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Review Article

Visualizing Generalizations of the Pythagorean Theorem

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The Pythagorean theorem is the most famous theorem. It was extended extensively over the ages, for instance as the Notrott-Ebisui's fivefold theorem or as the four hinged squares theorem. Here, visual proofs are presented for these generalizations, including a proof of the cosine rule in the style of Euclid's windmill proof.

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The Pythagorean Theorem

The Pythagorean theorem, the unrivaled $a^2 + b^2 = c^2$, is named after the Greek mathematician Pythagoras (c. 570–495 BCE). However, some scholars argue that Pythagoras may never have existed or that the theorem predates him. Evidence such as the 3,700-year-old 'Plimpton 322' clay tablet, discovered in Iraq in 1921, contains numerical examples that align with the theorem. In 2017, Australian mathematicians Daniel Mansfield and Norman Wildberger revisited this ancient artifact, drawing the attention of the international press to the theorem's history (see [1]).

The theorem owes much of its fame to Euclid's *The Elements* (circa 300 BCE), a text that was second only to the Bible in historical popularity (see [2]). Euclid's 'windmill proof' (see Fig. 1) has been studied by countless great minds, including Omar Khayyam (1048–1123; see Fig. 2a), Leonardo da Vinci (1452–1519; see Fig. 2b), and even self-taught scholar Abraham Lincoln, who referenced Euclid in at least one of his speeches. In 1968, E.S. Loomis compiled an impressive 370 proofs of the theorem (see [3]), highlighting its broad appeal. Today, the Internet is overflowing with all kinds of proofs, including animated ones.

For visual representations, artists often favor proofs by rearrangement, where the square on the hypotenuse is formed by the squares on the other sides, cut into pieces or with additional shapes (see Fig. 2d), but copyright restrictions prevent showing them (see [4][5]).

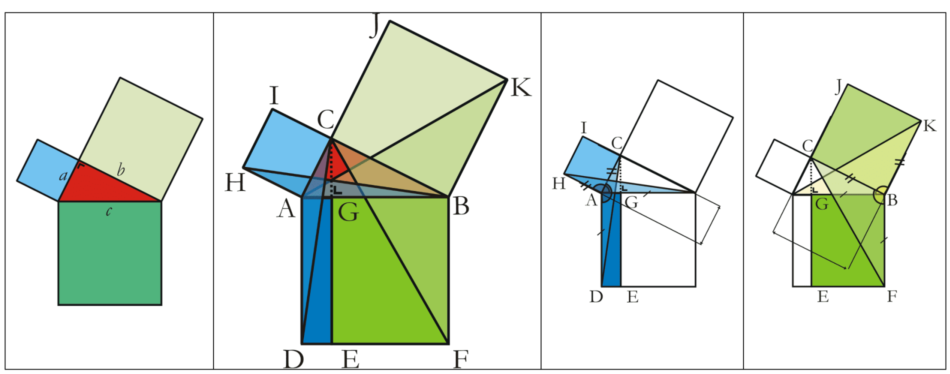

Figure 1. Euclid's 'windmill' proof: IHAC + JKBC = AGED + BGEF = ABFD, and details explaining why IHAC = 2.HAB = 2.CAD = AGED and JKBC = 2.KAB = 2.CFB = BGEF.

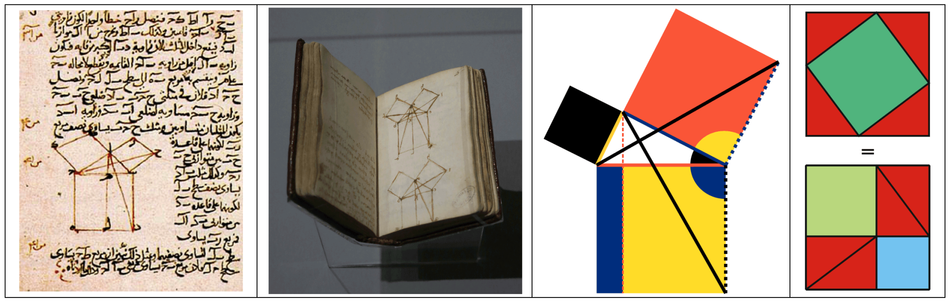

Figure 2. The Euclidean proof by Persian al-Tusi (see [6]), Leonardo (see [7]), Oliver Byrne (see [2]), and a proof by rearrangement (the colors refer to Fig. 1a).

A proof by rearrangement for the Ebisui-Notrott theorem

There are numerous generalizations of the Pythagorean theorem, some of which involve replacing the squares on the sides of the triangle with other geometric shapes, such as triangles, pentagons, or even semicircles, not to mention attempts in three dimensions. A more recent contribution to this area involves Austrian mathematician Gunter Weiss (born 1946), a former professor at the University of Vienna and TU Dresden and a key figure in the International Society for Geometry and Graphics (ISGG). Weiss became intrigued by the work of Hirotaka Ebisui, a Japanese amateur mathematician whom he met at a conference in Osaka.

Ebisui, described by Weiss as an eccentric with no formal mathematical training, little proficiency in English, and no connection to Japan's professional mathematical community, found over 4,000 'new' theorems. Recognizing the significance of Ebisui's work, Weiss brought attention to his findings. One of Ebisui's contributions is a generalization of the Pythagorean theorem that involves constructing squares on the convex hull of the original squares. He stated that the sum of the areas of two of these new squares is equal to five times the area of the third square. Furthermore, by connecting the vertices of these squares with additional squares, he again derived the original Pythagorean theorem, and so on, in a recursive manner. Ebisui referred to this result as the Pythagorean Fivefold Theorem.

Unbeknownst to Ebisui and Weiss at the time, J.C.G. Notrott, a Dutch mathematician, had previously published an even more general result in a Dutch magazine (see [8] and [9]). This independent discovery by Ebisui and Weiss, alongside Notrott's earlier work, highlights the universal appeal and richness of this generalization of the Pythagorean theorem.

Previously, a trigonometric proof (see [10]) and a vector-based proof (see [11]) were given for this result, while a proof with complex numbers can be found in [12]. Here, we present a visual proof of the Ebisui-Notrott theorem using the method of rearrangement.

Referring to Figure 3, observe the following: the right-angled red triangle and the two enclosed triangles (orange and purple) all have equal areas because they share a common base and an equal height. The light-yellow square is the combined area of the dark-blue square and four light-blue squares. Similarly, the dark-yellow square is the combined area of the light-blue square and four dark-blue squares. Adding these, the sum of the two yellow squares equals five times the combined areas of the light-blue and dark-blue squares. Since the sum of the areas of the blue squares equals the area of the magenta square, the proof is complete.

doi.org/10.32388/CKHBKO.2

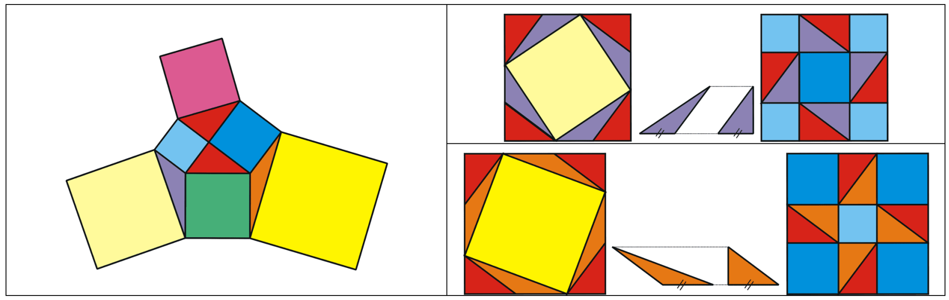

Figure 3. Visual proof for the first ring in the Ebisui-Notrott theorem: 5 times the magenta square equals the sum of the yellow squares.

Connecting the vertices of these squares, we note they can be regrouped around a (larger) right triangle by translations, as the squares have parallel edges. Thus, the classical Pythagorean theorem can be applied again, and so on.

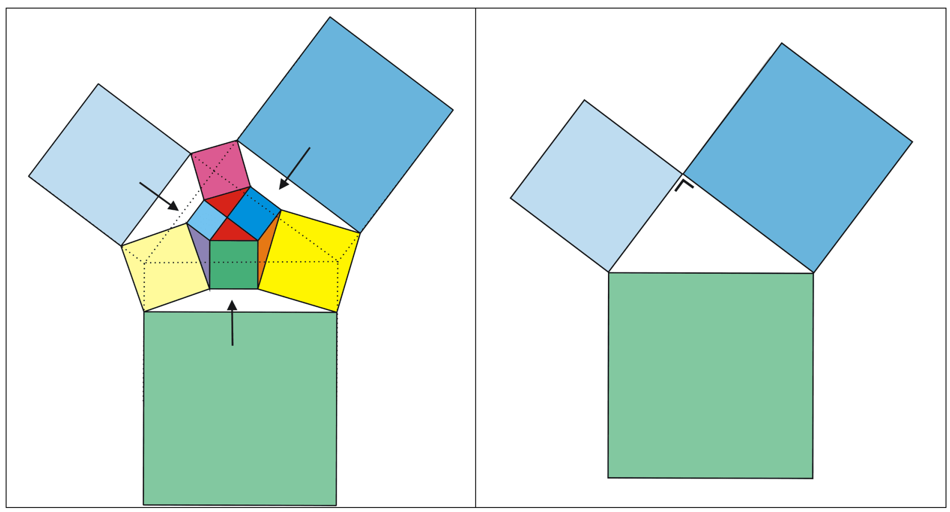

Figure 4. Visual proof for the second ring in the Ebisui-Notrott theorem: it again obeys the classical Pythagorean theorem.

A visual proof for the four (hinged) squares theorem

Another generalization of the Pythagorean theorem is the so-called 'four (hinged) squares theorem', yet another generalization. Consider four squares, grouped around two triangles with a common vertex as in Figure 5. The four squares theorem now states that the sum of the areas of the light green square ABFD and the dark green square HJRS on the non-equal sides of two triangles with two equal sides and supplementary enclosed angles is double the sum of the areas of the light blue square BCJK and the dark blue square ACHI on the equal sides.

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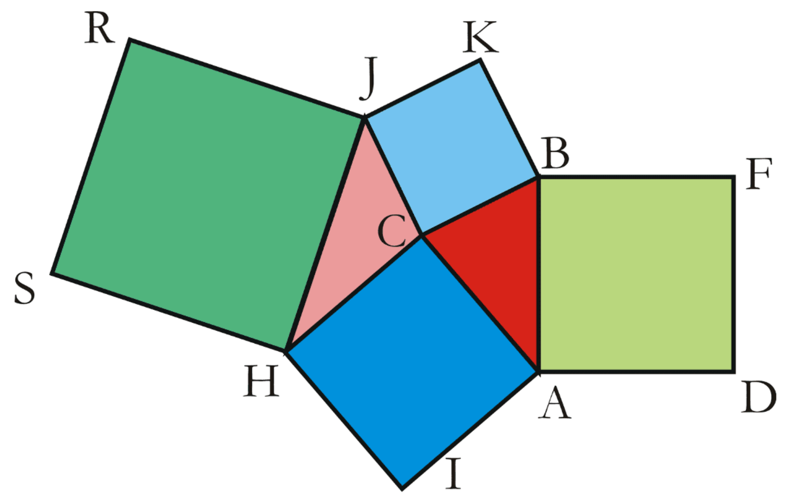
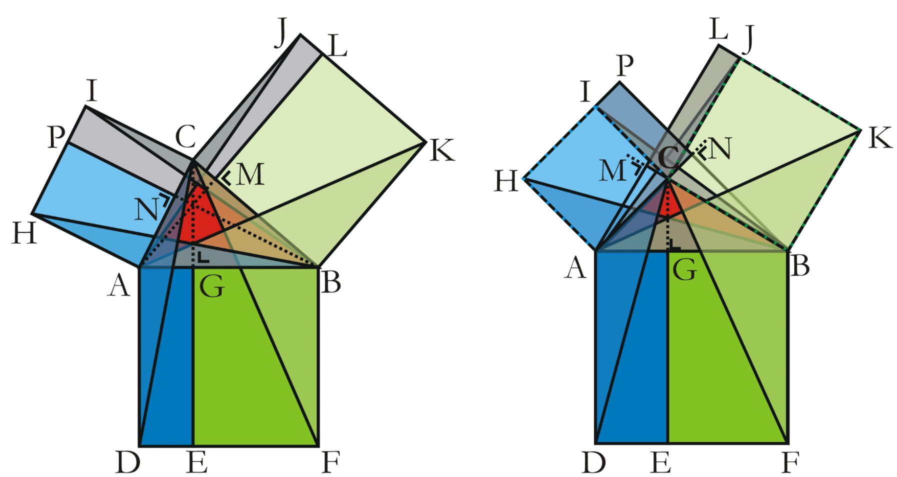


Figure 5. The 4 (hinged) squares theorem: the sum of the green squares is double the sum of the blue squares.

The trigonometric or vectorial proofs are neat (see [13]), but here we nevertheless propose a visual proof as it illustrates the use of Euclid's windmill theorem for an arbitrary triangle. First, we consider the case where the angle in C is acute (see Fig. 6a). As in Euclid's proof, PNAH = 2.BAH = 2.DAC = GEDA while LKBM = 2.KAB = 2.CFB = GBFE. Thus, 'Square ICAH' – ICNP + 'Square JKBC' – JLMC = 'Square ABFD'. However, using two other 'windmill constructions', it is seen that ICNP = 2.ICB = 2.AJC = JCML and thus:

$$square\ ABFD = square\ ICAH + square\ JKBC - 2.ICNP = square\ ICAH + square\ JKBC - 2.JCML.$$

Algebraically, this corresponds to $DF^2 = AC^2 + BC^2 - 2.AC.BC.cosC$, that is, the cosine rule. If the angle in C is obtuse, rectangles ICNP and JLMC must be added to the squares, which, algebraically, corresponds to cosC<0 in the law of cosines (see Fig. 6b). This visualization of the cosine rule is well-known (see, for instance, [14]). However, no reference could be found proving it using the generalization of Euclid's windmill proof given here, while there most probably is one, as Euclid's proof was the most popular proof for over a thousand years.



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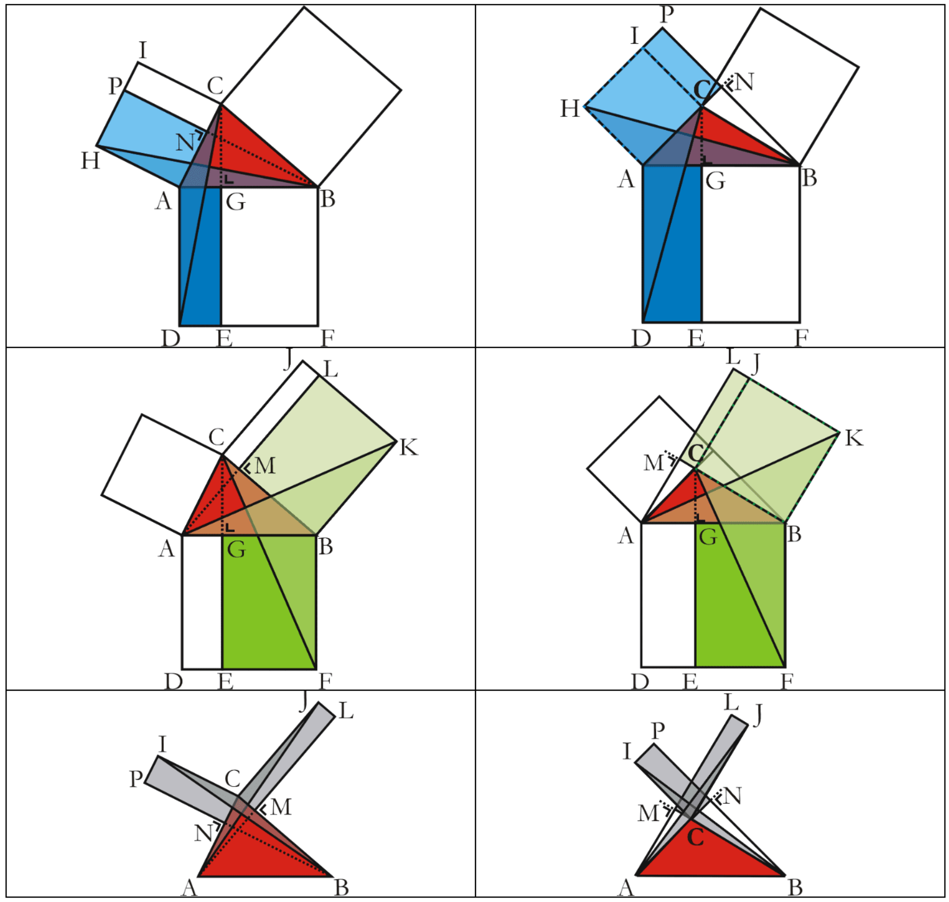


Figure 6. Euclid's windmill proof of the cosine rule, for a triangle with an acute angle C (left) and for a triangle with an obtuse angle C (right); note both gray rectangles have the same area in each case.

Applying this to the four-squares setting given above, ABFD = BCJK + ACHI – 2.CJLP for the acute triangle ABC, and HJRS = BCJK + ACHI + 2.CBUQ, for the obtuse triangle HJC. Using another 'windmill construction', CJLP = 2.CJA = 2.CBH = CBUQ. Thus, ABFD + HJRS = 2.BCJK + 2.ACHI (see Fig. 7).

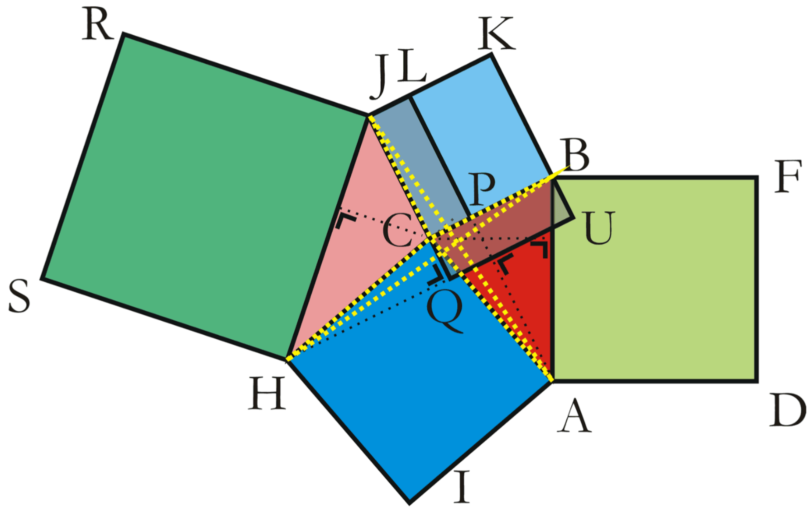


Figure 7. Visual proof of the four (hinged) squares theorem.

Note

The visualization of the cosine rule is well-known (see, for instance, [14]). However, no references could be found proving it using the generalization of Euclid's windmill proof given here (see Figure 8). Yet, there most probably is one, as Euclid's proof was the most popular proof for over a thousand years.

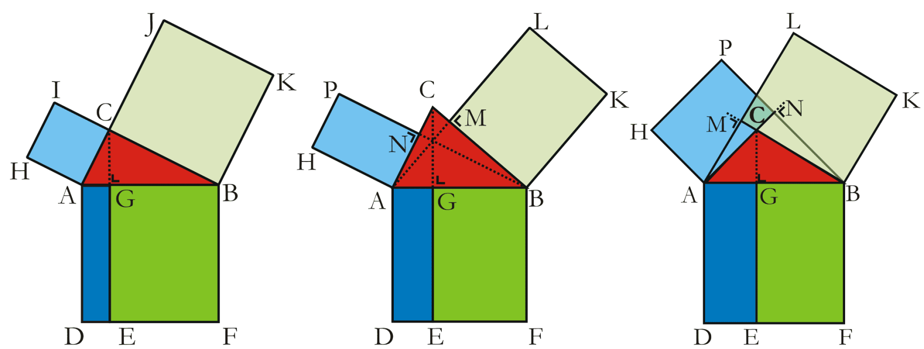
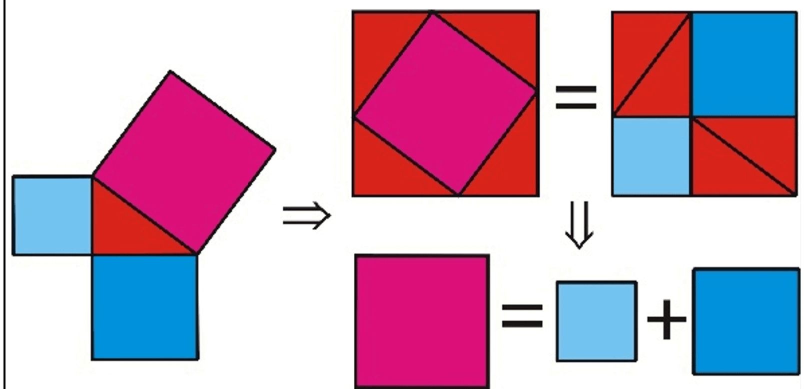
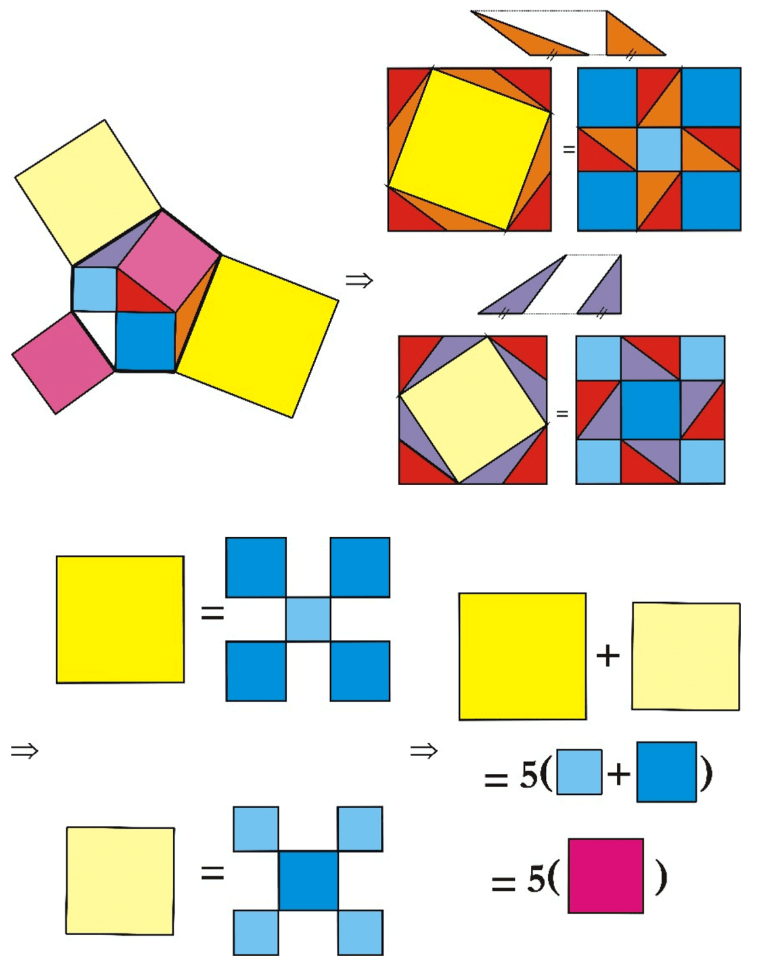
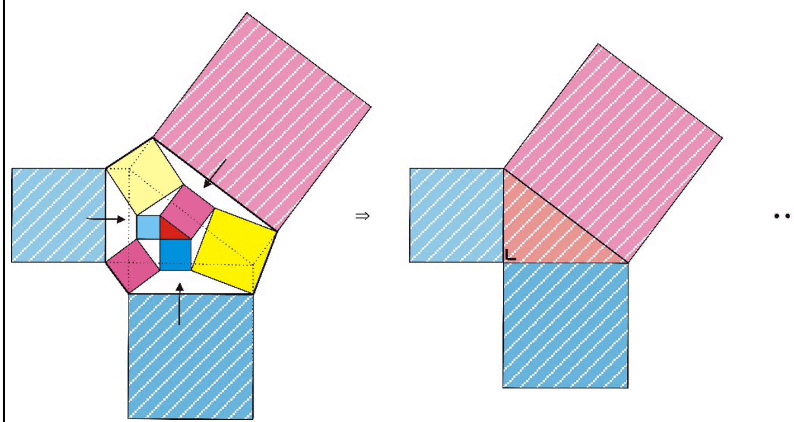


Figure 8. A comparison: a right (left), an acute (middle) and an obtuse (right) triangle. In each case, the sum of the light blue and green rectangles equals that of the darker ones.

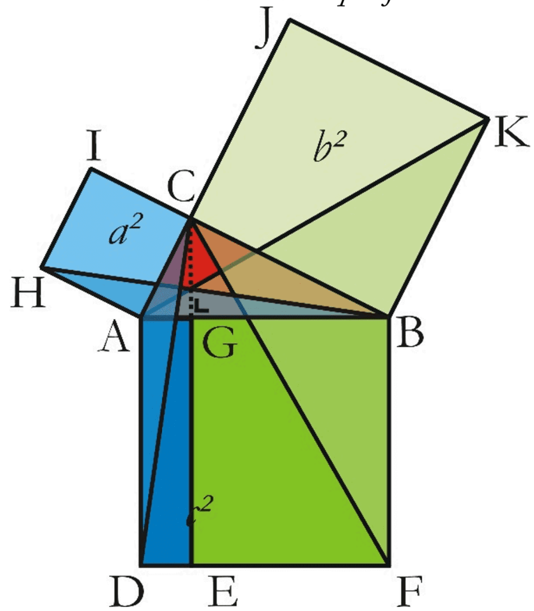
It seems audacious to think this visualization is new, but some, and not the least, have been remarkably bold too. According to the Encyclopædia Britannica (see [11]), the German mathematician Carl Friedrich Gauss (1777–1855) proposed in 1821 to create Euclid's windmill theorem with pine trees on a large open field so as to demonstrate the presence of intelligent life on Earth. The Encyclopædia regretted the project was never realized, “leaving undecided whether the inhabitants of Mars have no telescope, no geometry, or no existence”.

Summary

Pythagoras	
Notrott-Ebisui, 1	

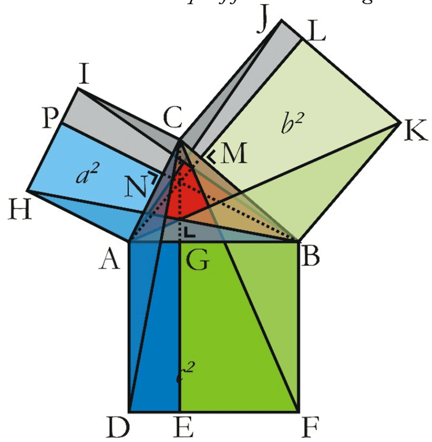
Notrott-Ebisui, 2	

Euclid's windmill proof



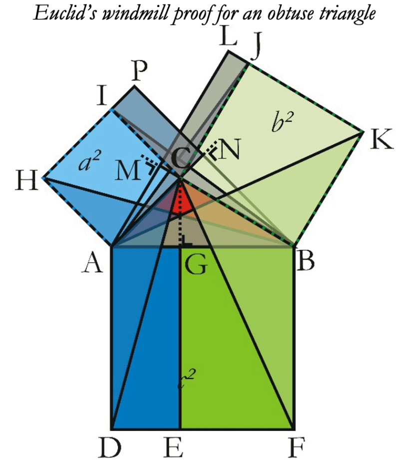
ICAH + JKBC = 2.HBA + 2.KBA
= 2.CDA + 2.CBF = AGED + GBFE = ABFD,
or $a^2 + b^2 = c^2$.

Euclid's windmill proof for an acute triangle



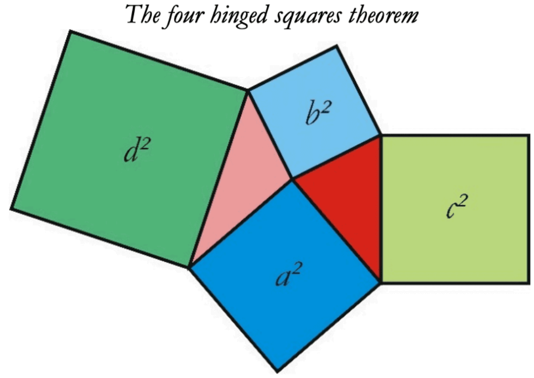
ICAH – ICNP + JKBC – JLMC
= PNAH + LKBM = 2.BAH + 2.KBA
= 2. CDA + 2.CBF = AGED + GBFE = ABFD.
ICNP = 2.ICB = 2.ACJ = JLMC
⇒ ICAH + JKBC – 2.ICNP
= ICAH + JKBC – 2.JLMC = ABFD,
or $a^2 + b^2 - 2ab\cos C = c^2$

Euclid's windmill proof for an obtuse triangle



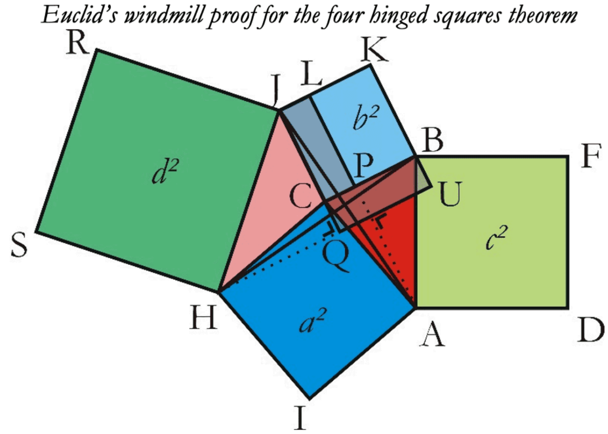
ICAH + ICNP + JKBC + JLMC
= PNAH + LKBM = 2.BAH + 2.KBA
= 2.CDA + 2.CBF = AGED + GBFE = ABFD.
IPNC = 2.ICB = 2.AJC = JCML
⇒ ICAH + JKBC + 2.ICNP
= ICAH + JKBC + 2.JLMC = ABFD,
or $a^2 + b^2 - 2ab\cos C = c^2$

The four hinged squares theorem



$c^2 + d^2 = 2.(a^2 + b^2)$

Euclid's windmill proof for the four hinged squares theorem



ABFD = AIHC + BCJK – 2.CJLP and HSRJ = AIHC + BCJK + 2.CBUQ
CJLP = 2.CJA = 2.CBH = CBUQ ⇒ ABFD + HSRJ = 2.AIHC + 2.BCJK
or $c^2 + d^2 = 2.(a^2 + b^2)$

Statements and Declarations

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed. This study presents theoretical proofs and historical context based on existing mathematical principles and cited literature.

Author Contributions

DH was the sole author of this work and is responsible for the conception, research, writing, and revision of the manuscript.

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Declarations

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Potential competing interests: No potential competing interests to declare.