their efficiency and correctness. Second, the point collocation strategy used cannot be generalized because it has not been verified in the formulation of higher-order numerical methods. Thirdly, the new methods become cumbersome correspondingly to the complexity of differential equation systems, as the derivative functions have to be provided. Tests on numerical examples, however, show that our obtained formulae are workable on first-order, second-order IVPs, and application difficulty in biology (SIR model).

For that reason, the present study is structured as follows: The proposed methods are derived in section two, the analysis of the numerical properties is shown in section three, the implementation strategy is presented in section four, the numerical experiment is shown in section five, the methods are applied in real-world scenarios in section six, and the conclusion and future research are presented in section seven.

## 2. Derivation of the seventh-derivative methods

Consider the following power series polynomial:


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$$y(x)=\sum_{j=0}^{k+8}a_jx^j, \tag{2}$$

with its derivatives given as:

$$y'(x)=\sum_{j=0}^{k+8}ja_jx^{j-1}=f(x,y), \tag{3}$$

$$y''(x)=\sum_{j=0}^{k+8}j(j-1)a_jx^{j-2}=g(x,y), \tag{4}$$

$$y'''(x)=\sum_{j=0}^{k+8}j(j-1)(j-2)a_jx^{j-3}=u(x,y), \tag{5}$$

$$y''''(x)=\sum_{j=0}^{k+8}j(j-1)(j-2)(j-3)a_jx^{j-4}=v(x,y), \tag{6}$$

$\vdots$

with the following as seventh-derivative:

$$y^{(7)}(x)=\sum_{j=0}^{k+8}j(j-1)(j-2)(j-3)(j-4)(j-5)(j-6)a_jx^{j-7}=q(x,y). \tag{7}$$

Note that:

$y'_{n+i}=f_{n+i}, i=0(1)k, y''_{n+i}=g_{n+i}, i=1,\dots k, y'''_{n+i}=u_{n+i}, i=k, y^{(4)}_{n+i}=v_{n+i}, i=k, y^{(5)}_{n+i}=w_{n+i}, i=k, y^{(6)}_{n+i}=m_{n+i}, i=k, y^{(7)}_{n+i}=q_{n+i}, i=k(1)$.
Where $a_{j's}\in\mathbb{R}$ in (2)–(7) are found using the Gaussian elimination method. Therefore, (2) and (2)–(7) are then interpolated and collocated at $x_n$ and $x_{n+l}, l=0(1)k$ (where $k$ is the step number and $k=2$) to give the following block figures:

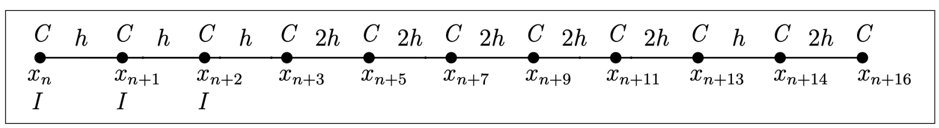


**Figure 1.** Seventh-derivative non-uniform two-step block figure for 7D2PIB1

Which yields the following equation system:

$$PX=Q, \tag{8}$$

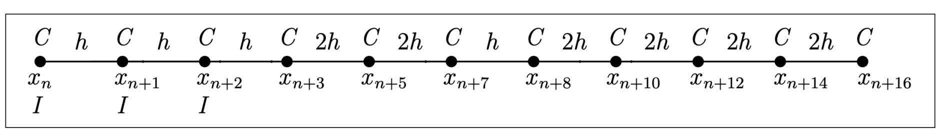


**Figure 2.** Seventh-derivative non-uniform two-step block figure for 7D2PIB2

Where,


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$$P = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} \\ 0 & 1 & 2\,x_{n+l} & 3\,x_{n+l}^2 & 4\,x_{n+l}^3 & 5x_{n+l}^4 & 6x_{n+l}^5 & 7x_{n+l}^6 & 8x_{n+l}^7 & 9x_{n+l}^8 & 10x_{n+l}^9 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 2 & 6x_{n+k} & 12x_{n+k}^2 & 20x_{n+k}^3 & 30x_{n+k}^4 & 42x_{n+k}^5 & 56x_{n+k}^6 & 72x_{n+k}^7 & 90x_{n+k}^8 \\ 0 & 0 & 0 & 6 & 24x_{n+k} & 60x_{n+k}^2 & 120x_{n+k}^3 & 210x_{n+k}^4 & 336x_{n+k}^5 & 504x_{n+k}^6 & 720x_{n+k}^7 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+k} & 360x_{n+k}^2 & 840x_{n+k}^3 & 1680x_{n+k}^4 & 3024x_{n+k}^5 & 5040x_{n+k}^6 \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+k} & 2520x_{n+k}^2 & 6720x_{n+k}^3 & 15120x_{n+k}^4 & 30240x_{n+k}^5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 720 & 5040x_{n+k} & 20160x_{n+k}^2 & 60480x_{n+k}^3 & 151200x_{n+k}^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5040 & 40320\,x_{n+1} & 181440x_{n+1}^2 & 604800x_{n+1}^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5040 & 40320\,x_{n+k} & 181440x_{n+k}^2 & 604800x_{n+k}^3 \end{pmatrix},$$

$$X = \left( a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10} \right)^\top,$$
$$Q = \left( y_n, f_n, f_{n+1}, f_{n+2}, f'_{n+k}, f^{(2)}_{n+k}, f^{(3)}_{n+k}, f^{(4)}_{n+k}, f^{(5)}_{n+k}, f^{(6)}_{n+1}, f^{(6)}_{n+k} \right)^\top.$$

After (8) is solved for $a_{j's} \in \mathbb{R}, j = 0(1)10$ by multiplying the inverse of matrix $P$ with $Q$ and substitution is made into (2), it gives the LMM below using the Maple 18 software environment:

$$\begin{aligned} y(x_{n+\xi}) = \alpha_0(\xi) y_n + h \sum_{j=0}^{k} \beta_j(\xi) f_{n+j} + h^2 \sum_{j=3} \beta_j(\xi) g_{n+k} + h^3 \sum_{j=4} \beta_j(\xi) u_{n+k} \\ + h^4 \sum_{j=5} \beta_j(\xi) v_{n+k} + h^5 \sum_{j=6} \beta_j(\xi) w_{n+k} + h^6 \sum_{j=7} \beta_j(\xi) m_{n+k} + h^7 \sum_{j=8}^{9} \beta_j(\xi) \sum_{i=1}^{k} q_{n+i}, \end{aligned} \tag{9}$$

Therefore, the parameters $\alpha_0(\xi)$ and $\beta_j(\xi)$ are obtained for $\xi = x - x_n$, so that the coefficients of $y_{n+1}$ and $y_{n+2}$ are normalized. Therefore,

$$\alpha_0 = 1, \tag{10}$$

$$\beta_0 = \xi - \frac{3}{2}\frac{\xi^2}{h} + \frac{2}{3}\frac{\xi^3}{h^2} + \frac{7\,\xi^4}{8\,h^3} - \frac{63\,\xi^5}{40\,h^4} + \frac{7}{6}\frac{\xi^6}{h^5} - \frac{1}{2}\frac{\xi^7}{h^6} + \frac{33\,\xi^8}{256\,h^7} - \frac{43\,\xi^9}{2304\,h^8} + \frac{3\,\xi^{10}}{2560\,h^9}, \tag{11}$$

$$\beta_1 = \frac{128\,\xi^3}{3\,h^2} - 112\,\frac{\xi^4}{h^3} + \frac{672\,\xi^5}{5\,h^4} - \frac{280\,\xi^6}{3\,h^5} + 40\,\frac{\xi^7}{h^6} - \frac{21}{2}\frac{\xi^8}{h^7} + \frac{14\,\xi^9}{9\,h^8} - \frac{1}{10}\frac{\xi^{10}}{h^9}, \tag{12}$$

$$\beta_2 = \frac{3}{2}\frac{\xi^2}{h} - \frac{130\,\xi^3}{3\,h^2} + \frac{889\,\xi^4}{8\,h^3} - \frac{5313\,\xi^5}{40\,h^4} + \frac{553\,\xi^6}{6\,h^5} - \frac{79\,\xi^7}{2\,h^6} + \frac{2655\,\xi^8}{256\,h^7} - \frac{3541\,\xi^9}{2304\,h^8} + \frac{253\,\xi^{10}}{2560\,h^9}, \tag{13}$$

$$\beta_3 = \frac{-5}{2}\xi^2 + 44\,\frac{\xi^3}{h} - \frac{441\,\xi^4}{4\,h^2} + \frac{525\,\xi^5}{4\,h^3} - 91\,\frac{\xi^6}{h^4} + 39\,\frac{\xi^7}{h^5} - \frac{1311\,\xi^8}{128\,h^6} + \frac{583\,\xi^9}{384\,h^7} - \frac{25\,\xi^{10}}{256\,h^8}, \tag{14}$$

$$\beta_4 = 2\,\xi^2 h - \frac{45\,\xi^3}{2} + \frac{217\,\xi^4}{4\,h} - \frac{1281\,\xi^5}{20\,h^2} + \frac{133\,\xi^6}{3\,h^3} - 19\,\frac{\xi^7}{h^4} + \frac{639\,\xi^8}{128\,h^5} - \frac{853\,\xi^9}{1152\,h^6} + \frac{61\,\xi^{10}}{1280\,h^7}, \tag{15}$$

$$\beta_5 = -\xi^2 h^2 + \frac{23\,\xi^3 h}{3} - \frac{419\,\xi^4}{24} + \frac{203\,\xi^5}{10\,h} - 14\,\frac{\xi^6}{h^2} + 6\,\frac{\xi^7}{h^3} - \frac{101\,\xi^8}{64\,h^4} + \frac{15\,\xi^9}{64\,h^5} - \frac{29\,\xi^{10}}{1920\,h^6}, \tag{16}$$

$$\beta_6 = \frac{1}{3}\xi^2 h^3 - \frac{17\,\xi^3 h^2}{9} + 4\,\xi^4 h - \frac{109\,\xi^5}{24} + \frac{28\,\xi^6}{9\,h} - \frac{4}{3}\frac{\xi^7}{h^2} + \frac{45\,\xi^8}{128\,h^3} - \frac{181\,\xi^9}{3456\,h^4} + \frac{13\,\xi^{10}}{3840\,h^5}, \tag{17}$$

$$\beta_7 = \frac{-1}{15}\xi^2 h^4 + \frac{14\,\xi^3 h^3}{45} - \frac{37\,\xi^4 h^2}{60} + \frac{41\,\xi^5 h}{60} - \frac{67\,\xi^6}{144} + \frac{1}{5}\frac{\xi^7}{h} - \frac{17\,\xi^8}{320\,h^2} + \frac{23\,\xi^9}{2880\,h^3} - \frac{\xi^{10}}{1920\,h^4}, \tag{18}$$

$$\beta_8 = -\frac{2\,h^5\xi^2}{315} + \frac{2\,h^4\xi^3}{105} - \frac{1}{36}h^3\xi^4 + \frac{11\,h^2\xi^5}{450} - \frac{h\xi^6}{72} + \frac{13\,\xi^7}{2520} - \frac{7\,\xi^8}{5760\,h} + \frac{\xi^9}{6048\,h^2} - \frac{\xi^{10}}{100800\,h^3}, \tag{19}$$

$$\beta_9 = \frac{2\,h^5\xi^2}{315} - \frac{5\,h^4\xi^3}{189} + \frac{1}{20}h^3\xi^4 - \frac{49\,h^2\xi^5}{900} + \frac{1}{27}h\xi^6 - \frac{9\,\xi^7}{560} + \frac{5\,\xi^8}{1152\,h} - \frac{121\,\xi^9}{181440\,h^2} + \frac{\xi^{10}}{22400\,h^3}. \tag{20}$$

At $\xi = h$ and $\xi = 2h$, evaluate (11) – (20), and substitute into (9) to obtain the newly developed seventh-derivative implicit block methods, abbreviated as "7D2PIB1 and 7D2PIB2″ correspondingly.

$$\begin{aligned} y_{n+1} = y_n + \tfrac{5639}{23040} h f_n + \tfrac{121}{45} h f_{n+1} - \tfrac{44551}{23040} h f_{n+2} + \tfrac{1289}{768} h^2 g_{n+2} - \tfrac{7687}{11520} h^3 u_{n+2} \\ + \tfrac{287}{1920} h^4 v_{n+2} - \tfrac{583}{34560} h^5 w_{n+2} + \tfrac{1}{5760} h^6 m_{n+2} - \tfrac{257}{604800} h^7 q_{n+1} + \tfrac{121}{907200} h^7 q_{n+2}, \end{aligned} \tag{21}$$

$$\begin{aligned} y_{n+2} = y_n + \frac{11}{45} h f_n + \frac{128}{45} h f_{n+1} - \frac{49}{45} h f_{n+2} + \frac{4}{3} h^2 g_{n+2} - \frac{26}{45} h^3 u_{n+2} + \frac{2}{15} h^4 v_{n+2} \\ - \frac{2}{135} h^5 w_{n+2} - \frac{2}{4725} h^7 q_{n+1} + \frac{2}{14175} h^7 q_{n+2}. \end{aligned} \tag{22}$$

In a similar manner, 7D2PIB2, the second formula, is obtained and given as follows:

$$\begin{aligned} y_{n+1} = y_n + \tfrac{1663}{11520} h f_n + \tfrac{121}{45} h f_{n+1} - \tfrac{21119}{11520} h f_{n+2} + \tfrac{2837}{1920} h^2 g_{n+2} - \tfrac{2687}{5760} h^3 u_{n+2} \\ - \tfrac{257}{5040} h^4 v_{n+1} + \tfrac{1343}{20160} h^4 v_{n+2} - \tfrac{113}{120960} h^5 w_{n+2} - \tfrac{37}{33600} h^6 m_{n+2} + \tfrac{121}{907200} h^7 q_{n+2}, \end{aligned} \tag{23}$$

$$\begin{aligned} y_{n+2} = y_n + \frac{13}{90} h f_n + \frac{128}{45} h f_{n+1} - \frac{89}{90} h f_{n+2} + \frac{17}{15} h^2 g_{n+2} - \frac{17}{45} h^3 u_{n+2} - \frac{16}{315} h^4 v_{n+1} \\ + \frac{16}{315} h^4 v_{n+2} + \frac{h^5 w_{n+2}}{945} - \frac{2}{1575} h^6 m_{n+2} + \frac{2}{14175} h^7 q_{n+2}. \end{aligned} \tag{24}$$

## 3. The stability analysis of the methods

With the proposed numerical techniques, this section provides the numerical properties and theorems (without proofs).

**Theorem 1**. *Convergence* [4]: The necessary and sufficient conditions for the linear multistep method (LMM) of (21)–(24) to be convergent are that it must be consistent and zero-stable.

**Theorem 2**. The necessary and sufficient condition for the method given by (21)–(24) to be zero-stable is that it satisfies the root condition (See [4]).

**Definition 1**. *Zero-stability* [29]
The numerical methods in (21) – (24) are said to be zero-stable if no root of the first characteristic polynomial has a modulus greater than one and if every root with modulus one is simple.

**Definition 2**. *A-stability*: A numerical method is said to be A-stable if the whole of the left-half plane $z : \Re(z) \leq 0$ is contained in the region $z : \Re(z) \leq 1$. Where $\Re(z)$ is the stability polynomial of the proposed method. (See [4]).

**Definition 3**. A($\alpha$)-*stability*: A numerical algorithm is said to be A($\alpha$)-stable for some $\alpha \in [0, \frac{\pi}{2}]$ if the wedge $S_\alpha = \{z : |Arg(-z)| < \alpha, z \neq 0\}$ is contained in its region of absolute stability. (See, [30]).

**Definition 4**. *Linear Multistep Method (LMM):* A linear multi-step method is a computational method for determining the numerical solution of initial value problems of ODEs which form a linear relation between $y_{n+j}$ and $f_{n+j}$. This is a method that requires starting values from several previous steps for the approximation of the solution at the current step. For instance, in the method $k-$step, the values of $y-$ computed at the previous $k-$step, that is, $x_{n+j} = x_n + jh, j = 1(0)k - 1$ are used to calculate $y_{n+k}$ (see [31]).

**Definition 5**. *Interpolation and collocation:* Collocation is the evaluation of the differential system of the basis or trial function at some selected grid points, while interpolation is the evaluation of the approximate solution also at some selected grid points. This collocation method is widely considered as a means of providing a numerical solution to ordinary differential equations (see [31]).

**Definition 6**. *Block method:* A block method can be seen as a set of linear multistep methods simultaneously applied to initial value problems and then combined to yield a better approximation. In other words, the set of new values derived by each application of the method is known as a block. That is, at each iteration of the algorithm, the values of $y_{n+1}, y_{n+2}, \ldots, y_{n+k}$ are computed simultaneously (see [4]).


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### 3.1. The Order of the 7D2PIB1 and 7D2PIB2

To determine the order of the derived methods, (21) – (24) are rewritten in block form to give the linear operator:

$$\begin{aligned} L(y(x);h) = A^{(1)}Y_m - A^{(0)}Y_{m-1} - h\left(B^{(0)}F_{m-1} - B^{(1)}F_m\right) - h^2C^1G_m - h^3C^2U_m \\ -h^4C^3V_m - h^5C^4W_m - h^6C^5M_m - h^7C^6Q_m, \end{aligned} \tag{25}$$

Where, =10mu

$$A^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^{(0)} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad B^{(0)} = \begin{pmatrix} 0 & \frac{5639}{23040} \\ 0 & \frac{11}{45} \end{pmatrix}, \quad B^{(1)} = \begin{pmatrix} \frac{121}{45} & \frac{128}{45} \\ -\frac{44551}{23040} & -\frac{49}{45} \end{pmatrix},$$

$$C^{(1)} = \begin{pmatrix} 0 & \frac{1289}{768} \\ 0 & \frac{4}{3} \end{pmatrix}, \quad C^{(2)} = \begin{pmatrix} 0 & -\frac{7687}{11520} \\ 0 & -\frac{26}{45} \end{pmatrix}, \quad C^{(3)} = \begin{pmatrix} 0 & \frac{287}{1920} \\ 0 & \frac{2}{15} \end{pmatrix},$$
$$C^{(4)} = \begin{pmatrix} 0 & -\frac{583}{34560} \\ 0 & -\frac{2}{135} \end{pmatrix},$$

$$C^{(5)} = \begin{pmatrix} 0 & -\frac{1}{5760} \\ 0 & -\frac{2}{135} \end{pmatrix} \quad C^{(6)} = \begin{pmatrix} -\frac{257}{604800} & \frac{121}{907200} \\ -\frac{2}{4725} & \frac{2}{14175} \end{pmatrix}, \quad Y_m = \begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix},$$
$$Y_{m-1} = \begin{pmatrix} y_{n-(k-1)} \\ y_n \end{pmatrix},$$

$$F(Y_m) = \begin{pmatrix} f_{n+1} \\ f_{n+k} \end{pmatrix}, \quad F(Y_{m-1}) = \begin{pmatrix} f_{n-(k-1)} \\ f_n \end{pmatrix}, \quad G_m = \begin{pmatrix} f'_{n+1} \\ f'_{n+k} \end{pmatrix}, \quad U_m = \begin{pmatrix} f^{(2)}_{n+1} \\ f^{(2)}_{n+k} \end{pmatrix},$$
$$V_m = \begin{pmatrix} f^{(3)}_{n+1} \\ f^{(3)}_{n+k} \end{pmatrix},$$

$$W_m = \begin{pmatrix} f^{(4)}_{n+1} \\ f^{(4)}_{n+k} \end{pmatrix}, \quad M_m = \begin{pmatrix} f^{(5)}_{n+1} \\ f^{(5)}_{n+k} \end{pmatrix}, \quad Q_m = \begin{pmatrix} f^{(6)}_{n+1} \\ f^{(6)}_{n+2} \end{pmatrix}.$$

Recall that $f_{n+l}, l = 0(1)k$ are the first-order derivative functions in $x, y$.
By comparing coefficients in powers of $h$ and $y$, using the Taylor series expansion of (25), gives:

$$\begin{aligned} L(y(x);h) = q_0y(x) + q_1hy'(x) + q_2h^2y''(x) + \ldots + q_ph^py^p(x) + \ldots \\ +q_{p+1}h^{p+1}y^{p+1}(x) + \ldots, \end{aligned} \tag{27}$$

In the form of (27), (21) and (22) are equivalent to:

$$q_i = \begin{bmatrix} \frac{(1)^i}{i!} - (1)\frac{(0)^i}{i!} - \frac{1}{(i-1)!}\left(\frac{5639}{23040}(0)^{i-1} + \frac{121}{45}(1)^{i-1} - \frac{44551}{23040}(2)^{i-1}\right) \\ -\frac{1}{(i-2)!}\left(0(0)^{i-2} + 0(1)^{i-2} + \frac{1289}{768}(2)^{i-2}\right) \\ -\frac{1}{(i-3)!}\left(0(0)^{i-3} + 0(1)^{i-3} - \frac{7687}{11520}(2)^{i-3}\right) \\ -\frac{1}{(i-4)!}\left(0(0)^{i-4} + 0(1)^{i-4} + \frac{287}{1920}(2)^{i-4}\right) \\ -\frac{1}{(i-5)!}\left(0(0)^{i-5} + 0(1)^{i-5} - \frac{583}{34560}(2)^{i-5}\right) \\ -\frac{1}{(i-6)!}\left(0(0)^{i-6} + 0(1)^{i-6} + \frac{1}{5760}(2)^{i-6}\right) \\ -\frac{1}{(i-7)!}\left(0(0)^{i-7} - \frac{257}{604800}(1)^{i-7} + \frac{121}{907200}(2)^{i-7}\right), \\ \frac{(2)^i}{i!} - (1)\frac{(0)^i}{i!} - \frac{1}{(i-1)!}\left(\frac{11}{45}(0)^{i-1} + \frac{128}{45}(1)^{i-1} - \frac{49}{45}(2)^{i-1}\right) \\ -\frac{1}{(i-2)!}\left(0(0)^{i-2} + 0(1)^{i-2} + \frac{4}{3}(2)^{i-2}\right) \\ -\frac{1}{(i-3)!}\left(0(0)^{i-3} + 0(1)^{i-3} - \frac{26}{45}(2)^{i-3}\right) \\ -\frac{1}{(i-4)!}\left(0(0)^{i-4} + 0(1)^{i-4} + \frac{2}{15}(2)^{i-4}\right) \\ -\frac{1}{(i-5)!}\left(0(0)^{i-5} + 0(1)^{i-5} - \frac{2}{5}(2)^{i-5}\right) \\ -\frac{1}{(i-6)!}\left(0(0)^{i-6} + 0(1)^{i-6} + 0(2)^{i-6}\right) \\ -\frac{1}{(i-7)!}\left(0(0)^{i-7} - \frac{2}{4725}(1)^{i-7} + \frac{2}{14175}(2)^{i-7}\right) \end{bmatrix} = 0, \quad i = 7(1)10.$$

Similarly, the 7D2PIB2 order is as follows:

$$q_i = \begin{bmatrix} \frac{(1)^i}{i!} - (1)\frac{(0)^i}{i!} - \frac{1}{(i-1)!}\left(\frac{1663}{11520}(0)^{i-1} + \frac{121}{45}(1)^{i-1} - \frac{21119}{11520}(2)^{i-1}\right) \\ -\frac{1}{(i-2)!}\left(0(0)^{i-2} + 0(1)^{i-2} + \frac{2837}{1920}(2)^{i-2}\right) \\ -\frac{1}{(i-3)!}\left(0(0)^{i-3} + 0(1)^{i-3} - \frac{2687}{5760}(2)^{i-3}\right) \\ -\frac{1}{(i-4)!}\left(0(0)^{i-4} - \frac{257}{5040}(1)^{i-4} + \frac{1343}{20160}(2)^{i-4}\right) \\ -\frac{1}{(i-5)!}\left(0(0)^{i-5} + 0(1)^{i-5} - \frac{113}{120960}(2)^{i-5}\right) \\ -\frac{1}{(i-6)!}\left(0(0)^{i-6} + 0(1)^{i-6} - \frac{37}{33600}(2)^{i-6}\right) \\ -\frac{1}{(i-7)!}\left(0(0)^{i-7} + 0(1)^{i-7} + \frac{121}{907200}(2)^{i-7}\right), \\ \frac{(2)^i}{i!} - (1)\frac{(0)^i}{i!} - \frac{1}{(i-1)!}\left(\frac{13}{90}(0)^{i-1} + \frac{128}{45}(1)^{i-1} - \frac{89}{90}(2)^{i-1}\right) \\ -\frac{1}{(i-2)!}\left(0(0)^{i-2} + 0(1)^{i-2} + \frac{17}{15}(2)^{i-2}\right) \\ -\frac{1}{(i-3)!}\left(0(0)^{i-3} + 0(1)^{i-3} - \frac{17}{45}(2)^{i-3}\right) \\ -\frac{1}{(i-4)!}\left(0(0)^{i-4} - \frac{16}{315}(1)^{i-4} + \frac{16}{315}(2)^{i-4}\right) \\ -\frac{1}{(i-5)!}\left(0(0)^{i-5} + 0(1)^{i-5} + \frac{1}{945}(2)^{i-5}\right) \\ -\frac{1}{(i-6)!}\left(0(0)^{i-6} + 0(1)^{i-6} - \frac{2}{1575}(2)^{i-6}\right) \\ -\frac{1}{(i-7)!}\left(0(0)^{i-7} + 0(1)^{i-7} + \frac{2}{14175}(2)^{i-7}\right) \end{bmatrix} = 0, \quad i = 7(1)10.$$

Note that $q_0 = q_1 = q_2 = q_3 = q_4 = q_5 = q_6 = 0, i = 0(1)6$, considering the initial terms in $q_i$ correspondingly.
Thus, if $q_0 = q_1 = q_2 = \ldots = q_p = 0$ and $q_{p+1} \neq 0$, then the linear operator $L(y(x);h)$ in (25) and the associated continuous linear multistep methods in (21)–(24) are said to be of order $p$. Given that $q_{p+1}$ is the error constant, the local truncation error is given by:

$$t_{n+k} = q_{p+1} h^{(p+1)} y^{(p+1)}(x_n) + 0(h^{p+2}). \tag{28}$$

Consequently, with $q_i$ above, the order and error constants for "7D2PIB1 and 7D2PIB2" are examined as follows:

| Method | Order, $p$ | Error constant $(q_{p+1})$ |
|---|---|---|
| 7D2PIB1 | 10 | $-\frac{5881}{7185024000} D^{(11)}(y)(x) h^{11} + O(h^{12})$ |
| | 10 | $-\frac{23}{28066500} D^{(11)}(y)(x) h^{11} + O(h^{12})$ |
| 7D2PIB2 | 10 | $-\frac{3931}{12573792000} D^{(11)}(y)(x) h^{11} + O(h^{12})$ |
| | 10 | $-\frac{31}{98232750} D^{(11)}(y)(x) h^{11} + O(h^{12})$ |

**Table 1.** *Order and error constants*

### *3.2. Zero-Stability*

The block methods in (21)–(24) have a zero-stability polynomial that may be stated by evaluating:

$$R(t) = \left|(A^{(0)} t - A^{(1)})\right|, \tag{29}$$

The characteristic roots for 7D2PIB1 and 7D2PIB2 are thus obtained by equating (29) to zero and solving for $t$ to give:
$t = 0, 1$. As a result, 7D2PIB1 and 7D2PIB2 in (21)–(24) are zero-stable according to Definition 3.1.

### *3.3. Consistency*

**Lemma 1**. The new linear multistep methods in (9) are said to be consistent if and only if:

1. $p \geq 1$,
2. $\sum_{j=0}^{k} \alpha_j = 0$,
3. $\sum_{j=0}^{1} j\,\alpha_j = \sum_{j=0}^{k} \beta_j$
4. $\eta'(t) = \beta(t)$. (See [6]).

Let $\alpha_0 = -1, \alpha_1 = 1, \beta_0 = \frac{5639}{23040}, \beta_1 = \frac{121}{45}, \beta_2 = -\frac{44551}{23040}$, $\eta(t) = t - 1$, $\eta'(t) = 1$ and $\sigma(t) = \dfrac{5639}{23040} + \dfrac{121}{45}t - \dfrac{44551}{23040}t^2$,
where $\sigma(t)$ and $\eta(t)$ are the second and first characteristic polynomials, since a first-order ODE is considered.

**Remark:**

1. Condition (i) is certainly satisfied, since the order $p$ of the methods is 10 each.
2. It is clear also that condition (ii) is satisfied with the developed methods.
3. Again, $\sum_{j=0}^{1} j\alpha_j = \sum_{j=0}^{k} \beta_j = 1$, upon evaluation of condition (iii), i.e., for the first scheme,
4. When $t = 1$, is substituted into condition (iv), it is also verified.

Hence, the methods in (21)–(24) are found to be consistent.

**Remark:** The above conditions are also verified and satisfied by the second 7D2PIB2 method.

### *3.4. Convergency*

In line with Theorem 3.1, since the newly derived block methods are consistent and zero-stable, they converge.
Let $y_i$ and $y(x_i)$ be the approximate and exact solutions of (1), respectively, then the absolute error is evaluated by using the formula:
$AbsErr = |(y_i)_t - (y(x_i))_t|, 1 \leq t \leq NS$,
where $NS$ is the total number of steps.

### *3.5. Linear stability*

The absolute stability polynomials are presented below in the light of [32], using the test equations:
$y' = \lambda h$, with $\lambda$ assumed to go through the negative eigenvalues of the Jacobian matrix. So that substituting the above into (21)–(24) yields:

$$\begin{aligned} M(w,z) = -A_1 w + A_0 + zB_0 + zB_1 w + z^2 B_2 w + z^3 B_3 w + z^4 B_4 w + z^5 B_5 w \\ + z^6 B_6 w + z^7 B_7 w, \end{aligned} \tag{30}$$

Where $z = \lambda h$ and $w(y_n) = y_{n+j}^{j}, j = 1(1)7$ is the difference equation shift operator. From this, we have the following expression as the stability polynomial:

$$\pi_i(w,z) = |M(w,z)|, \quad i = 1, 2. \tag{31}$$

The following stability functions for 7D2PIB1 and 7D2PIB2, respectively, are produced by evaluating (31) to determine the absolute stability regions:

$$\begin{aligned} \pi_1(w,z) = -\frac{w^2}{285768000}z^{14} + \frac{w^2}{13608000}z^{13} - \frac{23\,w^2}{27216000}z^{12} + \frac{w^2}{151200}z^{11} - \frac{67\,w^2}{1814400}z^{10} \\ + \frac{29\,w^2}{201600}z^9 + \left(-\frac{1291\,w^2}{3628800} - \frac{w}{3628800}\right)z^8 + \left(-\frac{127\,w^2}{604800} - \frac{w}{604800}\right)z^7 + \frac{11\,w^2}{1350}z^6 \\ -\frac{7\,w^2}{135}z^5 + \frac{19\,w^2}{90}z^4 - \frac{11\,w^2}{18}z^3 + \left(\frac{223\,w^2}{180} - \frac{7\,w}{180}\right)z^2 + \left(-\frac{8}{5}w^2 - \frac{2}{5}\,w\right)z + w^2 - w, \end{aligned} \tag{32}$$

$$\pi_2(w,z) = -\frac{w^2}{2381400}z^{11} + \frac{w^2}{113400}z^{10} - \frac{23\,w^2}{226800}z^9 + \frac{w^2}{1260}z^8 - \frac{367\,w^2}{75600}z^7 + \frac{1817\,w^2}{75600}z^6 + \left(-\frac{2923\,w^2}{30240} - \frac{w}{30240}\right)z^5 + \left(\frac{523\,w^2}{1680} - \frac{w}{5040}\right)z^4 - \frac{7\,w^2}{9}z^3 + \left(\frac{64\,w^2}{45} - \frac{w}{45}\right)z^2 + \left(-\frac{17\,w^2}{10} - \frac{3}{10}w\right)z + w^2 - w. \tag{33}$$

From this, $\pi_1$ and $\pi_2$ in (32) and (33) are then coded in a MATLAB R2023a software environment, and the region of absolute stability for each derived method is as shown in Figures 3 and 4 below.

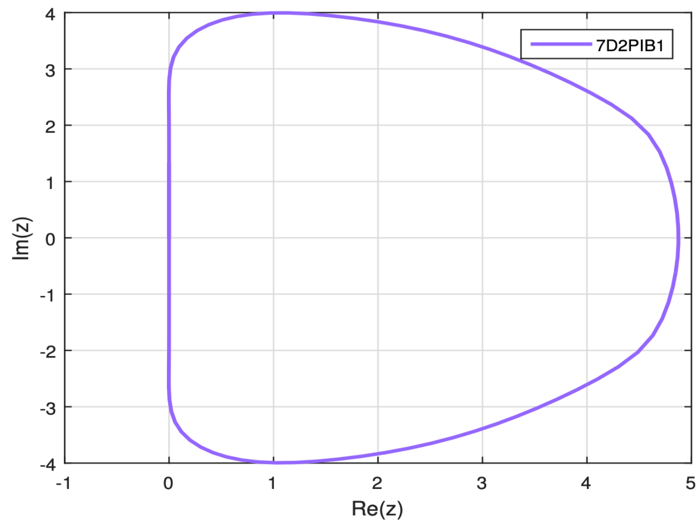


**Figure 3.** Absolute Stability Region of 7D2PIB1

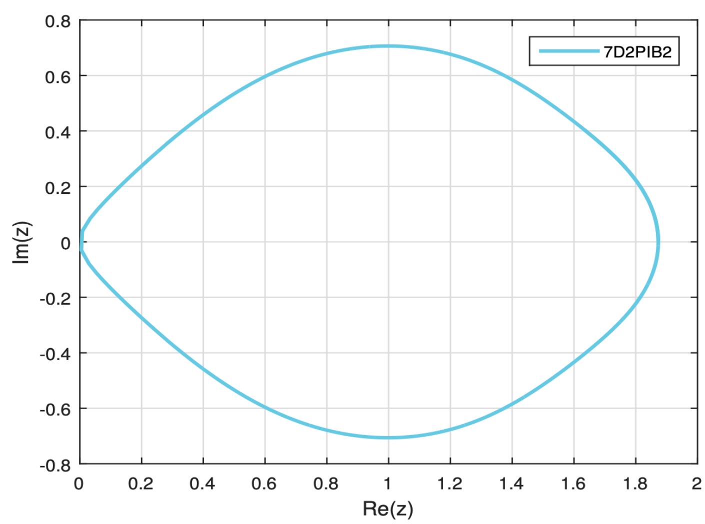


**Figure 4.** Absolute Stability Region of 7D2PIB2


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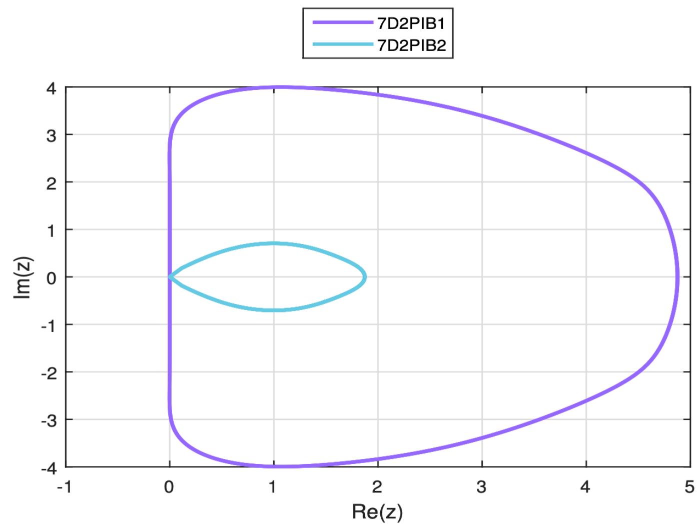


**Figure 5.** Compared absolute stability region of methods

Figures 3 and 4 indicate the region of absolute stability of the methods. The first method, 7D2PIB1, whose unstable region is the closed region, is larger than the second method, 7D2PIB2, as precisely shown in Figure 5; implying that 7D2PIB2 has an open region of larger stability than 7D2PIB1. However, both methods have regions of absolute stability that are left symmetric. Hence, both developed formulae are A-stable in line with Definition 3.2.

## 4. Implementation of the methods

The simultaneous approximation of $y_{n+l}$ in the new methods was done using Newton Raphson's techniques on the MATLAB (R2015a) software environment using: HP 655, Windows 8.1 Pro, processor: AMD E1-1200 APU with Radeon(tm) HD graphics 1.40GHz, Installed memory (RAM): $6.00$GB, $64-$bits operating system, x$64-$ based processor. Therefore, let $y_{n+l}^{(i+1)}$ be the $(i+1)^{th}$ iterations for approximating $y_{n+l}$ and $e_{n+1}^{j+1} = y_{n+l}^{(i+1)} - y_{n+l}^{(i)}$,

$$y_{n+l}^{(i+1)} = y_{n+l}^{(i)} - \frac{f(y_{n+l}^{(i)})}{f'(y_{n+l}^{(i)})}, \quad l = 1(1)k. \tag{34}$$

So that (35) can be rewritten as:

$$y_{n+l}^{(i+1)} - y_{n+l}^{(i)} = -[f(y_{n+l}^{(i)})][f'(y_{n+l}^{(i)})]^{-1}, \tag{35}$$

From which we get:

$$e_{n+1}^{j+1} = -[f(y_{n+l}^{(i)})][f'(y_{n+l}^{(i)})]^{-1}, \tag{36}$$

where,

$$f(y_{n+l}^{(i)}) = \left( \begin{array}{c} y_{n+1} - y_n - \frac{5639}{23040} h f_n - \frac{121}{45} h f_{n+1} + \frac{44551}{23040} h f_{n+2} - \frac{1289}{768} h^2 g_{n+2} \\ + \frac{7687}{11520} h^3 u_{n+2} - \frac{287}{1920} h^4 v_{n+2} + \frac{583}{34560} h^5 w_{n+2} - \frac{1}{5760} h^6 m_{n+2} \\ + \frac{257}{604800} h^7 q_{n+1} - \frac{121}{907200} h^7 q_{n+2} \\ y_{n+2} - y_n - \frac{11}{45} h f_n - \frac{128}{45} h f_{n+1} + \frac{49}{45} h f_{n+2} - \frac{4}{3} h^2 g_{n+2} + \frac{26}{45} h^3 u_{n+2} \\ - \frac{2}{15} h^4 v_{n+2} + \frac{2}{135} h^5 w_{n+2} + \frac{2}{4725} h^7 q_{n+1} - \frac{2}{14175} h^7 q_{n+2} \end{array} \right),$$

and


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$$
\begin{aligned}
A &= \begin{pmatrix} 1-\frac{121}{45}h\frac{\partial f_{n+1}}{\partial y_{n+1}}+\frac{257}{604800}h^7\frac{\partial q_{n+1}}{\partial y_{n+1}} & \begin{matrix}\frac{44551}{23040}h\frac{\partial f_{n+2}}{\partial y_{n+2}}-\frac{1289}{768}h^2\frac{\partial g_{n+2}}{\partial y_{n+2}}+\frac{7687}{11520}h^3\frac{\partial u_{n+2}}{\partial y_{n+2}}-\frac{287}{1920}h^4\frac{\partial v_{n+2}}{\partial y_{n+2}}\\ +\frac{583}{34560}h^5\frac{\partial w_{n+2}}{\partial y_{n+2}}-\frac{1}{5760}h^6\frac{\partial m_{n+2}}{\partial y_{n+2}}-\frac{121}{907200}h^7\frac{\partial q_{n+2}}{\partial y_{n+2}}\end{matrix} \\ -\frac{128}{45}h\frac{\partial f_{n+1}}{\partial y_{n+1}}+\frac{2}{4725}h^7\frac{\partial q_{n+1}}{\partial y_{n+1}} & \begin{matrix}1+\frac{49}{45}h\frac{\partial f_{n+2}}{\partial y_{n+2}}-\frac{4}{3}h^2\frac{\partial g_{n+2}}{\partial y_{n+2}}+\frac{26}{45}h^3\frac{\partial u_{n+2}}{\partial y_{n+2}}-\frac{2}{15}h^4\frac{\partial v_{n+2}}{\partial y_{n+2}}\\ +\frac{2}{135}h^5\frac{\partial w_{n+2}}{\partial y_{n+2}}-\frac{2}{14175}h^7\frac{\partial q_{n+2}}{\partial y_{n+2}}\end{matrix} \end{pmatrix}
\end{aligned}
$$

.

Hence, the approximations: $y_{n+l}^{(i+1)} = y_{n+l}^{(i)} + e_{n+1}^{j+1}$, as in step 6 of the algorithm below; while $B$ is a system of equations and $A$ is a $(2 \times 2)$ Jacobian matrix, $g, u, v, w, m$ and $q$ are the second, third, fourth, fifth, sixth, and seventh derivatives, respectively.

Since the new block is self-starting, it does not require a starting formula to incorporate all the initial values for the first-order IVPs. Therefore, approximate solutions $y_{n+l}$ are simultaneously generated.

**Algorithm 1** Methods Algorithm

**Input:** Define initial guess: $f(x), df(x), N, h$, $[a, b]$, where $f(x)$ is the problem to be solved and $df(x)$ is the derivative function, $e$ is the tolerance, $N$ total number of iterations and $h$ is the step-size and $[a, b]$ is the iterations interval.

**Output:** $y_{new} = y_{n+l}^{(i+1)}$, l=1,2.

1: **Set** $tol, N, n = 0, a \leq x \leq b$.
2: **Define** $x_n = a, y_n = y(a)$, $y_{new} = y_{n+l}^{(i+1)}$, $y_{old} = y_{n+l}^{(i)}$, $[a, b], h = \frac{(b-a)}{N}$,
3: **begin timing:** tic,
4: **for** $n = 1 : N - 1$, **do**
5: $\quad x(n) = x_0 + nh$
6: $\quad$ **while** $|(y_{old} - y_{new})| > tol$, **do**
7: $\quad\quad y_{new} = y_{old} - A^{-1}B$, where A and B are defined above,
8: $\quad\quad$ **Let** $y_{new} = y_{old}, i = 0(1)N - 1$,
9: $\quad\quad$ Goto 19
10: $\quad\quad$ **for** $n = n + 1$, **do**
11: $\quad\quad\quad$ Goto 5
12: $\quad\quad$ **end for**
13: $\quad\quad$ **if** $n \geq N$, **then**
14: $\quad\quad\quad$ Goto 6
15: $\quad\quad$ **end if**
16: $\quad$ **end while**
17: $\quad$ Goto 8
18: **end for**
19: **end timing:** toc.

## 5. Numerical Experiment

The performance of the new methods is tested on the following first-order initial-value problems, and where possible, comparisons are performed with a few chosen current methods of close or higher orders. The following notations are used:

SIR model $\longrightarrow$ Susceptible, infected, and recovered model.

LMM $\longrightarrow$ Linear multistep method.

ODEs $\longrightarrow$ Ordinary differential equations.

IVPs $\longrightarrow$ Initial value problems.

EINM $\longrightarrow$ Error in new method [33].

1 SHBM $\longrightarrow$ New one-step hybrid block method [34].

2 SHBM $\longrightarrow$ New two-step hybrid block method [34].

7D2PIB1 and 7D2PIB2 $\longrightarrow$ Seventh-derivative two-point implicit block 1 and 2 (derived methods),

$|y_n - y(x_n)| \longrightarrow$ Absolute error computed at the end of the mesh point over the chosen interval of integration.

**Problem 1**. Consider the first-order system of stiff initial-value problems:

$$
\begin{aligned}
y_1' &= -8y_1 + 7y_2, \quad y_1(0) = 1, \quad h = 0.1, \\
y_2' &= 42y_1 - 43y_2, \quad y_2(0) = 8,
\end{aligned}
$$


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**Exact Solution:**

$$y_1(t) = 2e^{-x} - e^{-50x}$$
$$y_2(t) = 2e^{-x} + 6e^{-50x}.$$

Source: [35]

| $x$ | Error in [35], p = 10 | | 7D2PIB1, p = 10 | | 7D2PIB2, p = 10 | |
|---|---|---|---|---|---|---|
| | $y_1$ | $y_2$ | $y_1$ | $y_2$ | $y_1$ | $y_2$ |
| **0.1** | 1.32e-06 | 8.10e-02 | 4.36e-03 | 2.62e-02 | 6.31e-04 | 3.79e-02 |
| **0.2** | 1.90e-08 | 5.50e-04 | 7.78e-05 | 4.67e-04 | 8.11e-06 | 4.870e-05 |
| **0.3** | 4.00e-09 | 3.70e-06 | 1.06e-06 | 6.37e-06 | 7.82e-08 | 4.69e-07 |
| **0.4** | 4.00e-09 | 2.10e-08 | 1.31e-08 | 7.88e-08 | 6.71e-10 | 4.02e-09 |
| **0.5** | 2.00e-09 | 3.00e-09 | 1.55e-10 | 9.28e-10 | 5.39e-12 | 3.24e-11 |
| **0.6** | 3.00e-09 | 2.00e-09 | 1.73e-12 | 1.07e-11 | 3.78e-14 | 2.55e-13 |
| **0.7** | 4.50e-09 | 2.90e-09 | 2.37e-14 | 1.68e-13 | 4.33e-15 | 6.77e-15 |
| **0.8** | 4.10e-09 | 3.70e-09 | 3.88e-14 | 4.44e-14 | 4.00e-15 | 4.11e-15 |
| **0.9** | 4.60e-09 | 4.00e-09 | 3.49e-14 | 3.81e-14 | 3.55e-15 | 3.78e-15 |
| **1.0** | 4.80e-09 | 4.60e-09 | 3.24e-14 | 3.59e-14 | 2.55e-15 | 2.44e-15 |

**Table 2.** *Comparison of Absolute Error for Problem 1 with* $h = 0.1$

**Problem 2**. Consider the first-order system of stiff initial-value problem:

$$y_1' = -9y_1 + 95y_2, \quad y_1(0) = 1, \quad h = 0.1,$$
$$y_2' = -y_1 - 97y_2, \quad y_2(0) = 1,$$

**Exact Solution:**

$$y_1(t) = \frac{95}{47}e^{-2x} - \frac{48}{47}e^{-96x}$$
$$y_2(t) = \frac{48}{47}e^{-96x} - \frac{1}{47}e^{-2x},$$

**Source:** [35]


doi.org/10.32388/C5IA9C.2

| $x$ | Error in [35], p = 10 | | 7D2PIB1, p = 10 | | 7D2PIB2, p = 10 | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $y_1$ | $y_2$ | $y_1$ | $y_2$ | $y_1$ | $y_2$ | | | |
| **0.1** | 1.74e-04 | 1.74e-04 | 3.70e-04 | 3.70e-04 | 9.22e-04 | 9.22e-04 | | | |
| **0.2** | 5.40e-08 | 5.30e-08 | 1.84e-07 | 1.84e-07 | 7.07e-07 | 7.07e-07 | | | |
| **0.3** | 1.00e-09 | 4.00e-11 | 8.01e-11 | 8.07e-11 | 5.94e-10 | 5.94e-10 | | | |
| **0.4** | 2.30e-09 | 3.50e-11 | 4.27e-13 | 2.99e-14 | 4.46e-13 | 4.95e-13 | | | |
| **0.5** | 2.20e-09 | 3.10e-11 | 3.69e-13 | 3.88e-15 | 3.71e-14 | 7.95e-16 | | | |
| **0.6** | 1.80e-09 | 2.70e-11 | 2.92e-13 | 3.08e-15 | 2.81e-14 | 2.93e-16 | | | |
| **0.7** | 1.60e-09 | 2.20e-11 | 2.34e-13 | 2.47e-15 | 1.93e-14 | 2.04e-16 | | | |
| **0.8** | 1.40e-09 | 2.00e-11 | 1.85e-13 | 1.95e-15 | 1.44e-14 | 1.49e-16 | | | |
| **0.9** | 1.20e-09 | 1.60e-11 | 1.45e-13 | 1.53e-15 | 9.44e-15 | 9.98e-17 | | | |
| **1.0** | 9.10e-10 | 1.40e-11 | 1.16e-13 | 1.22e-15 | 6.50e-15 | 6.68e-17 | | | |

**Table 3.** *Comparison of Absolute Error for Problem 2 with* $h = 0.1$

**Problem 3**. Consider the second-order initial-value problem:

$y'' = y', \quad y(0) = 0, \quad y'(0) = -1, \quad 0 \leq x \leq 1, \quad h = 0.1,$

**Exact Solution:**

$y(x) = 1 - e^x,$

**Source:** [33]

| $x$ | **EINM** [33], $p = 10$ | **7D2PIB1,** $p = 10$ | **7D2PIB2,** $p = 10$ |
|---|---|---|---|
| **0.1** | 4.462679e-11 | 0.000000e+00 | 2.220446e-16 |
| **0.2** | 9.864032e-11 | 1.387779e-16 | 0.000000e+00 |
| **0.3** | 1.635218e-10 | 5.551115e-17 | 1.110223e-16 |
| **0.4** | 2.409591e-10 | 1.110223e-16 | 2.220446e-16 |
| **0.5** | 3.328765e-10 | 0.000000e+00 | 2.220446e-16 |
| **0.6** | 4.414623e-10 | 0.000000e+00 | 2.220446e-16 |
| **0.7** | 5.692067e-10 | 1.110223e-16 | 4.440892e-16 |
| **0.8** | 7.189380e-10 | 0.000000e+00 | 4.440892e-16 |
| **0.9** | 8.938681e-10 | 0.000000e+00 | 6.661338e-16 |
| **1.0** | 1.097642e-09 | 0.000000e+00 | 8.881784e-16 |

**Table 4.** *Comparison of Absolute Error for Problem 3 with* $h = 0.1$

**Problem 4**. Consider the first-order initial-value problem:

$y' = -y, \quad y(0) = 1, \quad 0 \leq x \leq 1, \quad h = 0.1,$

**Exact Solution:**

$y(x) = e^{-x},$

**Source:** [34]


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| $x$ | 1 SHBM [34] | 2 SHBM [34] | EINM [33] | 7D2PIB1, |
|---|---|---|---|---|
| | $p=10$ | $p=18$ | $p=10$ | $p=10$ |
| **0.1** | 0.00e+00 | 4.10e-20 | 3.98e-11 | 0.00e+00 |
| **0.2** | 1.10e-20 | 6.10e-20 | 7.20e-11 | 0.00e+00 |
| **0.3** | 2.10e-20 | 8.10e-20 | 9.77e-11 | 1.11e-16 |
| **0.4** | 1.10e-20 | 1.11e-20 | 1.18e-10 | 0.00e+00 |
| **0.5** | 1.10e-20 | 1.21e-19 | 1.33e-10 | 2.22e-16 |
| **0.6** | 2.10e-20 | 1.31e-19 | 1.46e-10 | 2.22e-16 |
| **0.7** | 1.10e-20 | 1.41e-19 | 1.53e-10 | 1.67e-16 |
| **0.8** | 2.10e-20 | 1.41e-19 | 1.58e-10 | 0.00e+00 |
| **0.9** | 2.10e-20 | 1.51e-19 | 1.61e-10 | 5.55e-17 |
| **1.0** | 3.10e-20 | 1.41e-20 | 1.62e-10 | 0.00e+00 |

**Table 5.** *Comparison of Absolute Error for Problem 4 with* $h=0.1$

## 6. Application problem

**Problem 5**.

The SIR model is an epidemiological model that calculates the theoretical number of individuals infected with an infectious disease in a closed population over time, as detailed in [28]. The fact that these models incorporate coupled equations linking the number of susceptible individuals S(t), the number of infected individuals I(t), and the number of recovered individuals R(t) is where the name of this class of models originates. For many infectious diseases, this is an effective and straightforward paradigm; encompassing rubella, mumps, and measles. The model's flow chart is displayed as follows:

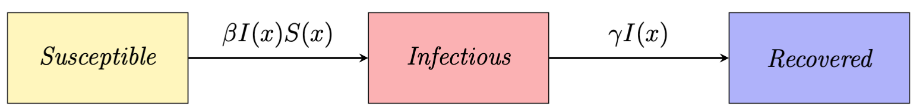


**Figure 6.** The flow chart of the SIR model

The nonlinear differential system describing the SIR model flow chart is given by three coupled equations below:

$$\left.\begin{array}{rcr} \frac{dS}{dt} & = & \mu(1-S)-\beta IS \\ \frac{dI}{dt} & = & -\mu I-\gamma I+\beta IS \\ \frac{dR}{dt} & = & -\mu R+\gamma I \end{array}\right\}, \tag{37}$$

Where $\mu$, $\gamma$ and $\beta$ are positive parameters to be determined. Therefore, let $y$ be given by:

$$y = S + I + R, \tag{38}$$

By taking the derivative of (38) and summing (37) and (38) to give the SIR model of the form:

$$y' = \mu(1-y), \quad 0 \le x \le 1, \quad h = 0.1, \tag{39}$$

Whose exact solution is:

$y(x) = 1 - 0.5e^{-0.5x}$.

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**Source:** [33]

| $x$ | Error in [33] | 7D2PIB1, $p = 10$ | 7D2PIB2, $p = 10$ |
|---|---|---|---|
| **0.1** | 3.198553e-13 | 0.000000e-00 | 0.000000e-00 |
| **0.2** | 6.086243e-13 | 0.000000e-00 | 0.000000e-00 |
| **0.3** | 8.685275e-13 | 0.000000e-00 | 0.000000e-00 |
| **0.4** | 1.101452e-12 | 1.1102230e-16 | 1.1102230e-16 |
| **0.5** | 1.309841e-12 | 0.000000e-00 | 3.330669e-16 |
| **0.6** | 1.495026e-12 | 1.1102230e-16 | 2.220446e-16 |
| **0.7** | 1.659228e-12 | 1.1102230e-16 | 2.220446e-16 |

**Table 6.** *Comparison of Absolute Error for Problem 5 in Equation (39) (SIR Model) with* $h = 0.1$

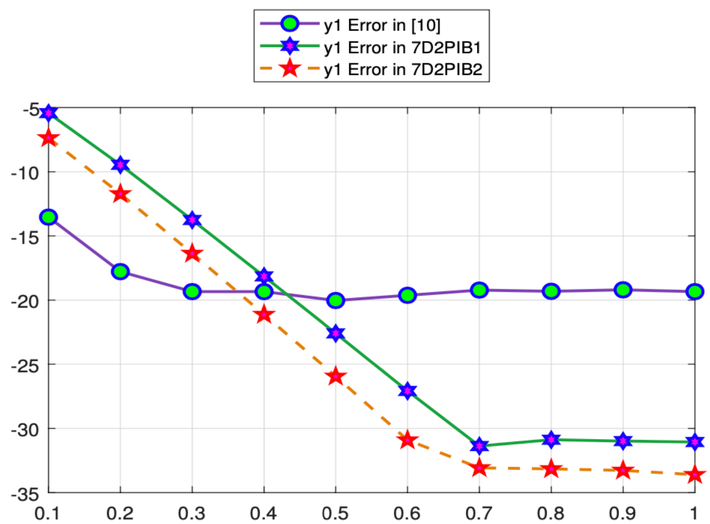


**Figure 7.** Efficiency curves for $y_1$ in Table 2


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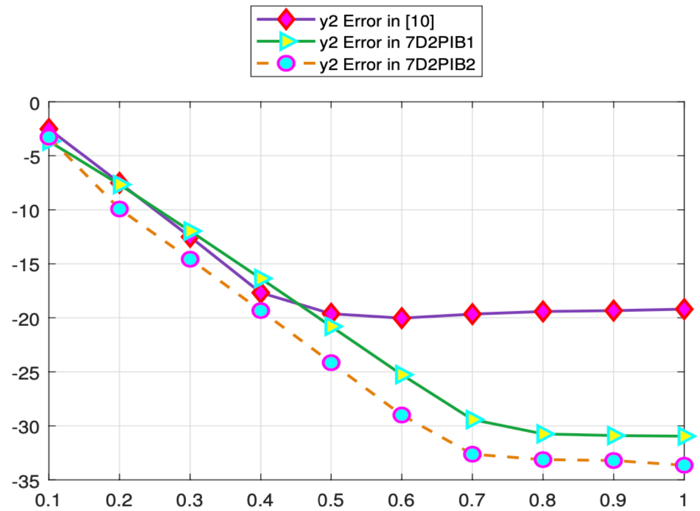


**Figure 8.** Efficiency curves for $y_2$ in Table 2

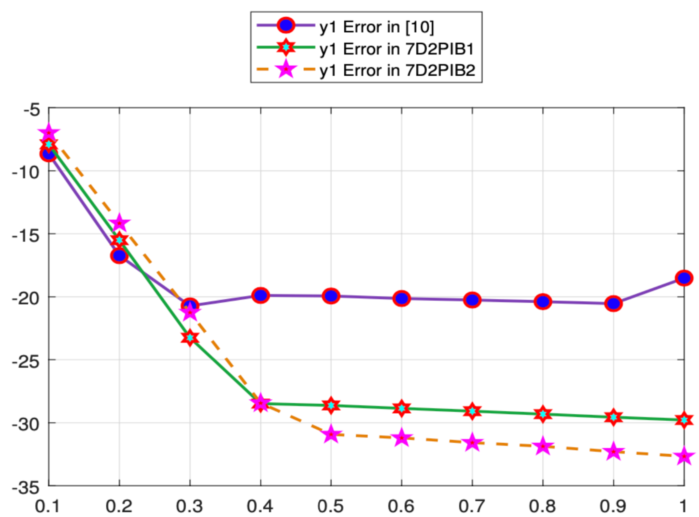


**Figure 9.** Efficiency curves for $y_1$ in Table 3

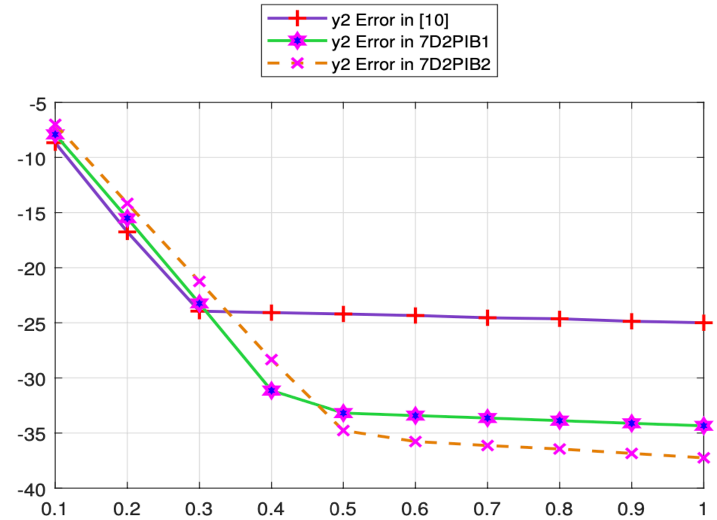


**Figure 10.** Efficiency curves for $y_2$ in Table 3

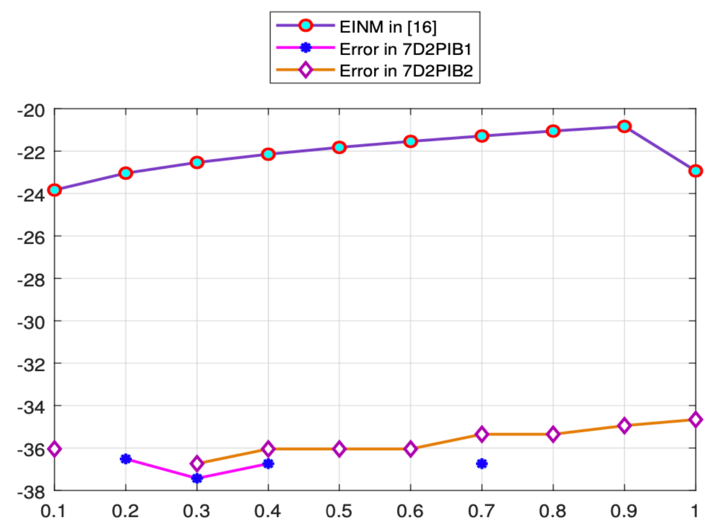


**Figure 11.** Efficiency curves of Table 4


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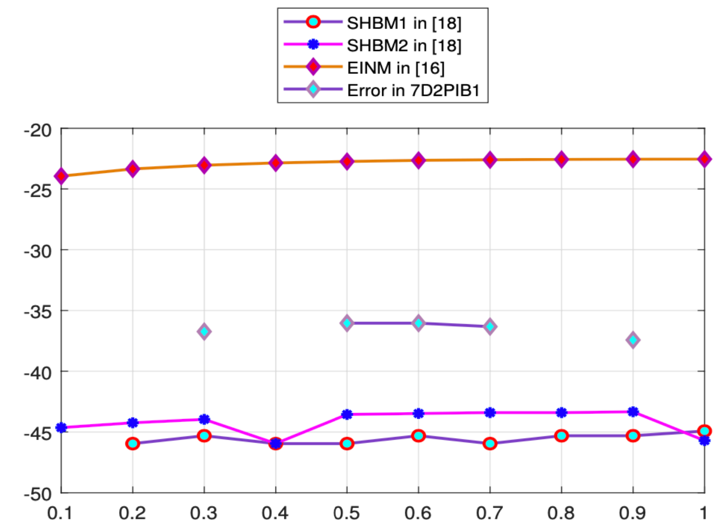


**Figure 12.** Efficiency curves for Table 5

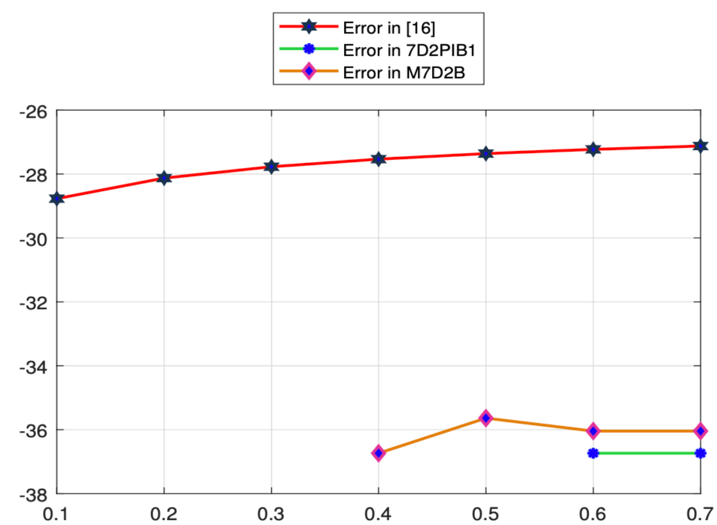


**Figure 13.** Efficiency curves of Table 6 (SIR Model)

## 7. Discussion of Results

The solutions to Problem 1 are displayed in Table 2, and their efficiency curves are shown in Figures 7 and 8. The figures demonstrate that the suggested approaches exhibit better accuracy than the method in [35] at several grid and approximate points of iterations. It is clear that the suggested methods show enhanced accuracy at $h = 0.1$. The accuracy of the derived methods suggests that they could have reduced absolute scale errors at smaller step sizes, which can lead to an approximate solution that approaches the genuine answer.

Similarly, [35] used an implicit block technique of uniform order $10$ to solve Problem 2. The suggested formulae are used to solve the same. The efficiency curves for the results are displayed in Figures 9 and 10, and the results are given in Table 3. It is evident that the suggested approaches perform more accurately with 7D2PIB2 than with 7D2PIB1 and similar methods in [35]. The data demonstrate that, as iterations go on, the suggested approaches exhibit decreasing scale absolute errors at numerous grid points, indicating consistency in terms of numerical attributes.

In [33], Problem 3 was examined using a uniform block order of $10$. Their approach was used straight away without any starting values. With the efficiency curves displayed in Figure 11, the outcomes of the obtained formulas are illustrated in Table 4. A straightforward comparison of our derived approaches shows that 7D2PIB1 outperformed such a method of order $10$ in [33], while outperforming 7D2PIB2 of the same order $10$, although with minor equivalent performance in accuracy. The competitive performances of 7D2PIB1, 7D2PIB2, and such a current approach in [33] are shown in Figure 10. It is evident that at many grid points of the iterations, 7D2PIB1 exhibits convergence.

For Problem 4 in [33], a half-step numerical model has been derived for solving first and second orders, respectively. The proposed approaches are implemented with a step size of $h = 0.1$. Numerical results, as shown in Table 5, indicate how well the proposed formulae (7D2PIB1 and 7D2PIB2) perform in comparison to that in [33] whose methods are of uniform order $10$. The convergence at average grid points of the iterations in 7D2PIB1 is shown in Figure 11, which outperforms, especially, EINM of uniform order $10$.

In [34], one-step and two-step hybrid block approaches of uniform order $10$ and $18$, respectively, were used to solve Problem 3. The suggested techniques are implemented using a step size of $h = 0.1$. Application results, as displayed in Table 5, demonstrate how well our initial formula, 7D2PIB1, performs in comparison to those in [34], specifically, 2 SHBM of order $18$, as well as EINM of order $10$, in [33]. The convergence at average grid points of the iterations in 7D2PIB1 is shown in Figure 12, which outperforms, especially for 1 SHBM and 2 SHBM of uniform order $10$ and $18$ respectively.

Using the suggested techniques, Problem 5 in (39) is resolved. The outcomes, as shown in Table 6, show the absolute errors and the discretized points over the integration interval. Figure 13 displays the efficiency curves, which are also the graph of the logarithm of the absolute errors against the logarithm of grid points. Figure 13 illustrates how the stiff nature of the modeled problem causes scale absolute errors inaccuracies, especially in the 7D2PIB2 inter-nodes. In contrast to 7D2PBI2, Table 6 also illustrates the convergence of 7D2PBI1 at most grid points. Reasonably, 7D2PBI1 offers better results over 7D2PBI2 for this specific Problem 5 in (39). Lastly, as Table 6 illustrates, the suggested methods offer increased accuracy over the compared method in [28].

## 8. Summary

In this research, a novel class of computational methods of uniform order $10$ with a seventh-derivative type, though of a first-order function, has been designed using an interpolation and collocation approach. The methods utilized the advantages of non-uniform points of collocation to improve effective time cost and accuracy in numerical method iterations. Again, the new methods use a seventh-derivative type, which is unique compared to other existing numerical methods and has proven to be computationally stable on an ample number of test examples, including an application problem. It is noted from the results that the non-uniformity and positioning of collocation points influence the accuracy of any given numerical method(s). Finally, since all problems solved used a large step size, $h = 0.1$ and results indicate efficiency and improved accuracy, it follows that smaller scale absolute errors are certainly possible with smaller step sizes, indicating close convergence or convergence of the methods.

## 9. Conclusion and future research

A new family of computational techniques, with a seventh-derivative type of implicit two-point block, for the direct approximation of first-order initial-value problems of uniform order 10 each, has been developed. Formulae were derived through interpolation and collocation techniques. The new methods considered uneven points of collocation. They require a seventh-derivative type, though of a first-order function. It is established that uneven points of collocation affect the accuracy of numerical schemes, in terms of absolute errors. The new methods are found to be A-stable and convergent. The convergence is shown through test problems on first-order and second-order IVPs, including a real-life problem such as the SIR model, and with comparison to such other existing methods. Results indicate that the new approaches showed different numerical behaviors on different problems solved, while outperforming such existing methods in the literature. Summarily, 7D2PIB2 displayed better accuracy for Problems 1 and 2 than 7D2PIB1. While for Problems 4, 3, and 5, 7D2PIB1 displayed improved accuracy compared to 7D2PIB2. This indicates that the non-uniform points of collocation in 7D2PIB1 give better accuracy than in 7D2PIB2. Our next future research will focus on developing and implementing efficient and robust numerical methods with uneven collocation points for real-life problems in chemical reactions in chemical engineering, models on drug magnetic nano-particle transport, population growth models, tumor immune interaction models, biomass transfer, nutrient flow in an aquarium, etc., and application to higher-order stiff IVPs may also be considered.

## Statements and Declarations


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### Data Availability

The data used to support the results of the study are duly enclosed in the paper.

### Conflicts of Interest

The authors declare that there are no conflicts of interest.


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### Author Contributions

Atabo, V. O.: Conceptualization, methodology, formal analysis, and software. Adee, S. O.: Investigation, review, and editing. Olatunji, P. O.: Validation and visualization. Yahaya, D. J.: Confirmation and original draft preparation. All content of the manuscript was written via the authors' contributions, read, and agreed to publish the final manuscript.

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