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Research Article

On a New Two-Point Taylor Expansion with Applications

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A new two-point Taylor series expansion is proposed. The expansion is slightly different than the classical definition. The coefficients are calculated as recursive relations in a general form. The two-point Taylor expansion is applied to several functions which are odd, even, neither odd nor even. Functions having finite interval of convergence or infinite interval of convergence are investigated. The conditions for convergence are derived and the results are compared with the results of single-point Taylor expansions as well as two-point Taylor expansions reported in the literature. It is found that for a finite radius of convergence, two-point Taylor expansions can have a single convergence interval as well as two separate convergence intervals. Generally speaking, two-point Taylor expansions better represent the real function when the series is truncated. The new two-point expansion and the classical two-point expansion produced identical results for all the problems treated. Based on the results of this analysis, the asymmetric two-point Taylor expansion presented here does not have an advantage compared to the classical symmetric expansion. An application of the series to solution of a variable coefficient differential equation is also treated.

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1. Introduction

Taylor and MacLaurin series are one of the fundamental topics in mathematics. A continuous and infinitely differentiable function may be expressed in terms of a polynomial series, the coefficient of which is determined by the derivatives at a given point. The series expansion may be convergent over the whole domain or may have a limited convergence interval which is determined by the radius of convergence. In order to better approximate the functions, instead of the single-point Taylor expansions, two-point Taylor expansions were also proposed in the literature. Rotational symmetric lens profiles were described by the two-point Taylor polynomials. It is shown that the two-point Taylor expansions better approximates the mapping functions [1]. Two-point Taylor expansions were employed in the area of finance to determine density and option price expansions [2]. Such expansions were considered in the complex domain also. Singular one dimensional boundary value problems were treated [3]. Another complex domain treatment of two-point Taylor expansions is presented in [4]. The two-point series solutions were applied to nonlinear partial differential equations [5]. Finally, elliptic boundary value problems were also analyzed [6]. For Taylor expansions and their link to perturbation solutions, see [7]. A nonlinear curve equation with constant acceleration components were examined by numerical, single-point Taylor expansion and perturbation methods [8]. An interesting recent paper discusses the properties of blends which can be used to approximate functions with two-point Taylor series [9].

In this work, a slightly different new version of the two-point Taylor expansions is proposed for the first time. The new version is compared with the single point Taylor expansion as well as the classical two-point Taylor expansion. Several functions which are odd, even or neither odd nor even are considered. Functions having finite radius of convergence as well as infinite radius of convergence are treated. The new asymmetric expansion and the classical symmetric two-point expansion produced identical results for all the problems considered. For the two-point expansions, the convergence interval is widened compared to a Taylor series with respect to the single-point expansion of the lower reference point whereas it is narrower with respect to the higher reference point of the single-point expansion. If the two reference points are sufficiently far away from each other, then the convergence intervals double with appearance of a divergent intermediate interval. For problems with infinite radius of convergence, the best results are obtained in the vicinity of the reference points and the error for truncated series may increase in between the reference points if the reference points are widely spaced. The criterion for convergence intervals is derived as well as for the doubling of convergence intervals for a specific problem. It is shown that two-point Taylor expansions can represent the solution between both sides of a vertical asymptote whereas the solutions of single-point expansions cannot cross the vertical asymptote. Finally, the new series solution is applied to a variable coefficient ordinary differential equation also.

2. Two-Point Taylor Series Expansions

The new proposed two-point Taylor series expression is given first.

Theorem 1. Given an analytical function $f(x)$ and the convergent polynomial approximation defined with respect to two reference points $x = x_0$ and $x = x_1$

$$f(x) = \sum_{m=0}^{\infty} a_{2m}(x-x_0)^m(x-x_1)^m + a_{2m+1}(x-x_0)^{m+1}(x-x_1)^m, \qquad (1)$$

the coefficients a_{2m} and a_{2m+1} are uniquely determined by the equations

$$f^{(k)}(x_0) = \frac{d^k}{dx^k}\left\{\sum_{m=0}^{\infty} a_{2m}(x-x_0)^m(x-x_1)^m + a_{2m+1}(x-x_0)^{m+1}(x-x_1)^m\right\}_{x=x_0} \qquad (2)$$

$$f^{(k)}(x_1) = \frac{d^k}{dx^k}\left\{\sum_{m=0}^{\infty} a_{2m}(x-x_0)^m(x-x_1)^m + a_{2m+1}(x-x_0)^{m+1}(x-x_1)^m\right\}_{x=x_1} \qquad (3)$$

$k = 0, 1, 2, \ldots$, where $f^{(k)} = \frac{d^k f}{dx^k}$ with the coefficients being calculated from the recursive relations

$$a_{2m} = \left[f^{(m)}(x_0) - \sum_{i=1}^{\left\lfloor\frac{m}{2}\right\rfloor}\binom{m}{i}(m-i)!\left(\prod_{k=m-2i+1}^{m-i}k\right)(x_0-x_1)^{m-2i}a_{2m-2i} - \sum_{i=1}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\binom{m}{i}(m-i)!\left(\prod_{k=m-2i}^{m-i-1}k\right)(x_0-x_1)^{m-2i-1}a_{2m-2i} - m!(x_0-x_1)^{m-1}a_{2m-1}\right] / m!(x_0-x_1)^m \qquad (4)$$

$$a_{2m+1} = \left[f^{(m)}(x_1) - \sum_{i=1}^{\left\lfloor\frac{m}{2}\right\rfloor}\binom{m}{i}(m-i)!\left(\prod_{k=m-2i+1}^{m-i}k\right)(x_1-x_0)^{m-2i}a_{2m-2i}\right.$$

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$$- \sum_{i=1}^{\left\lfloor \frac{m+1}{2} \right\rfloor} \binom{m}{m-i} (m-i)! \left(\prod_{k=m-2i+2}^{m-i+1} k \right) (x_1 - x_0)^{m-2i+1} a_{2m-2i+1} - m!(x_1 - x_0)^m a_{2m} \Bigg]$$

$/m!(x_1 - x_0)^{m+1}$. (5)

Proof

A straightforward calculation of the *m*'th derivatives at points $x = x_0$ and $x = x_1$ yield

$$
\begin{aligned}
f^{(m)}(x_0) = a_{2m} \binom{m}{0} m!(x_0 - x_1)^m \\
+ a_{2m-2} \binom{m}{1} (m-1)!(m-1)(x_0 - x_1)^{m-2} + \cdots \\
+ a_{2m-2i} \binom{m}{i} (m-i)!(m-i)(m-i-1) \\
\ldots (m-2i+1)(x_0 - x_1)^{m-2i} \\
+ a_{2m-1} \binom{m}{0} m!(x_0 - x_1)^{m-1} + a_{2m-3} \binom{m}{1} (m-1)!(m \\
- 2)(x_0 - x_1)^{m-3} + \\
\cdots + a_{2m-2i-1} \binom{m}{i} (m-i)!(m-i-1) \\
\ldots (m-2i)(x_0 - x_1)^{m-2i-1}
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
f^{(m)}(x_1) = {} & a_{2m}\binom{m}{m} m!(x_1-x_0)^m \\
& + a_{2m-2}\binom{m}{m-1}(m-1)!(m-1)(x_1-x_0)^{m-2} + \cdots \\
& + a_{2m-2i}\binom{m}{i}(m-i)!(m-i)(m-i-1) \\
& \ldots(m-2i+1)(x_1-x_0)^{m-2i} \\
& + a_{2m+1}\binom{m}{m} m!(x_1-x_0)^{m+1} + a_{2m-1}\binom{m}{m-1} m(m \\
& -1)!(x_1-x_0)^{m-1} + \cdots \\
& + a_{2m-2i+1}\binom{m}{m-i}(m-i)!(m-i+1)(m-i) \\
& \ldots(m-2i+2)(x_1-x_0)^{m-2i+1}.
\end{aligned} \tag{7}
$$

Solving a_{2m} from (6) and a_{2m+1} from (7), and using the summation and multiplication signs, the recursive relations (4) and (5) are obtained. The first twelve coefficients in explicit form are

$$a_0 = f(x_0) \tag{8}$$

$$a_1 = \frac{f(x_1)-a_0}{x_1-x_0} \tag{9}$$

$$a_2 = \frac{f'(x_0)-a_1}{x_0-x_1} \tag{10}$$

$$a_3 = \frac{f'(x_1)-(x_1-x_0)a_2-a_1}{(x_1-x_0)^2} \tag{11}$$

$$a_4 = \frac{f''(x_0)-2(x_0-x_1)a_3-2a_2}{2(x_0-x_1)^2} \tag{12}$$

$$a_5 = \frac{f''(x_1)-2(x_1-x_0)^2a_4-4(x_1-x_0)a_3-2a_2}{2(x_1-x_0)^3} \tag{13}$$

$$a_6 = \frac{f'''(x_0)-6(x_0-x_1)^2a_5-12(x_0-x_1)a_4-6a_3}{6(x_0-x_1)^3} \tag{14}$$

$$a_7 = \frac{f'''(x_1)-6(x_1-x_0)^3a_6-18(x_1-x_0)^2a_5-12(x_1-x_0)a_4-6a_3}{6(x_1-x_0)^4} \tag{15}$$

$$a_8 = \frac{f^{(4)}(x_0)-24(x_0-x_1)^3a_7-72(x_0-x_1)^2a_6-48(x_0-x_1)a_5-24a_4}{24(x_0-x_1)^4} \tag{16}$$

$$a_9 = \frac{f^{(4)}(x_1)-24(x_1-x_0)^4a_8-96(x_1-x_0)^3a_7-72(x_1-x_0)^2a_6-72(x_1-x_0)a_5-24a_4}{24(x_1-x_0)^5} \tag{17}$$

$$a_{10} = \frac{f^{(5)}(x_0)-120(x_0-x_1)^4a_9-480(x_0-x_1)^3a_8-360(x_0-x_1)^2a_7-360(x_0-x_1)a_6-120a_5}{120(x_0-x_1)^5} \tag{18}$$

$$a_{11} = \frac{f^{(5)}(x_1)-120(x_1-x_0)^5a_{10}-600(x_1-x_0)^4a_9-480(x_1-x_0)^3a_8-720(x_1-x_0)^2a_7-360(x_1-x_0)a_6-120a_5}{120(x_1-x_0)^6} \tag{19}$$

To prove the uniqueness of the coefficients, assume that there are some other b_{2m} and b_{2m+1} coefficients to express the same function

$$
\begin{aligned}
f(x) = {} & a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) \\
& + a_3(x-x_0)^2(x-x_1) + \ldots
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
f(x) = {} & b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) \\
& + b_3(x-x_0)^2(x-x_1) + \ldots
\end{aligned} \tag{21}
$$

Subtracting (20) from (21)

$$
\begin{aligned}
& a_0 - b_0 + (a_1-b_1)(x-x_0) + (a_2-b_2)(x-x_0)(x-x_1) \\
& + (a_3-b_3)(x-x_0)^2(x-x_1) \\
& + \ldots = 0.
\end{aligned} \tag{22}
$$

But the Wronskian of $(x-x_0)^m(x-x_1)^m$ and $(x-x_0)^n(x-x_1)^n$ is

$$
\begin{aligned}
W(x) = {} & \begin{vmatrix} (x-x_0)^m(x-x_1)^m & (x-x_0)^n(x-x_1)^n \\ \frac{d}{dx}[(x-x_0)^m(x-x_1)^m] & \frac{d}{dx}[(x-x_0)^n(x-x_1)' \end{vmatrix} \\
& = (n-m)(2x-a-b)(x-x_0)^{m+n-1}(x-x_1)^{m+n-1} \\
& \neq 0 \text{ for } m \neq n,
\end{aligned} \tag{23}
$$

which dictates that the different powers are linearly independent functions of each other. Therefore, in order (22) to identically equal to zero for each x, the coefficients should vanish leading to $a_i = b_i,\ i = 0,1,2\ldots$ which proves the uniqueness of the coefficients

The classical symmetric version (with respect to reference points) of the two-point Taylor expansion used in the literature

$$f(x) = \sum_{m=0}^{\infty}[b_m(x-x_0) + c_m(x-x_1)] [(x-x_0)(x-x_1)]^m, \tag{24}$$

is slightly different from the asymmetric expansion (1) leading to different coefficients

$$b_0 = \frac{f(x_1)}{x_1-x_0} \tag{25}$$

$$c_0 = \frac{f(x_0)}{x_0-x_1} \tag{26}$$

$$b_1 = \frac{f'(x_1)-b_0-c_0}{(x_1-x_0)^2} \tag{27}$$

$$c_1 = \frac{f'(x_0)-b_0-c_0}{(x_0-x_1)^2} \tag{28}$$

$$b_2 = \frac{f''(x_1)-2(2b_1+c_1)(x_1-x_0)}{2(x_1-x_0)^3} \tag{29}$$

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$$c_2 = \frac{f''(x_0) - 2(b_1 + 2c_1)(x_0 - x_1)}{2(x_0 - x_1)^3} \quad (30)$$

$$b_3 = \frac{f'''(x_1) - 6(3b_2 + c_2)(x_1 - x_0)^2 - 6(b_1 + c_1)}{6(x_1 - x_0)^4} \quad (31)$$

$$c_3 = \frac{f'''(x_0) - 6(b_2 + 3c_2)(x_0 - x_1)^2 - 6(b_1 + c_1)}{6(x_0 - x_1)^4} \quad (32)$$

$$b_4 = \frac{f^{(4)}(x_1) - 24(4b_3 + c_3)(x_1 - x_0)^3 - 24(3b_2 + 2c_2)(x_1 - x_0)}{24(x_1 - x_0)^5} \quad (33)$$

$$c_4 = \frac{f^{(4)}(x_0) - 24(b_3 + 4c_3)(x_0 - x_1)^3 - 24(2b_2 + 3c_2)(x_0 - x_1)}{24(x_0 - x_1)^5} \quad (34)$$

$$b_5 = \frac{f^{(5)}(x_1) - 120(5b_4 + c_4)(x_1 - x_0)^4 - 360(2b_3 + c_3)(x_1 - x_0)^2 - 120(b_2 + c_2)}{120(x_1 - x_0)^6} \quad (35)$$

$$c_5 = \frac{f^{(5)}(x_0) - 120(b_4 + 5c_4)(x_0 - x_1)^4 - 360(b_3 + 2c_3)(x_0 - x_1)^2 - 120(b_2 + c_2)}{120(x_0 - x_1)^6}. \quad (36)$$

It can be proven that this representation is a unique way of expressing an analytical convergent function, that is there exist only unique polynomial coefficients b_m and c_m for a given function.

It is well known that the single-point Taylor expression in the vicinity of $x = x_0$ is

$$f(x) = \sum_{m=0}^{\infty} d_m (x - x_0)^m \quad (37)$$

where

$$d_m = \frac{f^{(m)}(x_0)}{m!} \quad (38)$$

Numerical comparisons of the two-point Taylor expansions and the single-point expansion will be given in the next section.

3. Functional Approximations

In this section, two functions will be approximately expressed in terms of two-point Taylor series. The two-point Taylor series and the single-point Taylor series will be compared with the exact solution. All Figures are generated by the Matlab software.

3.1. The function y=1/(1+x)

The two point Taylor expression of

$$f(x) = \sum_{m=0}^{\infty} a_{2m} (x - x_0)^m (x - x_1)^m + a_{2m+1} (x - x_0)^{m+1} (x - x_1)^m, \quad (39)$$

is derived for arbitrary two points x_0 and x_1 $(x_0 < x_1)$. A straightforward calculation of (4) and (5) yields

$$a_{2m} = \frac{1}{(1 + x_0)^{m+1} (1 + x_1)^m}, \; a_{2m+1} = -\frac{1}{(1 + x_0)^{m+1} (1 + x_1)^{m+1}}. \quad (40)$$

For the classical two-point Taylor expression

$$f(x)=\sum_{m=0}^{\infty}[b_m\,(x-x_0)+c_m\,(x-x_1)][(x-x_0)(x-x_1)]^m, \tag{41}$$

the coefficients are

$$b_m=\frac{1}{(1+x_0)^m(1+x_1)^{m+1}(x_1-x_0)},\ c_m= -\frac{1}{(1+x_0)^{m+1}(1+x_1)^m(x_1-x_0)}. \tag{42}$$

Finally the single point Taylor expression about x_2 is

$$f(x)=\sum_{m=0}^{\infty}d_m(x-x_2)^m, \tag{43}$$

where

$$d_m=\frac{(-1)^m}{(1+x_2)^{m+1}}. \tag{44}$$

Using the ratio test for convergence of series, i.e. $\lim_{n\to\infty}\left|\frac{p_{n+1}}{p_n}\right|<1$, p_n being the *n*'th term of the polynomial expression, the convergence criterion for both of the two-point Taylor expansions turns out to be

$$|(x-x_0)(x-x_1)|<|1+x_0|\,|1+x_1|, \tag{45}$$

and for the single point expansion

$$|(x-x_2)|<|1+x_2|. \tag{46}$$

All expansions cease to be valid at the singular point of the function $x=-1$, no matter what the values of x_{0-2} is.

Condition (45) may lead to a single convergence interval as well as double convergence intervals. To the best of the author's knowledge, double convergence intervals were not discussed previously in the literature. Using the properties of the quadratic functions, the criterion for a single convergence interval turns out to be

$$x_1<3x_0+2+2\sqrt{2}(x_0+1),\ (x_1>x_0). \tag{47}$$

If the criterion is not satisfied, then the convergence interval splits into two. The above criterion is special to this function. For other functions with finite radius of convergences, the criterion for single convergence region should be derived based on the specific form of the relevant coefficients. The convergence intervals are given in Table 1 for a number of specific numerical values of the reference points.

x_0	x_1	**Convergence Interval**	**Criterion (47)**
0	2	(-1,3)	Satisfied
0	6	(-1,1.58) and (4.41,7)	Not Satisfied
1	4	(-1,6)	Satisfied
1	9	(-1,11)	Satisfied
1	19	(-1,3.59) and (16.40,21)	Not satisfied
3	4	(-1,8)	Satisfied
18	19	(-1,38)	Satisfied
-2	2	(- $\sqrt{7}$,-1) and (1, $\sqrt{7}$)	Not satisfied

Table 1. Convergence intervals for some reference points

In Figure 1, the two-point Taylor expansions are contrasted with the exact solution for the case of $x_0 = 0$ and $x_1 = 6$. Table 1 predicts two convergence regions for this case. As the number of terms increase, the intervals which predict the exact function closely widen converging to the intervals predicted by Table 1. Outside the convergence regions however, the predictions are worse as the number of terms increase.

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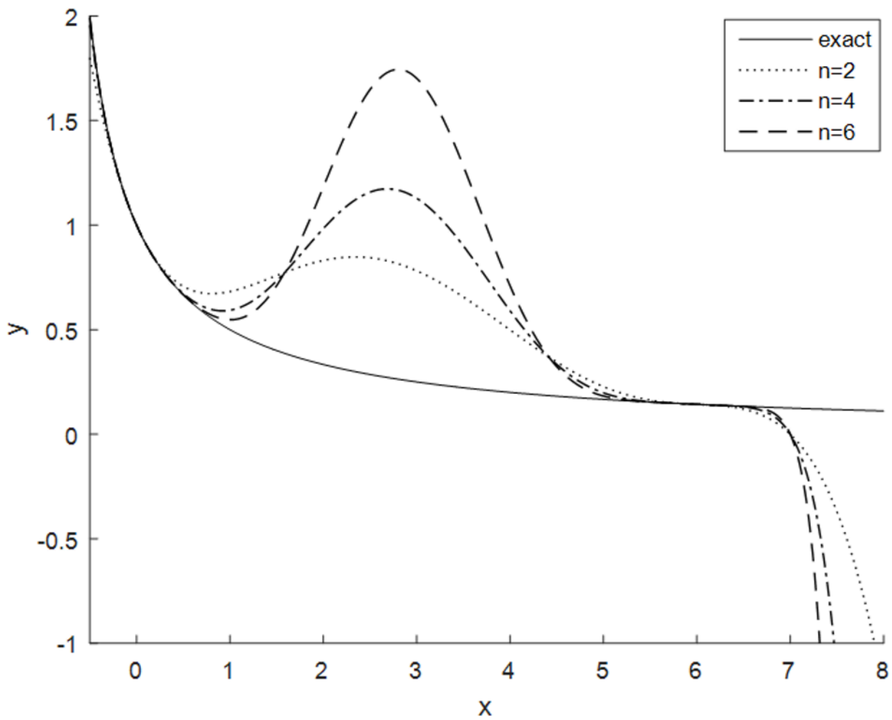

Figure 1. Two-point Taylor approximations of function y=1/(1+x) for $x_0 = 0$, $x_1 = 6$

Note that, although the formulations and coefficients are somewhat different, both two-point Taylor expansions produced identical results even for the finite truncations of the series.

In the case of single-point expansions, the convergence interval is (-1,1) for $x_2 = 0$, and

(-1,13) for $x_2 = 6$. That is, the two point Taylor expansion has a wider convergence region compared to the single point expansion in the vicinity of lower reference value and possesses a narrower convergence region compared to the single point expansion in the vicinity of higher reference value.

Another case in which there is only one convergence interval is shown in Figure 2. Four terms are taken in all expansions. It is obvious that two-point Taylor expansion has a wider convergence region compared to the single-point expansion in the vicinity of $x_2 = 1$, and a narrower convergence region compared to the single-point expansion about $x_2 = 4$ as predicted by the theory. Note that for finite number of truncations, the two-point Taylor expansion $(x_0 = 1,\ x_1 = 4)$ is better at the left than the single point expansion about $x_2 = 4$ and better at the right than the single point expansion about $x_2 = 1$.

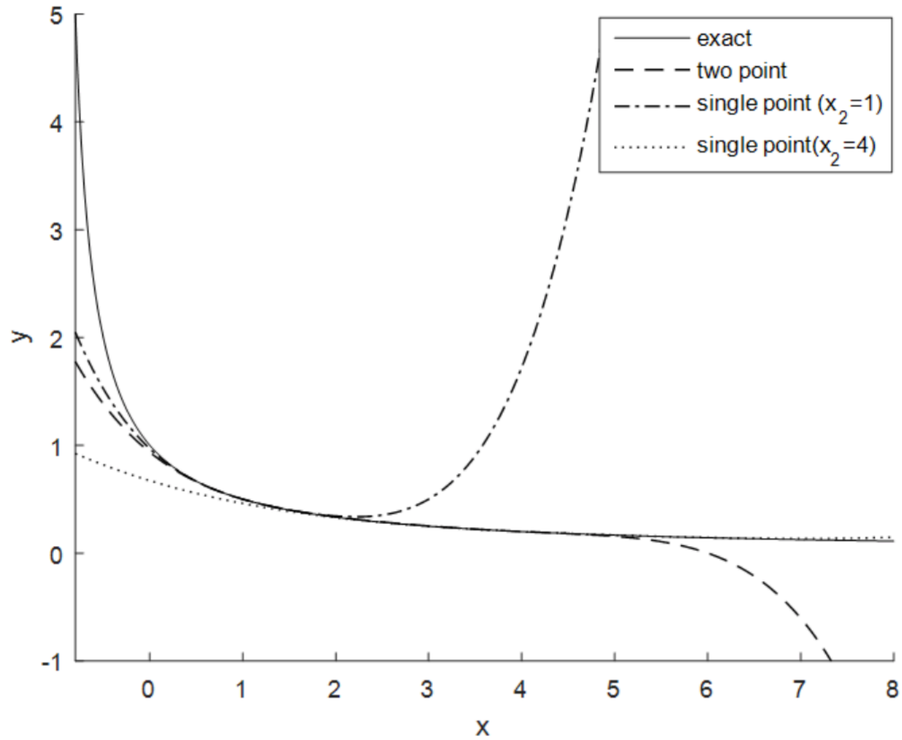

Figure 2. Comparison of two-point Taylor approximations ($x_0 = 1, x_1 = 4$) and single-point approximations ($x_2 = 1, x_2 = 4$) of function y=1/(1+x)

One of the advantages of the two-point Taylor expansions is that they can produce solutions at the right and left-hand sides of the singular point, i.e. $x_s = -1$ for this specific case. Single-point Taylor expansion solutions cannot cross the singularity points because they have only one convergence interval and the function ceases to be analytic at the singular point. Figure 3 is such an example in which the two reference points are $x_0 = -2$, $x_1 = 2$. Table 1 predicts two convergence intervals i.e., (-$\sqrt{7}$,-1) and (1, $\sqrt{7}$) which can be visualized from Figure 3 with 22 terms taken in the expansion.

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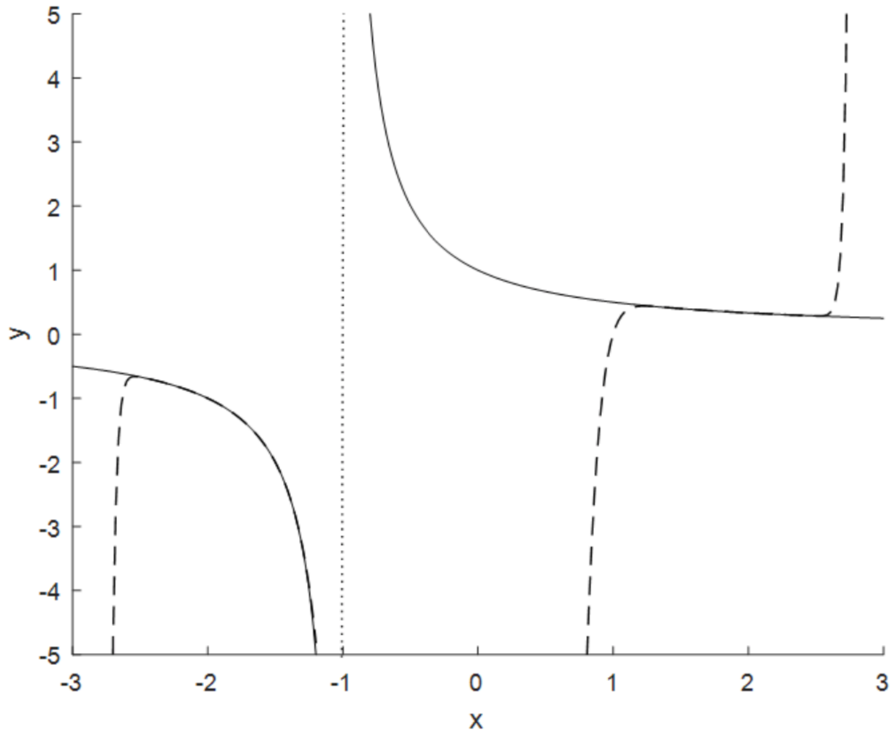

Figure 3. Comparison of the two-point Taylor approximation $(x_0 = -2, x_1 = 2)$ (dashed) and the exact function y=1/(1+x) (solid) about the singular point

3.2. The function y=exp(x)

This is a characteristic example where the convergence interval is infinity. The first twelve terms are taken in all the expansions. The performance of the two-point Taylor expansion about $x_0 = -2$ and $x_1 = 2$ is slightly better than the performance of the single expansion about $x_2 = 0$ and much better than the single expansion about $x_2 = 2$ (Figure 4). As the number of terms increases, all the solutions will converge to the real solution. However, there are differences between the performances of the extensions for finite mode truncations.

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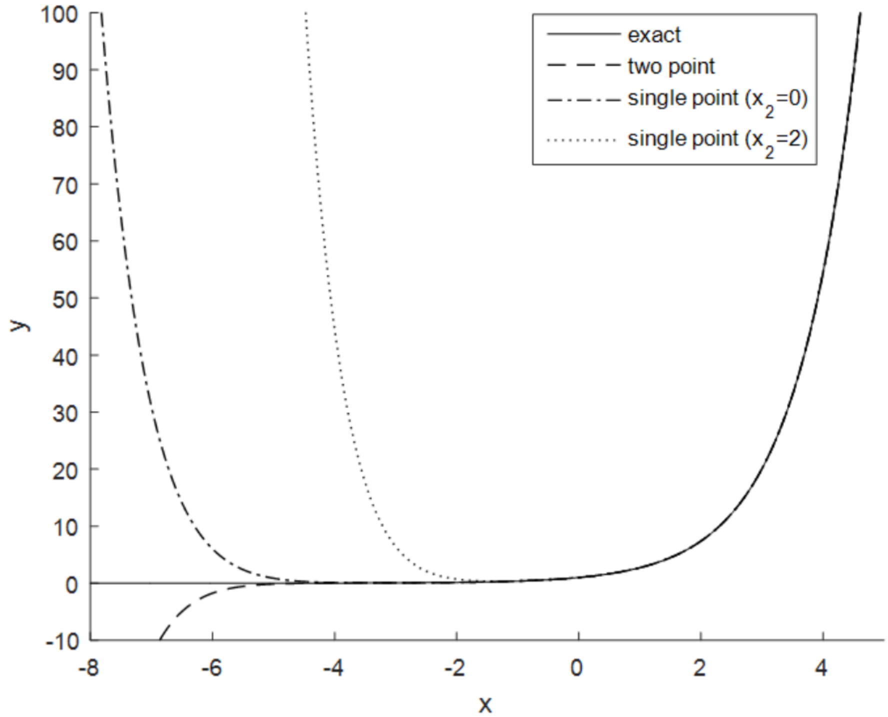

Figure 4. Comparison of the two-point Taylor approximation, single-point Taylor approximations and the exact function y=exp(x)

4. Further Examples And Comparisons

The two examples in the previous section are neither odd nor even functions. As mentioned earlier, results of the two variants of the two-point expansions are identical to each other. Two additional functions one being odd and the other even are used to further compare the performance of the different variants of the two-point Taylor expansions with the exact solutions.

4.1. The function y=sin(x)

The function sin(x) is an odd function and the aim is to test if the proposed asymmetric series has any advantage over the classical symmetric series for odd functions. The lower reference point is fixed to $x_0 = 0$ and the higher reference point is increased from 2 up to 8 in Figure 5. The first twelve terms are considered in both approximations. As the higher reference point is increased, the truncated series represents the function better in a wider region. Note that, the two variants produce exactly the same results as can be seen from the coincidence of the dashed and the dotted lines.

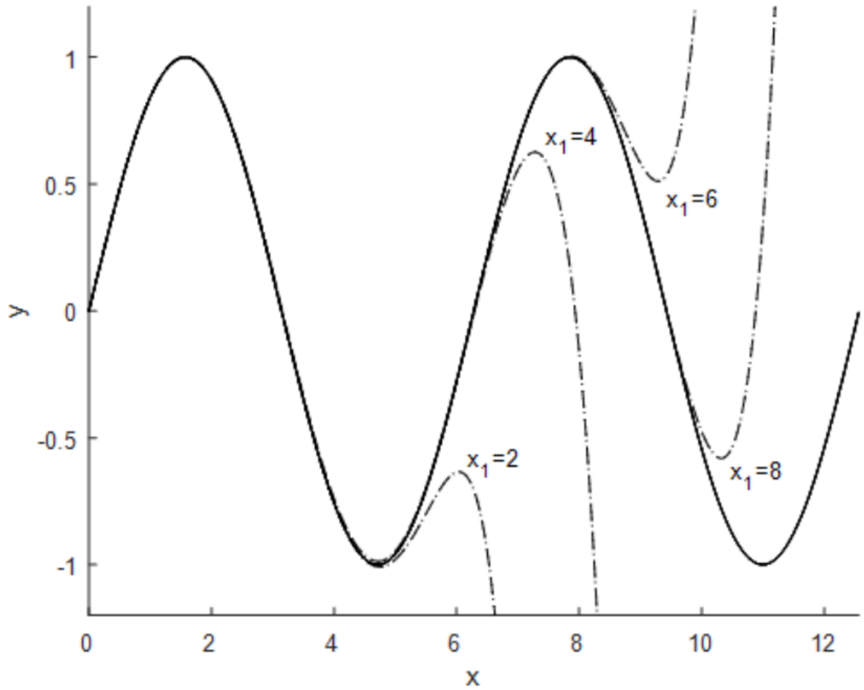

Figure 5. Comparison of the new two-point Taylor approximation (dashed), classical two-point Taylor approximation (dotted) and the exact function y=sin(x) (solid) for various right-hand side reference points ($x_0 = 0$)

The absolute error is defined as

$$err = |y_e - y_a|, \qquad (48)$$

where y_e stands for the exact solution and y_a for the approximate solution. Figure 6 is a comparison of the absolute errors of both methods.

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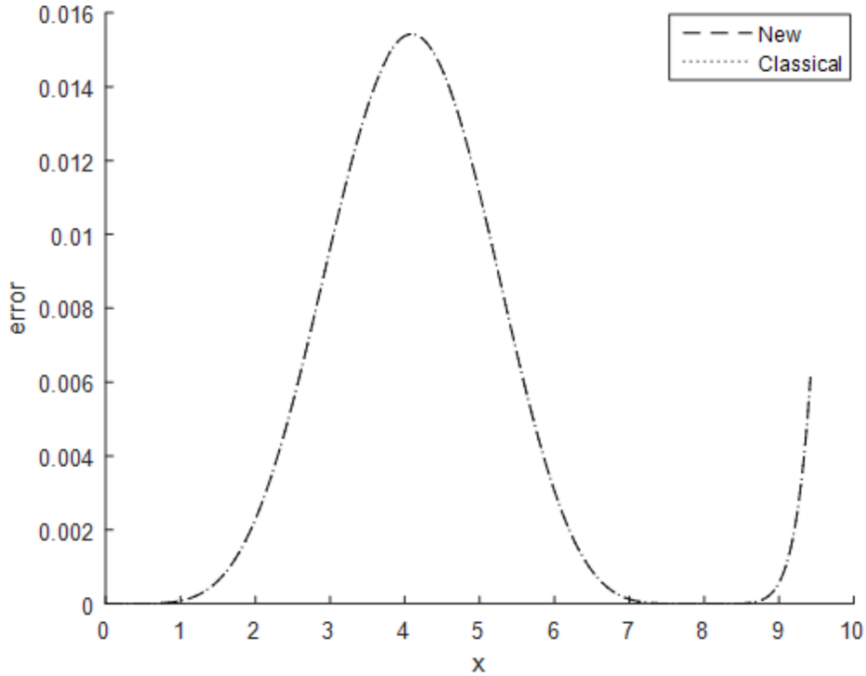

Figure 6. The absolute errors of the new two-point Taylor approximation (dashed) and the classical two-point Taylor approximation (dotted) for sine function $(x_0 = 0,\ x_1 = 8)$

The errors are minimized in the vicinity of the two reference points. Both approximations perform the same.

4.2. The function y=cos(x)

The analysis is repeated for the well-known even function cos(x). The lower reference point is fixed to $x_0 = 0$ and the higher reference point is increased from 4 to 8 in Figure 7. With an increase in the higher reference point, the truncated approximations (12 terms) have a wider range of validity. The two variants produce exactly the same results.

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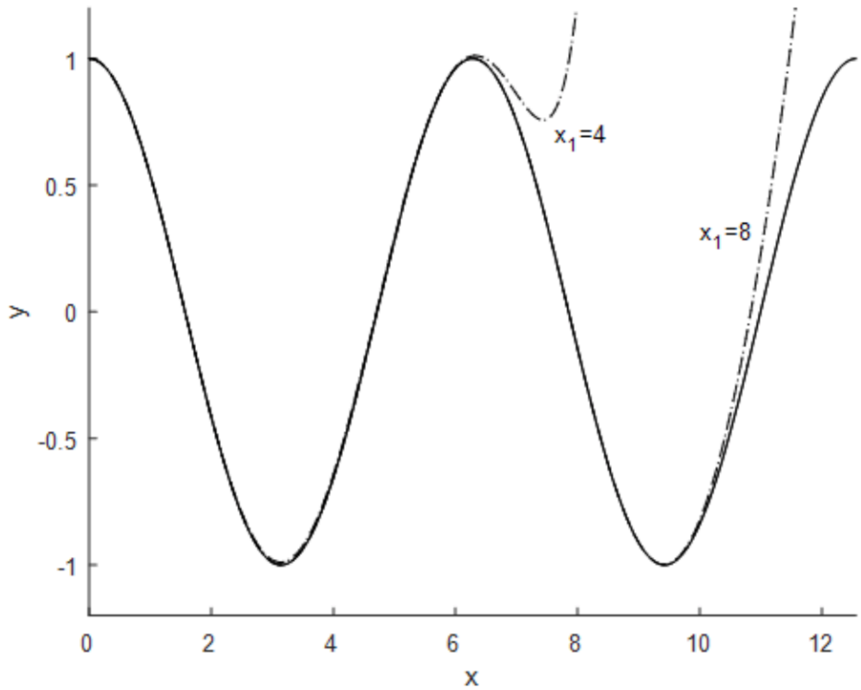

Figure 7. Comparison of the new two-point Taylor approximation (dashed), classical two-point Taylor approximation (dotted) and the exact function y=cos(x) (solid) for various right-hand side reference points $(x_0 = 0)$

The absolute error is given in Figure 8. The errors are the same over the whole domain of interest.

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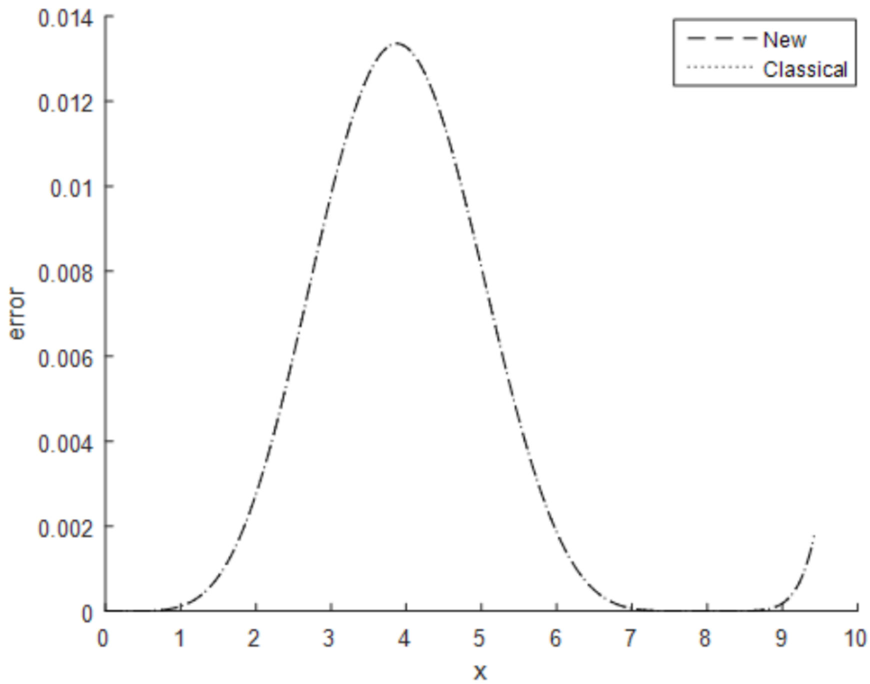

Figure 8. The absolute errors of the new two-point Taylor approximation (dashed) and the classical two-point Taylor approximation (dotted) for cosine function $(x_0 = 0,\ x_1 = 8)$

5. Differential Equations

One of the common analytical solution methods of differential equations is the Taylor series solutions. Two-point Taylor expansions can also be employed in search of approximate solutions. Consider the first order differential equation

$$y^{'} + (1 - 2x)y = 0,\ y(0) = 1, \tag{49}$$

which has an exact solution

$$y_e = \exp[x(x-1)]. \tag{50}$$

Assuming a two point series solution of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^m (x-1)^m + b_m x^{m+1} (x-1)^m, \tag{51}$$

substituting into (49) and grouping the terms, one finally has the recursive relations

$$a_{m+1} = \frac{a_m}{m+1},\ b_m = 0. \tag{52}$$

The general form of the coefficients is then

$$a_m = \frac{1}{m!}, \tag{53}$$

leading to the approximate two-point series solution

$$y(x) = \sum_{m=0}^{\infty} \frac{1}{m!} x^m (x-1)^m. \tag{54}$$

The success in achieving the above simple elegant solution may stem from the nature of the problem as the exact solution can be represented approximately as a two-point Taylor expansion. If the two reference points are not determined appropriately, there may be inconsistencies in deriving the recursive relations for coefficients. This topic needs further detailed investigation which is left as a potential area of research. The exact and approximate series solutions are contrasted in Figure 9. As the number of terms in the series solution increases, the approximate solution converges to the real solution.

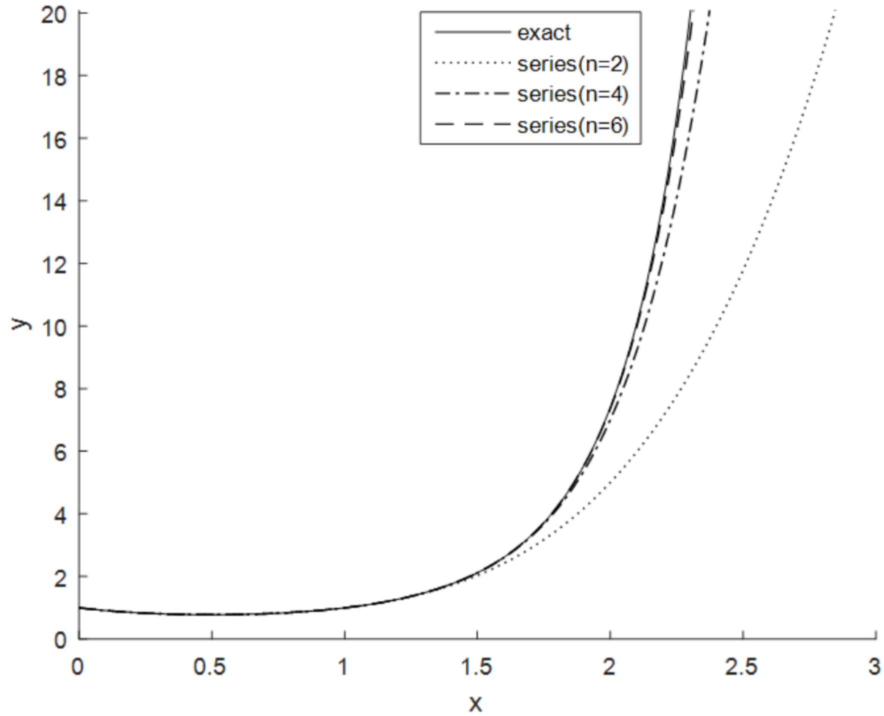

Figure 9. Comparison of the two-point series approximation with the exact solution of the differential equation

For variable coefficient equations as well as non-homogenous functions in the equation, the first step would be to expand the functions in a two-point Taylor expansion consistent with the assumed solution series. The coefficients are determined then by substituting the solution to the original differential equation. If inconsistencies appear in the recursive relations, then the assumed form may not be appropriate for the solution. The application of the two-point series to nonlinear equations needs further investigation. It has to be noted that the algebra involved in a single-point series solution is much less compared to the two-point series solution. The recursive relations and their solutions are not simple in the case of the two-point expansions. One should have a rational justification to resort to multiple-point solutions such as the nature of the equation, a wider range of validity etc.

6. Concluding Remarks

A new version of the two-point Taylor expansion is given. The new version produces identical results with the classical version reported in the literature. For problems with finite radius of convergence, two-point Taylor expansions may possess two different convergence intervals or a single convergence interval. When the selected two points are distant to each other, the single convergence interval may separate into two. A worked example is treated in detail. For problems of infinite radius of convergence, there is no separation of the convergence intervals. The two-point and the single-point expansions are compared with each other. For functions with finite radius of convergences, the two-point expansion is definitely advantageous compared to the single-point expansion about the lower reference point. However, the single-point expansion about the higher reference point may possess a wider convergence interval depending on the problem investigated. Despite the narrowing of the convergence interval, finite number of truncations of the series may produce better results compared to the single-point expansions. The two-point Taylor series can approximate the function at opposite sides of a singular point with two convergence intervals lying at the left and right of the singular point whereas single-point expansions cannot be valid at both sides. Based on the examples treated, the asymmetric new version does not have an advantage over the classical version. The point is that, unlike single-point expansions, there is no unique representation of the two-point expansions, that is they can vary in form. The proposed two-point expansion may be applied to solve approximately the differential equations. A variable coefficient linear differential equation with an exact solution is treated to demonstrate the application of the method. Although a wider range of validity for solutions can be achieved, the algebra is much more involved in the case of two-point expansions. The applications to non-homogenous equations, higher order equations and nonlinear equations may be a topic of further research.

Statements and Declarations

Availability of Supporting Data

There is no additional data associated with the paper.

Competing Interests

Author declares no competing interests.

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Declarations

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