Research Article

1D Self-Similar Fractals with Centro-Symmetric Jacobians: Asymptotics and Modular Data

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We establish the asymptotics of growing one-dimensional self-similar fractal graphs that are networks with multiple weighted edges between nodes. The asymptotics is described in terms of quantum central limit theorems for algebraic probability spaces in a pure state. We endow an additional structure upon the repeating units of centro-symmetric Jacobians in the adjacency of a linear graph, creating a self-similar fractal. The family of fractals induced by centrosymmetric Jacobians is formulated as orthogonal polynomials that satisfy three-term recurrence relations and support such limits. The construction proceeds with interacting Fock spaces and T-algebras endowed with a quantum probability space, corresponding to the Jacobi coefficients of the recurrence relations. When some elements of the centrosymmetric matrix are constrained in a specific way, we obtain, as the same Jacobian structure is repeated, quantum central limits. The generic formulation of Leonard pairs that form bases of conformal blocks and probabilistic Laplacians used in physics provide a choice of centrosymmetric Jacobians widening the applicability of the result. We establish that the T-algebras of these 1D fractals, as they form a special class of distance-regular graphs, are thin, and the induced association schemes are self-duals that lead to anyonic systems with modular invariance.

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1. Introduction

In a series of publications, we characterized anyons, systems that are the building blocks of topological quantum computing, in terms of interacting Fock spaces (IFS) and association schemes (AS)[1][2][3]. The key idea is to identify modular invariance, which is central to rational conformal field theories[4], in a class of self-dual association schemes and to represent them in terms of IFS, which is a very generic way to treat quantum systems[5]. This AS-IFS description of anyons is algebraic, in contrast to abstract modular tensor categories, employing related mathematical objects that encode fusion rules, conjugation, crossings in links, and braids.

We can start with an IFS corresponding to graphs to set up fusion rules of anyons via the induced association scheme. The linear combination of the classes that make up the basis of the Bose-Mesner algebra of the association scheme provides the matrix W that encodes the partition function of the spin system that can be set on a graph. The W matrix induces a commuting square from which we can derive a hyperfinite subfactor and the associated Temperley-Lieb algebra to describe the braidings of anyons. After we set up the background and establish the quantum central limit theorem (QLT), we discuss the modular data associated with the fractal graphs.

Algebraic (quantum) probability spaces are non-commutative generalizations of classical probability spaces, and in this paper, we focus on graph-induced *-algebras. We associate complex vector spaces with the vertices of the graph, and the resulting algebra is endowed with a state, which is a positive linear functional. The physical picture corresponds to a quantum particle whose configuration space is the vertices of the graph, evolving under the influence of a magnetic field.

Since we are concerned with graphs, an IFS is a subconstituent algebra of adjacency matrices with a state defined on it based on a fixed vertex. Our contribution in this work is to construct interacting Fock spaces from one-dimensional self-similar graphs and to establish central limit theorems. An IFS is a generalization of bosonic and fermionic Fock spaces, used in physics to describe the states of identical microscopic particles. It is N-graded, with each number corresponding to the number of particles, forming a disjoint union of Hilbert spaces. In the context of association schemes, interacting Fock spaces arise as subconstituent algebras with Hilbert space structures instead of vector spaces, further endowed with a real-valued linear functional defined with respect to a fixed vertex of the algebra.

The resulting structure is an algebraic probability space with adjacency matrices as non-commuting operator-valued random variables. This setting enables generalizing central limit theorems (CLTs) from classical probability spaces to non-commutative settings, with stochastic independence appropriately extended to the non-commutative context. We can start with an association scheme with a fixed number of classes and consider the operators of the corresponding T-algebra as quantum random variables. By increasing the diameter of this graph, we can consider a sequence of random variables, that leads to the question of their limits along the lines of CLTs. We denote them as QCLTs that have applications in quantum query complexity theory.

In this work, we considered 1D fractals whose adjacency matrices are irreducible tridiagonal, but the analysis can be extended to other fractals, including 2D graphs. Quantum central limit theorems, which provide the limiting spectral distribution of the adjacency matrix of a growing graph, are relevant to routing in quantum networks and social networks represented as growing graphs. Our results are significant in the context of fractal-graph-based information processing. Moreover, fractal graphs may be better suited for confining exotic phases of matter such as fractons[6], and our results have implications for topological quantum computing. We refer the reader to our previous three publications[1][2][3] for a detailed background on the materials used in this article. Our contribution, in the context of fractals, relates IFS which is geometric, with quantum states described by orthogonal projections, and AS that is topological with states encoded in the fusion spaces.

Let us now define the notions of an algebraic probability space in the context of subconstituent algebras induced by distance-regular graphs with a distinguished vertex.

Definition 1. Let o be the fixed vertex of the subconstituent algebra T, of complex valued functions defined on the vertices, endowed with an inner product $\langle . , . \rangle$ and a pure state as a linear functional satisfying

$$\rho_o(a) = \langle \delta_o, a\delta_o \rangle, a \in T.$$

The state and a notion of stochastic independence facilitate asymptotics of adjacency matrices of growing graphs via central limit theorems.

2. Association Schemes and T-algebras

An association scheme[2] is a collection of adjacency matrices of graphs with a set of $|\mathfrak{X}| = d$ vertices, which encodes 1-distance, 2-distance, ..., d-distance adjacency relations of the graph. Let X be a (finite) vertex set, and $\mathfrak{X} = \{A_j\}_{j=0}^{d}$ be a collection of $X \times X$ $\{0, 1\}$ matrices. The collection $\mathfrak{X}$ is an *association scheme* if the following conditions hold:

1. $A_0 = I$, the identity matrix;
2. $\sum_{j=0}^{d} A_j = J$, the all-ones matrix (In other words, the 1's in the A_j's partition $X \times X$);
3. For each j, $A_j^T \in \mathfrak{X}$; and

4. For each i, j, $A_i A_j \in \text{span}\mathfrak{X}$.

An association scheme is said to be *commutative* if it also satisfies

5. For each i, j, $A_i A_j = A_j A_i$.

Let $V = \mathbb{R}^X$ denote the real vector space of column vectors with coordinates indexed by X, with all entries in $\mathbb{R}$.

Let us now consider a commutative association scheme $\{A_j\}_{j=0}^d$. By the spectral theorem, there exists an alternative basis $E_0, \ldots, E_d$ consisting of projection matrices onto the maximal common eigenspaces of $A_0, \ldots, A_d$. Since the algebra $\mathcal{A}$ generated by the A_j is closed under the Hadamard (entrywise) product, forming what is known as a Bose-Mesner algebra, there exist scalars $q_{i,j}^k$ such that:

$$E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{d} q_{i,j}^k E_k \qquad (0 \leq i, j \leq d),$$

where the operator $\circ$ denotes the Hadamard product. The coefficients $q_{i,j}^k$ are known *Krein parameters* of the association scheme and induce the structure of a commutative hypergroup. Let $m_j = \text{rank} E_j$, and define $e_j = m_j^{-1} E_j$. Then the normalized Hadamard product takes the form:

$$e_i \circ e_j = \frac{1}{|X|} \sum_{k=0}^{d} \left(\frac{m_k}{m_i m_j} q_{i,j}^k \right) e_k.$$

The dual notion to Krein parameters is that of the intersection numbers p_{ij}^k, which arise from the matrix product $A_i \bullet A_j = \sum_k p_{ij}^k A_k$. For a distance-regular graph (ex: complete graphs, cycles, and odd graphs), the intersection number p_{ij}^K represents the number of paths between a pair of k-distant vertices via i-distant plus j-distant paths. This number is independent of the specific vertex pair chosen. In self-dual association schemes, the Krein parameters and intersection numbers coincide.

Definition 2. Terwilliger algebras (T-algebras)[7] A Terwilliger algebra is an algebra related to an association scheme. Given the Bose-Mesner algebra structure defined by the idempotents $\{E_i\}$, we can define a refined algebra by fixing a vertex $x \in X$ and considering the corresponding idempotents $\{E_i(x)\}$. The resulting algebra is called the Terwilliger algebra $T(x)$ with respect to the vertex x. A T-module for this algebra is a subspace $W \subseteq V$ such that $BW \subseteq W, \forall B \in T$. Every T-module admits a decomposition as an orthogonal direct sum of irreducible modules. When all irreducible T-modules have dimension one for every $x \in X$, the scheme is said to be thin. This thinness condition plays a key role in characterizing self-dual association schemes.

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There is a substantial body of literature on T-algebras, and in this work, we establish a connection between T-algebras and interacting Fock spaces following their foundational definitions[3]. This correspondence allows us to transfer techniques between these two independently developed frameworks.

Example 1. Let G be a finite abelian group that acts transitively on a finite set X. Then G also acts on $X \times X$ via $g \cdot (x, y) = (g \cdot x, g \cdot y)$ for $g \in G$ and $x, y \in X$. Let $R_0, \ldots, R_d \subseteq X \times X$ be the orbits of this action, labeled such that $R_0 = \{(x, x) : x \in X\}$. (This is indeed an orbit, since G acts transitively on X.) For each $j = 0, \ldots, d$, define the matrix $A_j \in \{0, 1\}^{X \times X}$ by

$$(A_j)_{x,y} = \begin{cases} 1, & \text{if } (x, y) \in R_j \\ 0, & \text{otherwise.} \end{cases}$$

Then, the set $\mathfrak{X} = \{A_j\}_{j=0}^{d}$ forms an association scheme called a translation scheme. It is commutative if and only if the action of G on X is *multiplicity free*- that is, the permutation representation of G associated with its action on X decomposes as a direct sum of irreducibles, with no irreducible repeated more than once (up to unitary equivalence).

3. Interacting Fock spaces

Quantum probability-based interacting Fock spaces (IFS) generalize symmetric and antisymmetric Fock spaces, which have wide-ranging applications from quantum optics in physics to graph theory. These noncommutative spaces extend classical probability spaces, which only admit a single notion of stochastic independence, by allowing multiple formulations of independence each leading to different central limit theorems. In quantum probability theory, notions of independence are essential for defining graph products, and based on the associated monadic operations, different forms of stochastic independence arise. In a quantum probability space $(\mathcal{A}, \phi)$, the usual commutative independence—for example, $\phi(bab) = \phi(a)\phi(b^2); a, b \in \mathcal{A}$ (as often assumed in quantum optics)—leads to conjugate Brownian motion (measured as quadratures) in the limit. The monotone independence, defined by $\phi(bab) = \phi(a)\phi(b)^2; a, b \in \mathcal{A}$, relevant in quantum walks leads asymptotically to arcsine-Brownian motion (i.e. a double-horn distribution). The other two common types of independence, free and boolean, are not the focus of this work. In the context of graphs, these notions of independence correspond to different graph product operations, each inducing a different type of limiting behavior.

Definition 3. [8] An IFS associated with the Jacobi sequences $\{\omega_n\}$, where $\omega_m = 0 \Rightarrow \omega_n = 0, \forall n \geq m$, and real parameters $\{\alpha_n\} \subseteq \mathbb{R}$, is a tuple

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$$(\Gamma \subseteq \mathcal{H}, B^{+}, B^{-}, B^{\circ}),$$

where $\{\Phi_n\}$ is a sequence of orthogonal polynomials and Γ is the subspace of a Hilbert space $\mathcal{H}$ formed as the disjoint union of degree-n polynomial subspaces. The operators B^{+}, B^{-} and B° act on Γ and satisfy the following relations:

$$\begin{aligned} B^{+}\Phi_n &= \sqrt{\omega_{n+1}}\Phi_{n+1}. \\ B^{-}\Phi_n &= \sqrt{\omega_n}\Phi_{n-1}; \qquad B^{-}\Phi_0 = 0. \\ B^{\circ}\Phi_n &= \phi_n. \end{aligned}$$

These operators recover the recurrence relation:

$$x\Phi_n(x) = \Phi_{n+1}(x) + \omega_n \Phi_{n-1}(x) + \alpha_{n+1}\Phi_n(x). \tag{1}$$

Given such an IFS, we can associate a graph whose adjacent matrix is tridiagonal, with the form $M = \begin{bmatrix} \alpha_1 & \sqrt{\omega_1} & & & & & \\ \sqrt{\omega_1} & \alpha_2 & \sqrt{\omega_2} & & & & \\ & \sqrt{\omega_2} & \alpha_3 & \sqrt{\omega_3} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \sqrt{\omega_{n-1}} & \alpha_n & \sqrt{\omega_n} & \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}$ This matrix admits a quantum decomposition $M = B^{+} + B^{-} + B^{\circ}$. The orthogonal polynomial sequence $\{\Phi_n\}$ can be interpreted as corresponding to a fixed vertex of the graph, with each Φ_n representing a stratum, that is, a level in the stratification of the graph based on distance from the chosen vertex. For example, in the case of the Spiderweb graph (Figure 1), the central vertex serves as the fixed point, and the stratification proceeds radially outward.

Let us now consider the example of orthogonal polynomials over the real numbers ($\mathbb{R}$) and demonstrate how they give rise to an IFS.

Example 2. A probability measure μ on the real line $\mathbb{R}$ is said to have a finite moment of order m if

$$\int_{-\infty}^{\infty} x^m \mu(dx) < \infty.$$

This is denoted by $M_m(\mu)$. Conversely, a sequence of real numbers $\{M_m\}$ corresponds to the moment sequence of a probability measure if either all terms are zero or, there exists some m such that $M_i > 0, 0 < i < m, M_j = 0, j > m$. This characterization arises from the classical determinate moment problem. Let P and Q be complex-valued polynomial functions in one real variable. We define the linear functional and the inner product with respect to the measure μ as

$$\mu(P) = \int_{\mathbb{R}} P(x)\mu(dx). \tag{2}$$

$$\langle P, Q \rangle = \mu(P^{*}Q). \tag{3}$$

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This μ inner product helps define an adjoint operation and along with a commutator (as shown in an example below) results in a *-Lie algebra of polynomials. An orthogonal basis, with respect to this measure, can be constructed and is denoted by $\{\Phi_n\}$. These orthogonal polynomials satisfy the following three-diagonal relation of the form:

$$x\Phi_n(x) = \Phi_{n+1}(x) + \alpha_n \Phi_n(x) + \omega_n \Phi_{n-1}(x), \tag{4}$$

which characterizes them as forming an IFS[5].

Example 3. The bosonic (symmetric) Fock space corresponds to the choice $\omega_n = n, \alpha_n = 0$. The fermionic (antisymmetric) Fock space has Jacobi parameters $\omega_1 = 1; \omega_n = 0$ for $n > 1; \alpha_n = 0$.

Example 4. The q-deformed 1-mode IFS is defined as follows. For $q \geq -1$, the Jacobi parameters satisfy:

$$\omega_n = \begin{cases} \sum_{k=0}^{n-1} q^k, & \text{if q > -1,} \\ 1, & \text{if q=-1 and n} \leq 1, \\ 0, & \text{if q=-1 and n} \geq 2. \end{cases} \tag{5}$$

This structure corresponds to the commutation relation:

$$aa^+ - qa^+a = 1,$$

which generalizes the canonical bosonic and fermionic cases.

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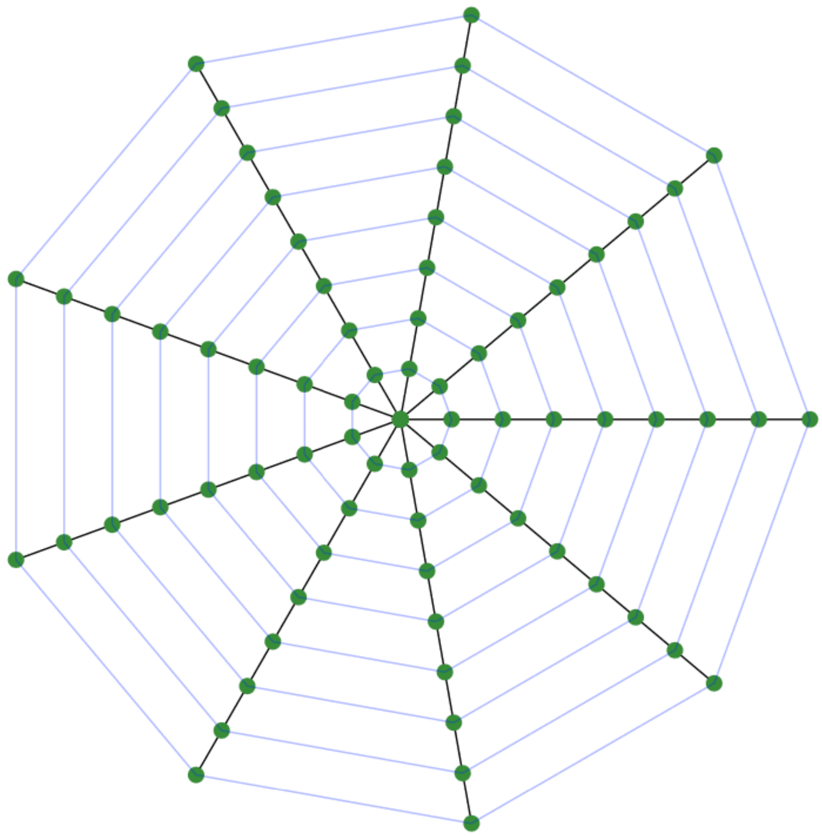

Figure 1. Spiderweb Diagram - an example of a stratified graph on which an IFS can be defined.

Let us now state and outline the proof of the quantum central limit (QCLT) for distance-regular graphs, following[8].

Theorem 1. *Let $\mathcal{G}^\nu = (V^\nu, E^\nu)$ be a growing distance-regular graph with an adjacency matrix A_ν. Let us denote the degree as $\kappa(\nu)$ and assume the following conditions in terms of intersection numbers hold:*

$$\omega = \lim_{\nu\to\infty} \overline{\omega_\nu} = \lim_{\nu\to\infty} \frac{p^n_{1,n-1}(\nu)p^n_{1,n}(\nu)}{\kappa(\nu)},$$
$$\alpha = \lim_{\nu\to\infty} \overline{\alpha_\nu} = \lim_{\nu\to\infty} \frac{p^{n-1}_{1,n-1}(\nu)}{\sqrt{\kappa(\nu)}}.$$

Let $\Gamma_{\{\omega_n\}} = (\mathcal{G}, \{\Phi_n\}, B^+, B^-)$ be an interacting Fock space associated with $\{\omega_n\}$ and $B^o = \alpha_{N+1}$ be the diagonal operator defined by $\{\alpha_n\}$, N be the number operator. Then we have

$$\lim_{\nu\to\infty} \frac{A^\epsilon_\nu}{\sqrt{\kappa(\nu)}} = B^\epsilon, \epsilon = \{o, +, -\}. \tag{6}$$

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in the sense of stochastic convergence with respect to the pure state, i.e,

$$\lim_{\nu\to\infty}\langle\Phi_0^\nu, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}}\cdots\frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}}\Phi_0^\nu\rangle = \langle\Psi_0, B^{\epsilon_m}\dots B^{\epsilon_1}\Psi_0\rangle, \epsilon\in\{+,-,o\}, m=1,2,\dots. \tag{7}$$

Proof. We have the following relations[8]:

$$\begin{aligned}
\frac{A_\nu^+}{\sqrt{\kappa(\nu)}}\Phi_n &= \sqrt{\overline{\omega_{n+1}}(\nu)}\Phi_{n+1}, \quad n=0,1,2,\dots \\
\frac{A_\nu^-}{\sqrt{\kappa(\nu)}}\Phi_0 &= 0; \quad \frac{A_\nu^-}{\sqrt{\kappa(\nu)}}\Phi_n = \sqrt{\overline{\omega_n}(\nu)}\Phi_{n-1}, \quad n=1,2,\dots \\
\frac{A_\nu^o}{\sqrt{\kappa(\nu)}}\Phi_n &= \overline{\alpha_n}(\nu)\Phi_n.
\end{aligned}$$

From the above it follows that $\frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}}\cdots\frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}}\Phi_0^\nu$ is a constant multiple of $\Phi^\nu_{\epsilon_1+\dots\epsilon_m}$ and the constant is a finite product of $\omega_n(\nu)$ and $\alpha_n(\nu)$. Therefore, the left side of the limit exists. Moreover, since the actions of A_ν^ϵ and B_ν^ϵ on the number vectors are given by the Jacobi coefficients $\{\overline{\omega_n}\}$, $\{\overline{\alpha_n}\}$ and $\{\omega_n\}$, $\{\alpha_n\}$, respectively, one may easily verify that the limit coincides with $\langle\Psi_0, B_m^\epsilon\dots B_1^\epsilon\Psi_0\rangle$. □

Example 5. [8] Let us consider a cyclic graph C_{2N+1} with $2N+1$ vertices. Then, the intersection numbers required to obtain the limits of the theorem are:

$$P^n_{1,n-1}(N) = \begin{cases} 1, & n=1,2,\dots N, \\ 0, & \text{otherwise.}\end{cases}$$

$$P^{n-1}_{1,n}(N) = \begin{cases} 2, & n=1, \\ 1, & n=2,\dots,N, \\ 0, & \text{otherwise.}\end{cases}$$

$$P^{n-1}_{1,n-1}(N) = \begin{cases} 1, & n=N+1, \\ 0, & \text{otherwise.}\end{cases}$$

It is easy to see that $\kappa = p^0_{11} = 0$ and so

$$\omega_n(N) = \begin{cases} 1, & n=1, \\ 1/2, & n=2,\dots N, \\ 0, & \text{otherwise.}\end{cases}$$

$$\alpha_n(N) = \begin{cases} 1/\sqrt{2}, & n=N+1, \\ 0, & \text{otherwise.}\end{cases}$$

This growing cyclic graph satisfies the conditions of the QCLT theorem. We will use similar techniques to constrain the Jacobians of the building blocks to get the asymptotics of self-similar graphs.

Recently Koheestani et al[9] have established the QCLT for large family of distance-regular graphs with classical parameters in the Gibbs state. We further extend the result to self-similar weighted graphs (Figure 2) in the pure state by constructing graphs that satisfy the conditions of the above theorem.

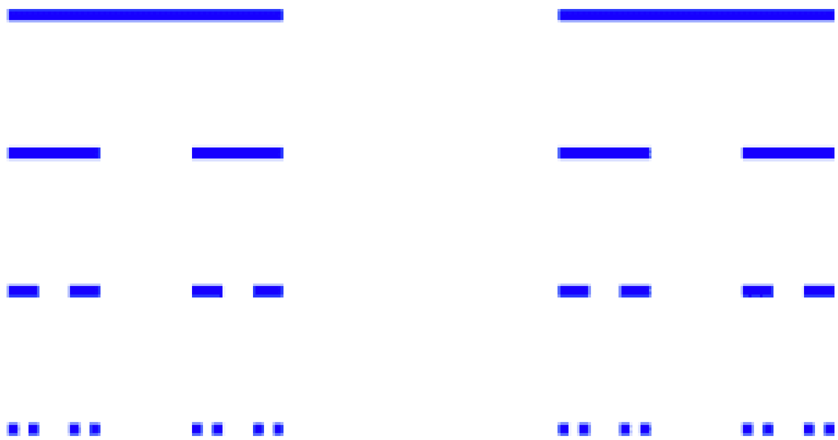

Figure 2. Cantor Set that is a self-similar graph

3.1. Self-similar p Laplacians on the half-integer lattice

In an earlier work, we considered a family of self-similar Laplacians on the half-line of integers and computed their spectra that are relevant to integer quantum Hall effects in physics[10]. This class of Laplacians investigated in[11] for the first time arises naturally when studying the unit interval endowed with a particular fractal measure. Here, we focus on Laplacian without potentials and consider the underlying graphs and their spectra.

In this context, we define the self-similar structure on the half-integer lattice with the origin serving the fixed vertex of our T-algebra. This self-similar structure describes a random walk on the half-line and gives rise to a class of self-similar probabilistic graph Laplacians Δ_p

Let $\mathbb{Z}_+$ be the set of nonnegative integers, and $\ell(\mathbb{Z}_+)$ be the linear space of complex-valued sequences $(f(x))_{x\in\mathbb{Z}_+}$. Let $p \in (0,1)$, for each $x \in \mathbb{Z}_+ \setminus \{0\}$, we define $m(x)$ to be the largest natural number m such that 3^m divides x. For $f \in \ell(\mathbb{Z}_+)$ we define a *self-similar Laplacian* Δ_p by,

$$(\Delta_p f)(x) = \begin{cases} f(0) - f(1), & \text{if } x = 0 \\ f(x) - (1-p)f(x-1) - pf(x+1), & \text{if } 3^{-m(x)}x \equiv 1 \pmod 3 \\ f(x) - pf(x-1) - (1-p)f(x+1), & \text{if } 3^{-m(x)}x \equiv 2 \pmod 3 \end{cases}. \tag{8}$$

We equip $\ell(\mathbb{Z}_+)$ with its canonical basis $\{\delta_x\}_{x\in\mathbb{Z}_+}$ where

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$$\delta_x(y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y. \end{cases} \tag{9}$$

The matrix representation of Δ_p with respect to the canonical basis has the following Jacobi matrix

$$jacobi_{+,p} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ p-1 & 1 & -p & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -p & 1 & p-1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & p-1 & 1 & -p & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & p-1 & 1 & -p & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & -p & 1 & p-1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & -p & 1 & p-1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & p-1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{10}$$

The case $p = \frac{1}{2}$ recovers the classical one-dimensional Laplacian (probabilistic graph Laplacian).

Definition 4. Let $G_0 = (V_0, E_0)$ be the graph shown in Figure We define the sequence of graphs $\{G_l\}_{l\in\mathbb{N}}$ inductively. Suppose $G_{l-1} = (V_{l-1}, E_{l-1})$ is given for some integer $l \geq 1$, where $V_{l-1} = \mathbb{Z}_+ \cap [0, 3^{l-1}]$. The graph $G_l = (V_l, E_l)$ is constructed according to the following *substitution rule*. We repeat the following steps for $i \in \{0, 1, 2\}$:

1. Insert a copy of G_{l-1} between the two vertices m_i and m_{i+1} of the protograph shown in the following sense. We identify the vertex 0 in G_{l-1} with the vertex m_i and similarly, we identify the vertex 3^{l-1} in G_{l-1} with the vertex m_{i+1}.
2. We substitute the edges $(0, 1)$ and $(3^{l-1}, 3^{l-1} - 1)$ in G_{l-1} with the corresponding directed weighted edges as indicated in the protograph.

Figure 3. Construction of self-similar graph from repeating units. (Top) A copy of the basic building block. The deleted edges correspond to the edges that are replaced when applying the substitution rule. (Bottom) The fractal graph is constructed by inserting the three copies of the building block in outer graph which is the 1D lattice While the vertices are labeled by the sequentially, the labeling of the edges represents the transition probabilities (off-diagonal entries in the self-similar Laplacian).

In a companion paper[12] we discuss a family of centro-symmetric Jacobians and used them as building blocks for constructing self-similar graphs. There is a three-term recurrence relation for the building block and another one for the main graph. Let us consider the outer graph as that plays a role in the asymptotics.

$$jacobi_{cs} := \frac{1}{m}\begin{pmatrix} b(0) & a(1) & 0 & \dots & 0 \\ a(n_0) & b(1) & a(2) & \dots & 0 \\ 0 & a(n_0-1) & b(2) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a(n_0) \\ 0 & 0 & 0 & a(1) & b(0) \end{pmatrix} \tag{11}$$

In the above we scale the centrosymmetric matrix by the inverse of the dimension of the matrix that will help with convergence later.

$$\begin{cases} P_0^D(x) = 1,\ P_1^D(x) = x - b(1) \\ P_k^D(x) = (x - b(k))P_{k-1}^D(x) - a(k)a(n_0+1-k)P_{k-2}^D(x),\ k \in \{2, \dots, n_0 - 1\}. \end{cases} \tag{12}$$

Let us now state and establish the main result.

Theorem 2. *Let $\mathcal{G}^\nu = (V^\nu, E^\nu)$ be a growing 1D fractal (bidirectional network) with an adjacency matrix A_ν. Let us denote the degree as $\kappa(\nu)$ and assume the following conditions hold:*

$$\omega = \lim_{\nu\to\infty} \frac{\bar{\omega}_\nu}{\kappa(\nu)}.$$
$$\alpha = \lim_{\nu\to\infty} \frac{\bar{\alpha}_\nu}{\kappa(\nu)}.$$

Let $\Gamma_{\{\omega_n\}} = (\mathcal{G}, \{\Phi_n\}, B^+, B^-)$ be an interacting fock space associated with $\{\omega_n\}$ and $B^o = \alpha_{N+1}$ be the diagonal operator defined by $\{\alpha_n\}$, N be the number operator. Then we have

$$\lim_{\nu\to\infty} \frac{A_\nu^\epsilon}{\sqrt{\kappa(\nu)}} = B^\epsilon, \epsilon = \{o, +, -\}. \tag{13}$$

in the sense of stochastic convergence with respect to the pure state, i.e,

$$\lim_{\nu\to\infty} \langle \Phi_0^\nu, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \dots \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \Phi_0^\nu \rangle = \langle \Psi_0, B^{\epsilon_m} \dots B^{\epsilon_1} \Psi_0 \rangle, \epsilon \in \{+, -, o\}, m = 1, 2, \dots. \tag{14}$$

Proof. We can rewrite the above equation (12) as to get the Jacobi coefficients:

$$xP_{k-1}^D(x) = b(k)P_{k-1}^D(x) + P_k^D(x) + a(k)a(n_0+1-k)P_{k-2}^D(x).$$
$$\omega_k = a(k)a(n_0+1-k).$$
$$\alpha_{k+1} = b(k).$$

The $a(i), b(j)$ terms in repeating units are bounded by 1 and there are only finitely many terms (finite moments) in equation (14) so the limit exists for the outer graph. Since, the elements $a(i)$ are probabilities the scaling constant m is not required. If we build the graph with probabilistic Laplacian then it is clear that we have the limits for the Jacobi coefficients as the numerators are probabilities and less than one and the denominator is $\kappa = 4$, and the QCLT theorem holds. The limiting measure can be obtained by the

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spectral decimation method[10] as the methods of applying continued fractions[8] are difficult in general for an arbitrary distance-regular graph.

Another class of systems can be constructed starting from any Leonard pairs[13] and taking the centro-symmteric Jacobians out of the pairs. For example the pairs where d is a any non-negative integer:

$$A = \frac{1}{m}\begin{pmatrix} 0 & d & & & & 0 \\ 1 & 0 & d-1 & & & \\ & 2 & . & . & & \\ & & . & . & . & \\ & & & . & . & 1 \\ 0 & & & & d & 0 \end{pmatrix}; \quad B = \mathrm{diag}(d, d-2, d-4, \ldots, -d); \tag{15}$$

When we normalize the above centro-symmetric matrix and build the self-similar graph then again we will have QCLT with spin Leonard pairs form the bases related by Krawtchouk polynmoials[14].

In the above examples we can replace the diagonal elements of the centro-symmetric Jacobian all zeros with an integer less than the degree $\kappa = 4$ and we will still have convergence. □

It is interesting to note that the adjacency matrix of our self-similar graphs are irreducible tridiagonal (each entry on the subdiagonal is nonzero) with nonnegative entries and thus has a bidirectional path and is described by a Q-polynomial[15].

4. Summary and Conclusions

We investigated graphs with weighted edges and endowed with self-similar fractal structures. We derived the QCLT for a family of graphs in pure state by constraining the centro-symmetric Jacobian that generate the fractals. The class of fractal graphs considered here lead to self-dual association schemes and thus encode modular invariance of RCFTs. This analysis sets the stage for exploring QCLT for fractals in coherent states that are relevant in physics and more general fractals in 2D such as the Sierpinski gasket.

Statements and Declarations

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Conflicts of interests

None.

Data availability

This manuscript has no associated data.

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