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Stellar dynamics

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Abstract

The stellar dynamics of virialised systems is analysed within the search of undamped oscillations. In the case of a time-dependent Hamiltonian qualified after a generic Terzic'-Kandrup potential, the Emrakov-Lewis-Leach invariant is specified; as a result, an infinite set of conservation laws defining constants of motion is demonstrated to be obtained. Two methodologies are applied: the small-time parameter series expansion, and the slowly-varying higher-orders expansion. The results are apt to be applied to the case of the Emrakov-Lewis invariant, of the Emrakov-Lewis adiabatic invariant, and of the generalised Guenther-Leach generalised invariant. The verification of the series-expansion infinitesimal parameter is envisaged. WKB calculations are studied to be feasible.

Key-words:

Stellar dynamics; virialised systems; conservation laws; invariants of motion; Emrakov-Lewis-Leach invariant; Emrakov-Lewis invariant; Emrakov-Lewis adiabatic invariant; generalised Guenther-Leach invariant.

I. INTRODUCTION

The presence of undamped oscillations in stellar systems is studied within the framework of the Hamiltonian formalism.
The Jeans theorem provides with a characterization of the phase space in terms of the integrals of motion in the case of the collisionless Boltzmann equations.
The stellar dynamics is qualified at the time of the virialisation corresponding to the epoch of non-Gaussianities.
The development of new techniques for the calculation of the new constants of motions provides with new expressions of the Hamiltonian velocities.
The new techniques allow for a comparison in the precision obtained with respect to the calculation of the Hamiltonian velocities.
The new analytical expressions allow for a juxtaposition with the cases of new stress-energy tensors.
In the present work, the *big-bang* scheme is considered, with an FLRW (Fridman-Lemai'tre-Robertson-Walker) metric, inflation scenario and the thermal history. The structure formation is framed within the big-bang model.

II. RELATIVISTIC STELLAR DYNAMICS

A. The virial theorem in General Relativity

Let $\bar{R}$ Be the gravitational *virialised radius* , i.e. the average measure of the linear dimension of the system, for instance, such as the radius of the sphere in a spherical system); and let M be the
virialised mass M, i.e.
the mass of the virialised system.
The gravitational potential energy $\mathcal{W}$ is given as

$$\mathcal{W} = -\frac{1}{2}\frac{GM^2}{\bar{R}}; \tag{1}$$

given the inertia-moment tensor $\mathcal{I}_{ij}$, the kinetic-energy tensor $\mathcal{T}_{ij}$, and the potential-energy tensor $\mathcal{W}_{ij}$, the total energy of the system $\mathcal{E}$ remains constant as

$$\frac{d^2\mathcal{I}}{dt^2} = 2\mathcal{T} + \mathcal{W} = 2\mathcal{E} - \mathcal{W}. \tag{2}$$

It now possible to analyse the evolution of the dynamics of the studied spherical system [1], [2].

B. The Jeans theorem

Be the *Asymmetric drift* the difference between the azhimutal velocity and the circular velocity at a given radius, averaged over a volume element at a fixed position in space for a steady state [3]:
i) Any steady-state solution of the collisionless Boltzmann equation depends on the 6-dimensional phase-space coordinates only through the 3 isolating integrals of motion descending from the galactic potential; and
ii) any function of the the integrals of motion yields a steady-state solution of the collisionless Boltzmann equation.
An *Isolating integral of motion* is an object that, for particular values, *isolates hyper-surfaces in phase space* [4].

C. Stellar Hamiltonian dynamics

The Poisson equation of self-gravitating stellar systems reads

$$\nabla^2\Phi = 4\pi G\tilde{\rho}(\vec{r},t) \equiv 4\pi G \int f(\vec{r},\vec{v},t)d^3v. \tag{3}$$

The acceleration from the Poisson equation is due to gravitational interaction within the stellar system; in particular, the inward gravitational acceleration $a(r)$ from circular velocity $v(r)$ at radius r is states as

$$a(r) = -\frac{v(r)}{r}. \tag{4}$$

The demonstration of the stellar dynamics is now therefore accomplished with a distribution function of negative energy [5], [6].
The collisionless Boltzmann distribution function is apt to the anakysis of stellar dynamics.
Collecting all the ingredients together, one finds that isolating integrals, the Jeans theorem and the distribution functions render the gravitating systems are parameterised after a distribution function, a mass density, related after a gravitational potential which obey the simplified Vlasov equations [7].

III. STELLAR OSCILLATIONS

A. Galactic oscillations

Oscillations in ancient star do reveal an ancient galactic merger: kinematically-distinct structures within the Milky Way can identify univoquely a stellar population which underwent accretion after the acquisition of multiple

smaller satellite galaxies. Such conclusions were drawn after the NASA Transiting Exoplanet Survey Satellite within a naked-eye visible-star observation experiment:
the age of star is dated after presence in the star of elements heavier than carbon which were produced within nuclear reactions involving He, and confronted with those Galactic-disk stars; furthermore, asteroseismic observations, spectroscopic, astrometric, and kinematic ones allow one to complete the investigation.
More in detail, the calculation of oscillation is obtained after the measurement of the positions and of the velocities; and after comparison with orbital integral calculations done for the populations with low and high Magnesium/Iron ratio: the resulting distributions of the eccentricity and that of maximum vertical excursion from the Galactic mid-plane are calculated to qualify the oscillations types [8].

B. Solar-System oscillations

Vertical oscillations within the Solar system can also be investigated: i.e. in the direction orthogonal to the galactic plane, due to the resultant of the vector sum of the gravitational accelerations from the regions of the Galaxy. More specifically, the vertical oscillation of the solar system with respect to the Galactic plane can be estimated from the mass distribution [9], [10], [11].

IV. OBSERVATIONAL EVIDENCES OF OSCILLATIONS AND CONSTANTS OF MOTIONS

The constants of motion are analytically calculated within *symplectic transformations.*
The data-analysis techniques are some based, nevertheless, on different approaches [12], such as the implementation of velocity-dispersion profiles [13], [14], in which the General-Relativistic phase space can be modified [15], specific initial conditions notwithstanding [16].

V. POTENTIALS

A. Mathematical outline

Regularity assumptions about the time-dependent-oscillators potentials have been recently summoned in [17].
The *Generalized time-dependent oscillator Hamiltonian* is spelled out as [18]

$$H = f(t)\frac{p^2}{2m} + \frac{1}{2}mg(t)w_0^2x^2 \tag{5}$$

for a point particle of mass m, i.e. for a *system in which both the kinetic term and the potential term are generalised as time-dependent* by means of the functions $f(t)$ and $g(t)$, respectively; furthermore, the *initial values for the two functions $f(t)$ and $g(t)$ are established.*

B. Generalised harmonic potentials: Astrophysical implementations

Among the generalised potentials used in Relativistic Astrophysics, the following are reviewed, and their peculiarities can be appreciated:
i) *(pulsed) Plummer potential* [19]

$$V(r,t) = -\frac{m(t)}{\sqrt{1+r^2}}; \tag{6}$$

ii) three *(pulsed) Dehnen potentials* [20]

$$V(r,t) = -\frac{m(t)}{2-\gamma}\left[1 - \frac{r^{2-\gamma}}{(1+r)^{2-\gamma}}\right], \tag{7}$$

with $\gamma = 0$, $\gamma = 1/2$ and $\gamma = 1$;
iii) *Terzic'-Kandrup potentials* [21]: spherically-symmetric potentials (which can be cast in the Mathieu equation) which lead to a periodic dynamics

$$V(r,t) = V(r)(1 + m_0 sin\omega t); \tag{8}$$

iv) *Eddington-Jeans potentials* [22]

$$V(r,t) = V(r)M_0 M^n(t); \tag{9}$$

v) *Jaffe potential* [24] and *Hernquist potential* [25] can be considered as particular cases of the Dehnen potential.

VI. COSMOLOGICAL MODEL WITH A GENERIC TERZIC'-KANDRUP PULSED POTENTIAL: THE ERMAKOV-LEWIS-LEACH INVARIANTS

It is our aim to write analytically the constants of motion descending from time-dependent potentials of one-dimensional generalised time-dependent harmonic oscillator of Hamiltonian

$$H = \ddot{x} + \Omega^2(t)x. \tag{10}$$

A generic Terzic'-Kandrup pulsed potential

$$V(r,t) \equiv V(r)(1 + m_0 sin\omega t). \tag{11}$$

is implemented.
The auxiliary function $\rho(t)$ obeying the equation

$$\ddot{\rho}(t) + \omega^2(t)\rho(t) - \frac{1}{\rho^3(t)} = 0 \tag{12}$$

defines the *Ermakov-Lewis-Leach invariant of motion* I_{ELL} [26], [27], [30] as

$$I_{ELL} = \frac{1}{2}\left[\rho^{-2}x^2 + (\rho p - \dot{\rho}x)^2\right] \tag{13}$$

VII. NEW SOLUTION TECHNIQUES

The *new asymptotic small-time- parameter expansion* is established as

$$\rho = \rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \epsilon^3\rho_3 + \epsilon^4\rho_4.., \tag{14}$$

with ϵ a parameter to be addressed in the following.
There descends a *new infinite series of conservation laws*

$$\frac{d}{dt}\dot{\rho}^2_{2n} = (-1)^{-1+n/2}2\dot{\rho}^2_{2n}f_{2n}(\rho_{2n}, \rho_{2n-2}, ..., \rho_2, \rho_0) \tag{15}$$

for the auxiliary variable ρ defining conservation laws (as they cannot be cast in the form of Abel equations nor in the form of modified Emden-Fowler equations [29]) *new invariant of motion*: *according to the new following two methodologies.*

A. New methodology 1: asymptotic small-time-parameters expansion

In the case of third-order approximation, the following expansion is set

$$\rho_0 \neq 0, \quad \Omega^2\rho_0 - \frac{1}{\rho_0^3} = 0, \tag{16a}$$

$$\Omega^2\rho_1 + 3\frac{\rho_1}{\rho_0^4} = 0, \tag{16b}$$

$$\ddot{\rho_0} + \Omega^2\rho_2 + 3\frac{\rho_0\rho_2 - 2\rho_1^2}{\rho_0^5} = 0, \tag{16c}$$

$$\ddot{\rho_1} + \Omega^2\rho_3 - \frac{12\rho_1\rho_2 - 10\rho_1^3 - 3\rho_3\rho_0^2}{\rho_0^6} = 0 \tag{16d}$$

which, at the considered orders in ϵ, is formally solved as

$$\Omega \equiv \pm \frac{1}{\rho_0^2}, \tag{17a}$$

$$\rho_1 = 0, \tag{17b}$$

$$\frac{d}{dt}\dot{\rho_0}^2 = 2\dot{\rho_0}\left(-\Omega^2\rho_2 - 3\frac{\rho_2}{\rho_0^4}\right), \tag{17c}$$

$$\rho_3 = 0. \tag{17d}$$

The new evaluation of the constants of motion at the time of non-Gaussianities, during a small time integration interval $\Delta t = t_f - t_i$ is found, during which *the virialised radius and the virialised mass are constant*, is displayed as

$$\rho_0 = A_0 + A_1(t-t_i) + A_2(t-t_i)^2 + ... \tag{18a}$$

$$\rho_1 = 0, \tag{18b}$$

$$\rho_2 = B_0 + B_1(t-t_i) + B_2(t-t_i)^2 + ... \tag{18c}$$

$$\rho_3 = 0; \tag{18d}$$

accordingly, the following expansion is found

$$2A_2 + 4B_0V(r)(1 + m_0 sin\omega t_i) + (4B_0V(r)m_0\omega cos(\omega t_i) + \tag{19}$$

$$+ 4B_1V(r)(1 + m_0 sin(\omega t_i)))(t - t_i) + (-2B_0V(r)m_0 sin(\omega t_i)\omega^2 +$$

$$+ 4B_1V(r)m_0 cos(\omega t_i)\omega + 4B_2V(r)(1 + m_0 sin(\omega t_i)))(t - ti)^2 + O((t - t_i)^3) = 0,$$

which is *newly solved as*

$$A_0 \equiv \frac{1}{(V(r)(1 + m_0 sin\omega t_i))^{1/4}}, \tag{20a}$$

$$A_1 \equiv -\frac{1}{4}\frac{m_0 cos\omega t_i}{(V(r)(1 + m_0 sin\omega t_i)^{1/4}(1 + m_0 sin\omega t_i)}, \tag{20b}$$

$$A_2 \equiv \frac{1}{32}\frac{m_0\omega^2(m_0 cos^2\omega t_i + 4sin\omega t_i + 4m_0)}{(V(r)(1 + m_0 sin\omega t_i))^{1/4}\,(-m_0^2cos^2\omega t_i + 2m_0 sin\omega t_i + m_0^2 + 1)} \tag{20c}$$

$$B_0 \equiv -\frac{1}{64}\frac{m_0\omega^2(m_0 cos^2\omega t_i + 4sin\omega t_i + 4m_0)}{(V(r)(1 + m_0 sin\omega t_i))^{5/4}\,(-m_0^2cos^2\omega t_i + 2m_0 sin\omega t_i + m_0^2 + 1)}, \tag{20d}$$

$$B_1 \equiv -\frac{1}{64}\frac{m_0^2\omega^3V(r)cos\omega t_i(m_0 cos^2\omega t_i + 4sin\omega t_i + 4m_0)}{(V(r)(1 + m_0 sin\omega t_i))^{9/4}\,(-m_0^2cos^2\omega t_i + 2m_0 sin\omega t_i + m_0^2 + 1)}, \tag{20e}$$

$$B_2 \equiv -2\omega\tilde{A}\neg 2m_0V(r)sin\omega t_i - \frac{1}{64}\frac{m_0\omega^2(m_0 cos^2\omega t_i + 4sin\omega t_i + 4m_0)}{(V(r)(1 + m_0 sin\omega t_i))^{5/4}\,(-m_0^2cos^2\omega t_i + 2m_0 sin\omega t_i + m_0^2 + 1)} + \tag{20f}$$

$$+ 4m_0\omega cos\omega t_i\frac{1}{64}\frac{m_0^2\omega^3V(r)cos\omega t_i(m_0 cos^2\omega t_i + 4sin\omega t_i + 4m_0)}{(V(r)(1 + m_0 sin\omega t_i))^{9/4}\,(-m_0^2cos^2\omega t_i + 2m_0 sin\omega t_i + m_0^2 + 1)}. \tag{20g}$$

B. New methodology 2: slowly-varying higher orders

Given the solution of $\rho_0 \forall \rho_j, \quad j = 1, 2, ...$, i.e.

$$\rho_0 = \frac{1}{(1 + m_0 sin(\omega t))^{1/4}} \tag{21}$$

the *new assumption* is taken, that the higher orders are slowly-varying; the conservation laws imply, at the chosen order,

$$\rho_2(t) \equiv \tilde{\rho}_2(t) \tag{22}$$

changing slowly during the integration time interval as

$$\tilde{\rho_2} = \frac{(1+m_0 sin(\omega t))^{1/4}}{\omega^2\sqrt{1-m_0^2}}\Big[2\omega t arctan\left(\frac{m_0 + tan\frac{\omega t_i}{2}}{\omega\sqrt{1-m_0^2}}\right) - 2\omega t_i arctan\left(\frac{m_0 + tan\frac{\omega t_i}{2}}{\omega\sqrt{1-m_0^2}}\right) +$$

$$-2\omega(t-t_i)arctan\left(\frac{m_0 + tan\frac{\omega t_i}{2}}{\omega\sqrt{1-m_0^2}}\right) - i\omega t Log\frac{1 - ie^{i\omega t}m_0 + \sqrt{1-m_0^2}}{1+\sqrt{1-m_0^2}} +$$

$$+ i\omega t Log\frac{-1 - ie^{i\omega t}m_0 + \sqrt{1-m_0^2}}{-1+\sqrt{1-m_0^2}} - i\omega t_i Log\frac{1 - ie^{i\omega t_i}m_0 + \sqrt{1-m_0^2}}{1+\sqrt{1-m_0^2}} + i\omega t_i Log\frac{-1 - ie^{i\omega t_i}m_0 + \sqrt{1-m_0^2}}{-1+\sqrt{1-m_0^2}} + PolyLog\left[2, -\frac{ie^{i\omega t}m_0}{-1+\sqrt{1-m_0^2}}\right] - PolyLog\left[2, -\frac{ie^{i\omega t_i}m_0}{-1+\sqrt{1-m_0^2}}\right] -$$

$$PolyLog\left[2, \frac{ie^{i\omega t}m_0}{1+\sqrt{1-m_0^2}}\right] + PolyLog\left[2, \frac{ie^{i\omega t_i}m_0}{1+\sqrt{1-m_0^2}}\right]\Big] \quad (22a).$$

VIII. NEXT ISSUES

The *verification* of the small time asymptotic expansion parameter ϵ is therefore in order.
As soon as the *Ermakov-Lewis* (EL) *invariant* [30], [31], I_{EL} is calculated, a new class of conservation laws of the constants of motions ruled after the generic Terzic'-Kandrup pulsed potential is found as

$$H = \frac{1}{2\eta}\left[\ddot{x} + \Omega^2(t)x\right], \tag{23}$$

where the auxiliary function $\sigma(t)$ obeying the definition

$$\eta^2\ddot{\sigma}(t) + \omega^2(t)\sigma(t) - \frac{1}{\sigma^3(t)} = 0 \tag{24}$$

leads to the the I_{EL} *invariant of motion*

$$I_{EL} = \frac{1}{2}\left[\sigma^{-2}x^2 + (\sigma p - \eta\dot{\sigma}x)^2\right]; \tag{25}$$

as a further result, the *new asymptotic small-times asymptotic expansion* of the auxiliary variable σ according to the infinitesimal parameter ϵ names the *generalised constants of motion*, which depend on the virialised radius, and bringing new light on the assumptions underlying the virialisation of the mass.

As a new further calculations, the *the Ermakov-Lewis adiabatic invariant* [32], [33], I_{ELa} is found: *a new class of adiabatic invariants ruled after the generic Terzic'-Kandrup pulsed potential* are stated: the series decomposition of I_{ELa} is obtained with the auxiliary variable σ being decomposed as positive powers of η.
A *new generalisation to three-dimensional models* including the anisotropic problem, and the singular quadratic perturbation problem is found after [34].

IX. PERSPECTIVES

WKB methods can be applied, such as the linear WKB method for linear oscillator: the distribution of the energy is always the arcsine distribution [35]; and the non-linear WKB method of time-dependent power law potentials, in which the isolating integral Energy is not constant: the technique is applied for homogeneous power law potentials which includes also the quartic oscillator [36].
The *characterization of the variable* $\dot{x}$ from a Hamiltonian point of view can be possible [37]. The prospective analyses [38] allow one to improve the investigation of the conservation laws of the constant of motion.

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