Quaternion Quantum Mechanics: The Baryons, Quarks, and Their q-Potentials

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Abstract: The results presented here are based on the Planck-Kleinert crystal concept. The rigorous use of quaternion algebra allows postulating the scalar, vectorial, and quaternion propagators in the ideal elastic continuum. The propagators are used in constructing the proton, electron, and neutron 2nd order partial differential equation systems, PEDS. The results generate the two 2nd order PEDS for the *u* and *d* quarks from the *up* and *down* groups. It was verified that both the proton and the neutron obey experimental findings and are formed by three quarks. The proton and neutron are formed by *d-u-u* and *d-d-u* complexes, respectively. All particle PEDS comply with the Cauchy equation of motion and can be considered as stable particles. The u and d quarks do not meet the relations of the Cauchy equation of motion. The inconsistencies of the quarks' PEDS with the quaternion forms of the Cauchy equation of motion account for their lifetime and the observed Quarks Chains. That is, they explain the Wilczek phenomenological paradox: "Quarks are Born Free, but Everywhere They are in Chains".

Keywords: q-potentials; vectorial potential; proton; quarks chain

1. Introduction

The focus here is on quaternion quantum mechanics, QQM, and quaternionic field theory, QFT. The quaternion algebra is attributed to many physical systems and laws, sporadically to quantum mechanics. Lanczos's dissertation was on a quaternionic field theory of classical electrodynamics [1,2]. In his derivation of Dirac's equation [3], there is a doubling in the number of solutions and several concepts that still remain at the front of the fundamental theory. These articles were unnoticed by contemporaries; Lanczos abandoned quaternions and never returned to quaternionic field theory. Fueter demonstrated that the Cauchy-Riemann type conditions in the quaternion representation are identical in shape to the vacuum equations of electrodynamics [4]. Yefremov described Newtonian mechanics in a rotating frame of reference [5] and the motion of non-inertial frames [6]. Adler shows that the Dirac transition amplitudes are quaternion valued [7]. Christianto derived an original wave equation from the correspondence between the Dirac equation and Maxwell's electromagnetic equations via the biquaternionic representation [8].

The Adler's method of quaternionizing quantum mechanics was avoided in the Harari-Shupe preon model for the composite quarks and leptons [9]. However, the composite fermion states were later identified with the quaternion real components [10]. In spite of the lack of progress in advancing the Harari-Shupe scheme, substantial progress in QQM and QFT was made [11,12].

The evolution of the P-KC model and the subsequent development of the QQM are shown [13,14,15]. The QQM presented here is ontological in the sense that it starts with being, that is, the Planck-Kleinert ideal regular crystal [14,16]. The basic categories of being and their relations are governed by the quaternion algebra [14]. The stress tensor of the Planck-Kleinert crystal is given by

$$\sigma' = \sigma''/m_P = \left(\lambda_L/m_P\,\mathbf{trD}\right)\mathbf{1} + 2\mu_L/m_P\,\mathbf{D} \quad , \tag{1}$$

where D denotes the deformation tensor (the symmetrical part of the strain tensor) and λ_L and μ_L are the Lamé coefficients of an ideal regular crystal. It was shown by Cauchy and Saint Venant that if the

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particles composing a regular crystal interact pairwise through central forces, then there is an additional symmetry requiring C44 = C12 that implies the Poisson ratio 0.25 and $\lambda_L = \mu_L$ [17]

$$\sigma' = \left(\lambda_L / m_P\, \mathbf{trD}\right)\mathbf{1} + 2\lambda_L / m_P\, \mathbf{D} \qquad . \tag{2}$$

Using the identity: $\operatorname{grad}\operatorname{div}\mathbf{u} = \operatorname{div}\operatorname{grad}\mathbf{u} + \operatorname{rot}\operatorname{rot}\mathbf{u}$, the stress tensor in the Planck-Kleinert crystal becomes [16]:

$$\operatorname{div}\sigma' = 2\lambda_L / m_P \operatorname{grad}\operatorname{div}\mathbf{u} + \lambda_L / m_P \operatorname{div}\operatorname{grad}\mathbf{u} = 3\lambda_L / m_P \operatorname{grad}\operatorname{div}\mathbf{u} - \lambda_L / m_P \operatorname{rot}\operatorname{rot}\mathbf{u}\,. \tag{3}$$

The motivation for writing this paper was to explicate the stress field origin of the QQM and QFT. The Standard Model of elementary particles lacks an adequate description of the mechanism of quark charges. It is showed here that the quark particle waves do exist, and two their PEDS are presented. Further studies in order to verify or refute those propositions are suggested.

1.1. Quaternions

The elements of the quaternion algebra used in the QQM and QFT were already presented in previous papers [13-15]. Only the two definitions are recalled here. In the ideal elastic continuum, the quaternion potential, i.e., the deformation four-potential, is defined by

$$\begin{matrix} \sigma & = & \sigma_0 & + & \hat{\phi} \\ \begin{bmatrix} q\text{-potential,} \\ \text{deformation} \end{bmatrix} & = & \begin{bmatrix} \operatorname{div}\mathbf{u}_0 \\ \text{compression} \end{bmatrix} & + & \begin{bmatrix} \operatorname{rot}\mathbf{u}_\phi \\ \text{twist pseudovector} \end{bmatrix} \end{matrix}, \tag{4}$$

where $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_\phi$ denotes the displacement, $\sigma = \sigma_0 + \hat{\phi} \in §$ is the q-potential, and the constraint $\operatorname{div}\hat{\phi} = 0$ holds.

We use the Cauchy–Riemann operator D in $\circ^4$ acting on the quaternion-valued functions

$$D\sigma = \left(-\operatorname{div}\hat{\phi}\right) + \operatorname{grad}\sigma_0 + \operatorname{rot}\hat{\phi}, \quad \sigma = \sigma_0 + \hat{\phi}. \tag{5}$$

Under the constraint: $\operatorname{div}\hat{\phi} = 0$, D corresponds physically to the nabla operator in $\circ^3$:

$$D\sigma = \operatorname{grad}\sigma_0 + \operatorname{rot}\hat{\phi}. \tag{6}$$

The exponent function has its trigonometrical representation

$$\exp(\sigma) = \left(\cos|\hat{\phi}| + \hat{\phi}/|\hat{\phi}|\sin|\hat{\phi}|\right)\exp(\sigma_0). \tag{7}$$

1.2. The critical review of the earlier results.

The Cauchy equation of motion and the overall energy density of the deformation field in the quaternion formulation equal [14]

$$\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\sigma - 2c^2\Delta\sigma_0 = 0, \tag{8}$$

$$\rho_E = \rho_P\left(\tfrac{1}{2}\dot{\hat{u}}\cdot\dot{\hat{u}}^* + \tfrac{1}{2}c^2\, \sigma\cdot\sigma^* + c^2\sigma_0^2\right), \tag{9}$$

where ρ_E and ρ_P denote the deformation energy and the mass densities in the P-KC respectively, $\dot{\hat{u}} = \dot{u}_1 i + \dot{u}_2 j + \dot{u}_3 k$ is the mass velocity in the quaternion representation:

$$\hat{u} = -\frac{\hbar}{m}\mathrm{D}\sigma. \tag{10}$$

The overall energy of the particle wave in an arbitrary volume Ω follows from Eq. (9) and is given by the integral:

$$\int_\Omega \rho_E(t,x)\,\mathrm{d}x = \int_\Omega \rho_P\left(\frac{1}{2}\hat{u}\cdot\hat{u}^* + \tfrac{1}{2}c^2\ \sigma\cdot\sigma^* + c^2\sigma_0^2\right)\mathrm{d}x = mc^2, \tag{11}$$

where for the sake of clarity, the external potential, $V(x)$, is not shown [14].

In the previous paper [15] upon substituting $\tilde{\sigma}_0 = \sqrt{3}\sigma_0$, we introduced in (11) the transformed, *q*-potential, $\tilde{\sigma} = \tilde{\sigma}_0 + \hat{\phi} = \sqrt{3}\sigma_0 + \hat{\phi}$, and expressed the particle mass by the symmetrical relation

$$m = \frac{\rho_P}{2c^2}\int_\Omega\left(\hat{u}\cdot\hat{u}^* + c^2\tilde{\sigma}\cdot\tilde{\sigma}^*\right)\mathrm{d}x. \tag{12}$$

The combination of (10) and (12) resulted in the energy functional and allowed consideration of the existence of the stable particle *m* in the potential field $V(x)$. Subsequently, the quaternionic particle density was defined,

$$\psi = \sqrt{\rho_P/m}\,\tilde{\sigma} \tag{13}$$

and it was proved that ψ satisfies the time-independent Schrödinger equation [14]

$$-\frac{\hbar^2}{2m}\Delta\psi + V(x)\,\psi = E\psi\,. \tag{14}$$

The quaternionic particle density ψ is also called the quaternionic probability because the relation $\int_\Omega \psi\cdot\psi^*\mathrm{d}x = 1$ holds [15].

Remark. The q-potential definition, $\tilde{\sigma} = \sqrt{3}\sigma_0 + \hat{\phi}$, is incompatible with the derived quaternionic oscillator formula where only integral coupling coefficients *n* are allowed, e.g., $\tilde{\sigma} = (1-n)\sigma_0 + \hat{\phi}$ in [15].

The 2nd order boson PEDS presented in [13,14] are based on the postulate of the scalar propagator, $G_0(m)\sigma\cdot\sigma^*$, providing the coupling between the longitudinal and transverse waves. The coupling is evident upon expressing the quaternionic Klein-Gordon system in the equivalent form, e.g.,

$$\begin{cases}\left(\dfrac{\partial^2}{\partial t^2} - c^2\Delta\right)\hat{\phi} = 0,\\ \left(\dfrac{\partial^2}{\partial t^2} - 3c^2\Delta\right)\sigma_0 = 0,\\ 2c^2\Delta\sigma_0 + G_0(m)\sigma\cdot\sigma^* = 0,\end{cases} \quad\Leftrightarrow\quad \begin{cases}\left(\dfrac{\partial^2}{\partial t^2} - c^2\Delta\right)\sigma + G_0(m)\sigma\cdot\sigma^* = 0,\\ 2c^2\Delta\sigma_0 + G_0(m)\sigma\cdot\sigma^* = 0.\end{cases} \tag{15}$$

Above two systems are identical, five equations and five unknowns: $\sigma_0, \phi_1, \phi_2, \phi_3$ and *m*, see definition (4). If mass *m* is unknown, it may be treated as the parameter in the Poisson equation above.

In [15], we further developed the propagator concept and postulated the family of second-order quaternionic wave equations:

$$\begin{cases}\left(\dfrac{\partial^2}{\partial t^2} - c^2\Delta\right)\hat{\phi} = 0,\\ \left(\dfrac{\partial^2}{\partial t^2} - 3c^2\Delta\right)\sigma_0 = 0,\\ 2(1-n)c^2\Delta\sigma_0 + G_0(m)\sigma\cdot\sigma^* = 0 \text{ where } n = 0,2,3,\ldots\end{cases} \tag{16}$$

where *n* is a coupling coefficient.

It's evident that at *n* = 0, the coupling (15) for boson particles follows. The propagator term $G_0(m)\sigma\cdot\sigma^*$ in (16) corresponds to the density of the rate of the momentum change, and the $G_0(m)$ term is referred to as the power of the harmonic oscillator. The coupling coefficient n can be elucidated as the radius R of the quaternionic oscillator in the Cauchy crystal, expressed in Planck length: $R = n l_P$. In the system (16), the generalized q-potential can be introduced

$$\tilde{\sigma}_n = (1-n)\sigma_0 + \hat{\phi} \quad \text{where} \quad n = 0, 2, 3, \ldots \tag{17}$$

Upon the $\tilde{\sigma}_n$ substitution into the system (16), the two 2nd order PEDS are evident:

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - c^2\Delta\right)\tilde{\sigma}_n + G_0(m)\sigma\cdot\sigma^* = 0, \\ \left[n\dfrac{\partial^2}{\partial t^2} - (n+2)c^2\Delta\right]\sigma_0 - G_0(m)\sigma\cdot\sigma^* = 0. \end{cases} \tag{18}$$

Remark. The problem of the coupling coefficients $n > |1|$ and the resulting different particles is not presented here.

The harmonic oscillator controls the acceleration of q-potential in the particle wave. The acceleration of the scalar part σ_0 of the q-potential was estimated in [15]:

$$\left\langle \frac{\partial^2 \sigma_0}{\partial t^2} \right\rangle = 4\pi^2 f_P f \,. \tag{19}$$

Using the equipartition theorem and the common frequency postulate for all four q-potential components: $\sigma_0, \phi_1, \phi_2, \phi_3$, the relation (19) was extended to ~ 4:

$$\left\langle \frac{\partial^2 \sigma}{\partial t^2} \right\rangle = 4\left\langle \frac{\partial^2 \sigma_0}{\partial t^2} \right\rangle = 16\pi^2 f_P f \,. \tag{20}$$

The acceleration of the q-potential will be called the power of the quaternionic oscillator in the particle wave:

$$G_0(f) = 16\pi^2 f_P f \,, \tag{21}$$

where f is an unknown particle frequency that may be postulated or computed.

Remark. The power of the oscillator $G_0(f)$, Equation (21), does not take into account both the constraint $\operatorname{div}\hat{\phi} = 0$ and the pseudovector character of twist $\hat{\phi}$.

The 1st order particle wave equation in the quaternion formulation obtained in [15] is consistent with its form in the Dirac algebra formalism. However, the 1st order system is generated by the invalidated substitution $n = \sqrt{3}$ in [14]. The 2nd order PEDS, following the schema (18), where $n = \sqrt{3}$ and $\tilde{\sigma} = \sqrt{3}\sigma_0 + \hat{\phi}$, equal:

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - c^2\Delta\right)\tilde{\sigma} + 2G_0\, \sigma^*\cdot\sigma = 0, \\ \left(\dfrac{\sqrt{3}-1}{3-\sqrt{3}}\dfrac{\partial^2}{\partial t^2} + c^2\Delta\right)\sigma_0 + \dfrac{2}{3-\sqrt{3}}G_0\, \sigma^*\cdot\sigma = 0. \end{cases} \tag{22}$$

The system (22) consists of the two 2nd order scalar PEDS while the vector potential is not present. In the next sections, the solutions of the presented problems are given.

Remark. The Equation (22) is mistaken and will be reformulated in the next sections.

2. The Baryons, Quarks, and Their q-Potentials

2.1. The Quaternionic Oscillator

The coupling of the transverse and longitudinal waves takes place in the PKC elementary cell, i.e., at the Planck scale. The quaternionic oscillator controls the acceleration of all the q-potential constituents in the particle wave in Ω: $\ddot{\sigma}_0, \ddot{\phi}_1, \ddot{\phi}_2, \ddot{\phi}_3$. The function $G_0 \in$ ° is called the power of the quaternionic oscillator. In earlier papers, we neglected the facts that twists ϕ_1, ϕ_2 and ϕ_3 form the pseudovector $\hat{\phi} = \phi_1 i + \phi_2 j + \phi_3 k$ [18] and that the constraint $\mathrm{div}\,\hat{\phi} = 0$ holds. Thus, the relation (19) for the scalar q-potential component σ_0 in $\sim^1$ extended for $\sigma_0, \hat{\phi}$ in $\sim^4$ (20) must be corrected and the two independent q-potential constituents, σ_0 and $\hat{\phi}$, considered:

$$\left\langle \frac{\partial^2 \sigma}{\partial t^2} \right\rangle = 2\left\langle \frac{\partial^2 \sigma_0}{\partial t^2} \right\rangle = 8\pi^2 f_P f \tag{23}$$

and the power of the quaternionic oscillator equals

$$G_0(f) = 8\pi^2 f_P f \,. \tag{24}$$

The particle wave frequency depends on the particle mass, $f = f(m)$, and follows from the $\sim^1$ schema, see Fig. 1 in [15]. The sum of moments of all the Planck masses forming the particle wave in Ω (at the arbitrary time t and solely due to the particle wave) equals the momentum of the particle m itself. To simplify, we may estimate the average momentum of the arbitrary single Planck mass m_P during the whole particle cycle $T = f^{-1}$. The complete cycle implies that every Planck mass returns to its initial conditions: $\mathbf{u}_P(t) = \mathbf{u}_P(t+T)$ and $\dot{\mathbf{u}}_P(t) = \dot{\mathbf{u}}_P(t+T)$. The overall distance that the arbitrary mass m_P passes during the wave cycle T equals $2\pi l_P$. The average momentum of the Planck mass $\bar{p}(m_P)$ during the particle wave cycle equals

$$\bar{p}(m_P) = m_P \frac{2\pi l_P}{T} = 2\pi m_P l_P f. \tag{25}$$

The momentum of the particle m results in the same way from the particle wave propagation velocity, e.g., c in the system (15):

$$p(m) = mc\,. \tag{26}$$

The moment (25) and (26) must equal, and the frequency of the particle wave becomes:

$$f = \frac{mc}{2\pi m_P l_P} \times \frac{c}{c} = \frac{mc^2}{2\pi\hbar} \text{ where } \hbar = m_P c l_P. \tag{27}$$

Combining the relations (24), (27), and the definition $f_P = 1/t_P$, the overall power of the quaternionic oscillator when the particle mass is known equals:

$$G_0(m) = 4\pi mc^2 / (\mathrm{h} t_P)\,. \tag{28}$$

By substituting $mc^2 = E_0$ in (27), the Planck–Einstein relation follows: $E_0 = hf$, where $h = 2\pi\hbar$. The family of the scalar 2nd order quaternionic wave equations, when the corrected propagator is used, becomes:

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - c^2\Delta\right)\hat{\phi} = 0, \\ \left(\dfrac{\partial^2}{\partial t^2} - 3c^2\Delta\right)\sigma_0 = 0, \\ nc^2\Delta\sigma_0 + G_0(m)\sigma\cdot\sigma^* = 0, \end{cases} \Leftrightarrow \begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - c^2\Delta\right)\hat{\phi} + 2G_0(m)\sigma\cdot\sigma^* = 0, \\ \left((n-1)\dfrac{\partial^2}{\partial t^2} - (n-3)c^2\Delta\right)\sigma_0 + 2G_0(m)\sigma\cdot\sigma^* = 0, \end{cases} \tag{29}$$

where n denotes an integer and $n \neq 0$.

It's evident that at n = 1, the coupling in (15) for the boson particle follows. The corrected propagator $G_0(m)\sigma\cdot\sigma^*$ results in symmetry of the coupling equation in system (29). I t does not have an effect on the scalar 2nd order PEDS and the gravitational constant in (15)-(18).

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2.2. The baryon particles formed by the odd number of quarks

The strong coupling only is considered here, e.g., $n=1$ in the system (29). The quaternionic oscillator $G_0(m)$ allows postulating three propagators: the scalar, $G_0(m)\sigma\cdot\sigma^*$, the vectorial, $G_0(m)\hat{\phi}$, and the quaternion, $G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)$.

The term $G_0(m)\hat{\phi}$ fixes the density of the rate of twist change and is called the vectorial propagator. We postulate the vectorial Poisson equation in system (29): $-c^2\Delta\hat{\phi}+G_0(m)\hat{\phi}=0$. Upon the rearrangement of the new system, the particle wave (electron) and the vectorial Poisson equations are evident:

$$\begin{cases}\left(\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\hat{\phi}=0,\\ \left(\dfrac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma_0=0,\\ -c^2\Delta\hat{\phi}+G_0(m)\hat{\phi}=0,\end{cases} \Leftrightarrow \begin{cases}\left(\dfrac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma+2G_0(m)\hat{\phi}=0,\\ -c^2\Delta\hat{\phi}+G_0(m)\hat{\phi}=0.\end{cases} \tag{30}$$

Note that the wave propagation velocity in system (30) equals the velocity of longitudinal waves in the Cauchy continuum: $c_L=\sqrt{3}\,c$ [16]. By adding equations in system (30), it is clear that it complies with the Cauchy equation of motion (8):

$$\begin{cases}\left(\dfrac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma+2G_0(m)\hat{\phi}=0,\\ -c^2\Delta\hat{\phi}+G_0(m)\hat{\phi}=0,\end{cases} \Rightarrow \left(\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma-2c^2\Delta\sigma_0=0. \tag{31}$$

The above vectorial Poisson equation hints at Equation (30) as the 2nd order PEDS for electrons. Note that the wave propagation velocity in the electron system in Equation (31) equals the velocity of longitudinal waves in Cauchy Equation (15): $c_L\sqrt{3}c$

In the quaternion propagator, $G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)$, the vectorial, $G_0(m)\hat{\phi}$, and scalar, $G_0(m)\sigma\cdot\sigma^*$, propagators are "merged" and form the strongly coupled system. The rearrangements of system (32) are shown below and display different forms of the 2nd order PEDS:

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\hat{\phi}=0, \\ \left(\dfrac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma_0=0, \\ -c^2\Delta\hat{\phi}+G_0(m)\hat{\phi}=0, \\ c^2\Delta\sigma_0+G_0(m)\sigma\cdot\sigma^*=0, \end{cases} \Leftrightarrow \begin{cases} \left(\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\hat{\phi}=0, \\ \left(\dfrac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma_0=0, \\ c^2\Delta\left(\sigma_0-\hat{\phi}\right)+G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)=0, \end{cases} \Leftrightarrow \begin{cases} \left(\dfrac{\partial^2}{\partial t^2}-2c^2\Delta\right)\sigma+G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)=0, \\ c^2\Delta\left(\sigma_0-\hat{\phi}\right)+G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)=0. \end{cases} \quad (32)$$

The comparison of the scalar, vectorial, and quaternionic propagators shows that the q-propagator offers the strongest coupling, Eq. (32). The quaternionic Poisson equation in (32) reveals that it is the 2nd order PEDS for a proton. The sum of equations in (32) shows that the system complies with the Cauchy equation of motion (8):

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2}-2c^2\Delta\right)\sigma+G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)=0, \\ c^2\Delta\left(\sigma_0-\hat{\phi}\right)+G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)=0. \end{cases} \Rightarrow \left(\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma-2c^2\Delta\sigma_0=0. \quad (33)$$

The scrupulous assessment of systems (15), (30), and (32) allows postulating the 2nd order PEDS for the quarks from the *up* and *down* groups. Explicitly, the 2nd order PEDS of the *u* quark from the *up* group equals:

$$\begin{cases} \left(\dfrac{1}{3}\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma+\dfrac{2}{3}G_0(m)\hat{\phi}=0, \\ -c^2\dfrac{2}{3}\Delta\hat{\phi}-\dfrac{2}{3}G_0(m)\hat{\phi}=0, \end{cases} \quad (34)$$

and the 2nd order PEDS of the *d* quark from the *down* group:

$$\begin{cases} \dfrac{1}{3}\dfrac{\partial^2\sigma}{\partial t^2}+G_0(m)\left(\sigma\cdot\sigma^*\right)-\dfrac{1}{3}G_0(m)\hat{\phi}=0, \\ c^2\Delta\left(\sigma_0+\dfrac{1}{3}\hat{\phi}\right)-G_0(m)\left(\sigma\cdot\sigma^*-\dfrac{1}{3}\hat{\phi}\right)=0. \end{cases} \quad (35)$$

The sum of equations in the quark systems (34) and (35) does not comply with the Cauchy equation of motion (8) and may indicate their short lifetime.

2.3. The quarks

There are two groups of hadrons: **baryons (containing three quarks or three antiquarks**); and mesons (containing a quark and an antiquark). In the following, we show that systems (30) - (35) comply with the experimental findings shown in Table 1.

Table 1. The basic properties of the quarks in baryons.

Group	Quarks	Charge	Spin
up	*u,c,t*	2/3	1/2
down	*d,s,b*	-1/3	1/2

The terms $\frac{2}{3}G_0(m)\hat{\phi}$ and $-\frac{1}{3}G_0(m)\hat{\phi}$ in systems (34) and (35) respectively, are related to the charge, see Table 1. A proton is formed by the two up quarks and the single down quark: $d-u-u$. Thus, by computing the sum of two systems (34) and one system (35), we may expect a proton, system (32):

$$\begin{cases} \dfrac{1}{3}\dfrac{\partial^2\sigma}{\partial t^2}+G_0(m)\left(\sigma\cdot\sigma^*\right)-\dfrac{1}{3}G_0(m)\hat{\phi}=0, \\ c^2\Delta\left(\sigma_0+\dfrac{1}{3}\hat{\phi}\right)+G_0(m)\left(\sigma\cdot\sigma^*-\dfrac{1}{3}\hat{\phi}\right)=0 \end{cases} +2\times\begin{cases} \left(\dfrac{1}{3}\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma+\dfrac{2}{3}G_0(m)\hat{\phi}=0, \\ -c^2\dfrac{2}{3}\Delta\hat{\phi}+\dfrac{2}{3}G_0(m)\hat{\phi}=0, \end{cases} \tag{36}$$

and the result is in agreement with equation (33):

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2}-2c^2\Delta\right)\sigma+G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)=0, \\ c^2\Delta\left(\sigma_0-\hat{\phi}\right)+G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)=0. \end{cases} \tag{37}$$

A neutron is formed by the one up quark and the two down quarks: $d-d-u$

$$2\times\begin{cases} \dfrac{1}{3}\dfrac{\partial^2\sigma}{\partial t^2}+G_0(m)\left(\sigma\cdot\sigma^*\right)-\dfrac{1}{3}G_0(m)\hat{\phi}=0, \\ c^2\Delta\left(\sigma_0+\dfrac{1}{3}\hat{\phi}\right)+G_0(m)\left(\sigma\cdot\sigma^*-\dfrac{1}{3}\hat{\phi}\right)=0, \end{cases} + \begin{cases} \left(\dfrac{1}{3}\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma+\dfrac{2}{3}G_0(m)\hat{\phi}=0, \\ -c^2\dfrac{2}{3}\Delta\hat{\phi}+\dfrac{2}{3}G_0(m)\hat{\phi}=0, \end{cases} \tag{38}$$

and the result is in agreement with the neutron system (15):

$$\begin{cases} \dfrac{\partial^2\sigma}{\partial t^2}-c^2\Delta\sigma+2G_0(m)\sigma\cdot\sigma^*=0, \\ c^2\Delta\sigma_0+G_0(m)\sigma\cdot\sigma^*=0. \end{cases} \tag{39}$$

The systems (30), (37), and (39) represent coupled 2nd order PEDS and show the different coupling strengths. The strongest coupling of the proton, Equation (37), is related to its enormously long lifetime.

3. The Quaternion Schrödinger Equation

The vectorial Poisson equation indicates that it's the 2nd order PEDS for the electron. We will apply this schema in the system (30) in the integral form of the energy conservation. We treat the wave as a particle in an arbitrary volume Ω [14]. The energy per mass unit, *e*, in the volume occupied by the particle wave defines its overall energy: $E_O=E_p+E_V=\int_\Omega \rho_p e\,\mathrm{d}x$,

$$e=\tfrac{1}{2}\hat{u}\cdot\hat{u}^*+\tfrac{1}{2}c^2\sigma\cdot\sigma^*+c^2\sigma_0^2 \quad \text{where} \quad \sigma^*=\sigma_0-\hat{\phi}\ , \tag{40}$$

where E_P and E_V denote energies of the particle and of its force field, respectively, ρ_P is the Planck mass density.

The 1st step in deriving the Schrödinger equation is the choice of the symmetrization scheme for the particle energy, E_P. Equation (40) can be written in the equivalent form:

$$e=\frac{1}{2}\hat{u}\cdot\hat{u}^*+\frac{3}{2}c^2\sigma\cdot\sigma^*-c^2\hat{\phi}\cdot\hat{\phi}^*, \tag{41}$$

Upon comparing with the system (30), we separate the E_P and E_V terms in the integral formula.

$$E_p+E_V=\rho_P\int_\Omega\left(\frac{1}{2}\hat{u}\cdot\hat{u}^*+\frac{3}{2}c^2\sigma\cdot\sigma^*-c^2\hat{\phi}\cdot\hat{\phi}^*\right)\mathrm{d}x \Leftarrow \begin{cases} E_p=\dfrac{1}{2}\rho_P\displaystyle\int_\Omega\left(\hat{u}\cdot\hat{u}^*+3c^2\sigma\cdot\sigma^*\right)\mathrm{d}x, \\ E_V=\rho_P\displaystyle\int_\Omega\left(-c^2\hat{\phi}\cdot\hat{\phi}^*\right)\mathrm{d}x. \end{cases} \tag{42}$$

The mass of the particle, $m=E_p/c^2$, follows from the particle wave energy in (42).

$$m=\frac{1}{2}\rho_P\int_\Omega\left(3\sigma\cdot\sigma^*+\frac{\hat{u}\cdot\hat{u}^*}{c^2}\right)\mathrm{d}x. \tag{43}$$

The terms $3\sigma\cdot\sigma^*$ and $\hat{u}\cdot\hat{u}^*/c^2$ oscillate and depend on the time and position. The symmetry in (43) allows normalizing the deformation and mass velocity with respect to the overall particle mass:

$$\int_\Omega \frac{3\rho_P}{m}\sigma\cdot\sigma^*\,\mathrm{d}x = \int_\Omega \psi\cdot\psi^*\,\mathrm{d}x = 1, \text{ where } \psi = \sqrt{\frac{3\rho_P}{m}}\sigma,$$
$$\int_\Omega \frac{\rho_P}{mc^2}\hat{u}\cdot\hat{u}^*\,\mathrm{d}x = \int_\Omega \psi\cdot\psi^*\,\mathrm{d}x = 1, \text{ where } \psi = \sqrt{\frac{\rho_P}{m}}\frac{\hat{u}}{c}. \tag{44}$$

The quaternionic particle mass density ψ can be called the quaternionic probability because the relation $\int_\Omega \psi\cdot\psi^*\,\mathrm{d}x = 1$ in (44) is satisfied. Obviously, terms $\psi = \sqrt{3\rho_P/m}\,\sigma(t,x)$ and $\psi\cdot\psi^*$, vary in time.

We analyze the evolution of the wave as in relations (42) and (43) in the time-invariant potential field, e.g., the particle wave in the field generated by other particles. The overall particle energy is now a sum of the ground and excess energy Q,

$$. E = E_p + Q = \int_\Omega \left(\frac{3}{2}\rho_P c^2\sigma\cdot\sigma^* + \frac{1}{2}\rho_P\hat{u}\cdot\hat{u}^* + V(x)\psi\cdot\psi^*\right)\mathrm{d}x. \tag{45}$$

We consider the low excess energies, and the impact of Q on the overall particle mass in (43) is marginal. Thus, the relation (45) becomes

$$\begin{aligned} E = E_p + Q &= \int_\Omega \left(\frac{1}{2}mc^2\psi\cdot\psi^* + \frac{1}{2}\rho_P\hat{u}\cdot\hat{u}^* + V(x)\psi\cdot\psi^*\right)\mathrm{d}x \\ &= \frac{1}{2}mc^2 + \int_\Omega \left(\frac{1}{2}\rho_P\hat{u}\cdot\hat{u}^* + V(x)\psi\cdot\psi^*\right)\mathrm{d}x. \end{aligned} \tag{46}$$

Both the E_p and m are constant; thus, it is enough to minimize the relation

$$Q = \int_\Omega \left(\frac{1}{2}\rho_P\,\hat{u}\cdot\hat{u}^* + V(x)\psi\cdot\psi^*\right)\mathrm{d}x. \tag{47}$$

The above relation contains two unknowns: $\hat{u} = \partial\hat{u}/\partial t$ and ψ. By relating the local lattice velocity $\hat{u}$ to the force, specifically to the normalized Cauchy–Riemann derivative of the deformation: $l_P D\sigma$, one gets

$$\hat{u} = \frac{\hat{p}}{m} = -\frac{\hbar}{m}D\sigma. \tag{48}$$

By introducing (48) and the normalization (44), the relation (47) becomes the functional

$$Q[\psi] = \int_\Omega \left(\frac{\hbar^2}{6m}(D\psi)\cdot(D\psi)^* + V(x)\,\psi\cdot\psi^*\right)\mathrm{d}x. \tag{49}$$

The functional $Q[\psi]$, Eq. (49), was minimized with respect to a quaternion function such that ψ satisfies the normalization introduced in relation (44). We follow the schema used in [14]. In simple terms, we seek a differential equation that has to be satisfied by the ψ function to minimize the energies allowed by (49). Given the functional (49) and the constraint $\mathrm{div}\hat{\phi} = 0$, the conditional extreme is found using the Lagrange coefficients method and the Du Bois Reymond variational lemma [19]. In such a case, ψ satisfies the time-invariant Schrödinger equation satisfied by the particle wave in the ground state of the energy E

$$-\frac{\mathrm{h}^2}{2m}\Delta\psi + V(x)\,\psi = \lambda\psi, \tag{50}$$

where a constant factor on the right-hand side can be considered as the extra energy of the particle in the presence of the field $V = V(x)$. For $E = \lambda$, Equation (50) is clearly the time-independent Schrödinger equation satisfied by the particle in the ground state of the energy E,

$$-\frac{\mathrm{h}^2}{2m}\Delta\psi + V(x)\,\psi = E\psi\ . \tag{51}$$

It has to be satisfied together with the condition

$$\mathrm{div}\hat{\psi} = 0 \text{ where } \psi = \psi_0 + \hat{\psi}. \tag{52}$$

Upon using the NIST data [20] for Planck's natural units m_P, l_P, t_p, and the light velocity c, the constant $\hbar$ introduced in relation (48) equals the Planck constant [16].

The particle mass center equals its wave energy center. The "space-localized" particle is defined in the sense given by the Bodurov definition [21]: *"A singularity-free multi-component function $\sigma=(\sigma_0,\phi_1,\phi_2,\phi_3)\in\S$ of the space $x=(x_1,x_2,x_3)$ and time t variables will be called space-localized if $\|\sigma(t,x)\|\to 0$ sufficiently fast when $\|x\|\to\infty$, so that its Hermitean norm*

$$\left\langle\sigma,\sigma^*\right\rangle=\int_\Omega\left(\sigma_0^2+\sum_{l=1}^3\phi_l\cdot\phi_l^*\right)\mathrm{d}x=\int_\Omega\sigma\cdot\sigma^*\mathrm{d}x<\infty \tag{53}$$

remains finite for all time."

4. The First-Order PDE in the P-KC

Operator quantum mechanics is based on the complex number algebra, the matrices, and the matrix algebra. Canonical quantization starts from classical mechanics and assumes that the point particle is described by a "probabilistic wave function." Dirac applied complex combinations of the displacements and velocities in the linear problem of secondary quantization [22] and replaced the second-order Klein–Gordon equation by an array of first-order equations. He also recognized the problem of a medium for the transmission of waves: *"It is necessary to set up an action principle and to get a Hamiltonian formulation of the equations suitable for quantization purposes, and for this the aether velocity is required"* [23].

In the earlier work [15], we did not separate the Planck and the particle time scales in the quaternionic oscillator $G_0(m)$, i.e., both the Planck and the particle frequencies were running the oscillator. In the following, we derive the proper formula of the quaternionic oscillator $G_\lambda(m)$ for the 1st order PEDS and the separated time scales. We base our work on the concept of the medium as a solid "aether" [16] and implement quaternion algebra [14]. The 2nd order particle wave equations in QQM, e.g., in the system (31), contain two characteristic terms:

$$\begin{array}{ccccc}\left(\dfrac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma & + & 2G_0(m)\hat{\phi} & = & 0,\\ \left[\begin{array}{c}2^{\text{nd}}\text{ order wave term }\sigma^\mu\sigma_\mu\text{: variable }\sigma\\ \text{and constant wave velocity }c_L=\sqrt{3}\,c\end{array}\right] & + & \left[\begin{array}{c}\text{Propagator with oscillator }G_0(m)\\ \text{that runs at two frequencies}\end{array}\right] & = & 0.\end{array} \tag{54}$$

We will comply with the above schema for the 1st order PEDS:

$$\begin{array}{ccccc}\left(\dfrac{\partial}{\partial t}-\sqrt{3}c\,D\right)\dfrac{\hat{u}}{c} & + & 2G_\lambda(m)\dfrac{\hat{u}}{c} & = & 0,\\ \left[\begin{array}{c}1^{\text{st}}\text{ order wave term }\sigma_\mu\text{: variable }\hat{u}/c,\\ \text{constant wave velocity }c_L=\sqrt{3}\,c\end{array}\right] & + & \left[\begin{array}{c}\text{Propagator with oscillator }G_\lambda(m)\\ \text{that runs at particle wave frequency}\end{array}\right] & = & 0.\end{array} \tag{55}$$

4.1. The 1st order wave term.

We consider the system (30) and the relation between the wave velocity and the Cauchy–Riemann derivative, $\mathrm{D}\sigma=-\frac{m}{\hbar}\dot{\hat{u}}$. The expression for the overall particle energy, Equation (42), implies:

- the displacement velocity as the alternative variable:

$$\frac{\dot{\hat{u}}}{c}=-\frac{\hbar}{mc}\mathrm{D}\sigma, \tag{56}$$

- the longitudinal wave velocity as the wave propagation velocity:

$$c_L=\sqrt{3}\,c. \tag{57}$$

The stable particle is considered, thus its wave is at a quasi-steady state. The 2nd order time derivative of the q-potential in (55) we express as follows:

$$\frac{\partial^2 \sigma}{\partial t^2} = \frac{\partial}{\partial t} \cdot \left(\frac{\partial \sigma}{\partial t} \right). \tag{58}$$

The term in the bracket on the right-hand side is the rate of change of the q-potential. We want to express this term by a new variable, i.e., separate the time scales. The rate of change of the deformation potential $\partial\sigma/\partial t$ is due to the wave propagation within the particle space. The propagation process must follow the extremum principle, i.e., it is the brachistochrone problem [24]. A good example of the "local principle" approximation is given by Derbes [25].

We know that the wave path fulfills the extremum principle, i.e., the wave path follows its unique trajectory given by the Cauchy–Riemann derivative $\mathrm{D}\sigma$. The trajectory which has the minimum property globally in the whole volume Ω occupied by the particle must have the same property locally. This path grants the shortest possible travelling time for the waves identified in QQM. Consequently, from (57) we postulate the following:

$$\begin{cases} c_L = \sqrt{3}c, \\ \mathrm{D}\sigma = -\dfrac{mc}{\hbar}\dfrac{\hat{u}.}{c}, \end{cases} \quad \Rightarrow \quad \frac{\partial \sigma}{\partial t} = c_L D\sigma = \sqrt{3}\,\frac{mc^2}{\hbar}\frac{\hat{u}}{c}. \tag{59}$$

From the relation (56), we get

$$\mathrm{D}\sigma = -\frac{m}{\hbar}\hat{u} \quad \Rightarrow \quad \Delta\sigma = -DD\sigma = \frac{mc}{\hbar} D \frac{\hat{u}}{c}. \tag{60}$$

Combining the relations (59) and (60), we get the 1st order particle wave term consistent with the 2nd order formula (54):

$$\frac{\partial^2 \sigma}{\partial t^2} - 3c^2 \Delta\sigma \Leftrightarrow \left(\sqrt{3}\,\frac{mc^2}{\hbar}\frac{\partial}{\partial t} - \sqrt{3}\,\frac{mc^2}{\hbar} D \right) \frac{\dot{\hat{u}}}{c} = \sqrt{3}\,\frac{mc^2}{\hbar} \left(\frac{\partial}{\partial t} - cD \right) \frac{\dot{\hat{u}}}{c}. \tag{61}$$

Thus, the 1st order particle wave term in (55) equals:

$$\left(\frac{\partial}{\partial t} - D \right) \frac{\dot{\hat{u}}}{c} = 0. \tag{62}$$

4.2. The 1st order quaternionic oscillator.

The power of the 2nd order quaternionic oscillator, Equation (28), follows from two time scales in PK-C, namely from the relations (24) and (27): $G_0(f) = 8\pi^2 f_P f$ and $f = mc^2/(2\pi\hbar)$. Combining the relations (24), (27), and removing the Planck frequency results in the power formula of the 1st order quaternionic oscillator when the particle mass is known:

$$G_\lambda(m) = 4\pi f = 2\frac{mc^2}{\hbar} = 2\frac{m}{m_P t_P}. \tag{63}$$

By introducing the relations (62) and (63) in the schema (55), the 1st order PDE for the electron equals

$$\left(\frac{\partial}{\partial t} - cD \right) \frac{\dot{\hat{u}}}{c} - 2\frac{m}{m_P t_P}\frac{\hat{u}}{c} = 0. \qquad \left(\frac{\partial}{\partial t} - cD \right) \frac{\dot{\hat{u}}}{c} - 2\frac{m}{m_P t_P}\frac{\hat{u}}{c} = 0. \tag{64}$$

Relation (44), $\psi = \sqrt{\rho_P/m}\,\hat{u}/c$, implies that by multiplying the particle wave equation (64) by $\psi = \sqrt{\rho_P/m}$, it will be expressed as a function of probability

$$\left(\frac{\partial}{\partial t} - cD \right) \psi - 2\frac{m}{m_P t_P}\psi = 0 \tag{65}$$

or

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - D \right) \psi - 2\frac{m}{m_P l_P}\psi = \left(\partial_\mu - 2\frac{m}{m_P l_P} \right) \psi = 0. \tag{66}$$

Equation (66) may require the σ_0 time dependence. This dependence results from the continuity equation presented in [15]. The comparison of the first-order wave equations in quaternion formulation, Equation (66), with the form in the Dirac algebra formalism:

$$\begin{aligned} &\text{Dirac:} && \left(i\gamma^{\mu}\partial_{\mu}-\frac{m}{m_P l_P}\right)\psi(t,x)=0 \ \text{ where } \ \gamma^{\mu}\partial_{\mu}=\frac{1}{c}\frac{\partial}{\partial t}+\alpha_1\frac{\partial}{\partial x}+\alpha_2\frac{\partial}{\partial y}+\alpha_3\frac{\partial}{\partial z}, \\ &\text{Quaternion:} && \left(\partial_{\mu}-\frac{m}{m_P l_P}\right)\psi(t,x)=0. \end{aligned} \tag{67}$$

5. Conclusions

The new results of the QQM and QFT make firmer the concept of the P-KC. The fine-tuning of our model allowed obtaining new results and next targets:

- The symmetrical formula of the scalar force field: $nc^2\Delta\sigma_0+G_0(m)\sigma\cdot\sigma^*=0$, is consistent with the scalar coupling between transverse and longitudinal waves in [13] and [14].
- The quaternion, $G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)$, scalar, $G_0(m)\sigma\cdot\sigma^*$, and vectorial, $G_0(m)\hat{\phi}$ propagators are postulated and used to generate the 2nd order partial differential equation systems, PEDS, for the proton, electron, and the neutron
- The scrupulous assessment of the 2nd order PDE systems allows postulating the two 2nd order PEDS for the *u* and *d* quarks from the *up* and *down* groups.
- It was verified that both the proton and the neutron obey experimental findings and are formed by three quarks. Namely, the proton and neutron are formed by $d-u-u$ and $d-d-u$ complexes, respectively. All the above systems comply with the Cauchy equation of motion (8) and can be considered as stable particles.
- The *u* and *d* quarks do not meet the relations of the Cauchy equation of motion. Also, experimental efforts to find the individual quarks were without success. Observed were the bound states of the three quarks – the baryons – and a quark and an antiquark – the mesons. Wilczek calls it the phenomenological paradox: *"Quarks are Born Free, but Everywhere They are in Chains"* [26]. The inconsistency of the quarks' PEDS with the quaternion forms of the Cauchy equation of motion might account for the observed *Quarks Chains.*
- The gravitational waves propagate at the velocity of the transverse wave in the Cauchy continuum, c.
- The electron waves propagate at the velocity of the longitudinal wave in the Cauchy continuum, √3c.

The results indicate the following targets for immediate future:

- The particles and quarks in the case of higher coupling coefficients: $n>|1|$.
- The ratios between the constants for the different force fields.
- The rigorous derivation of the 1st order PEDS basing on the extremum principle.
- The multivalued coordinate transformation to determine the properties of space with curvature and torsion produced by 2nd order PEDS of the QFT [27].

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Abbreviations

P-KC	Planck-Kleinert crysta
PDE	partial differential equation

PEDS	partial differential equation systems
QQM	quaternion quantum mechanics
QFT	Quaternion field theory
D	deformation tensor
λ_L, μ_L	Lamé coefficients;
σ', σ''	stress tensors
ρ_E	density of the deformation energy
$\mathbf{u}(u_1, u_2, u_3)$	displacement in $\sim^3$
$\sigma(\sigma_0, \phi_1, \phi_2, \phi_3)$	*q-potential* in $\sim^4$, the quaternion deformation potential
$\sigma^* \cdot \sigma$	strain energy density
G_0	power of the quaternionic oscillator
$G_0\, \sigma^* \cdot \sigma$	density of the rate of momentum change, i.e., the quaternionic scalar propagator
$G_0\, \hat{\phi}$	quaternionic vector propagator
$G_0(m)\left(\sigma \cdot \sigma^* + \hat{\phi}\right)$	quaternionic q-potential propagator
$\psi = \sigma\sqrt{\rho_P/m}$	quaternionic particle density, i.e., the particle wave function
$\psi \cdot \psi^*$	probability, i.e., the normalized particle mass density
n	coupling coefficient in the propagator
l_P	Planck length
$f_P = 1/t_P$	Planck frequency, inverse of the Planck time
m_P	Planck mass
$c = l_P/t_P$	transverse wave velocity in elastic continuum
$c_L = \sqrt{3}\, c$	longitudinal wave velocity in elastic continuum
$\rho_P = 4m_P/l_P^3$	Planck density, i.e., the mass density of the PK-C
ρ	mass density of the particle $\rho = \rho_E/c^2$, as the equivalent of the energy density ρ_E in the PK-C
$\hbar$	Planck constant in terms of angular frequency
h	Planck constant, $h = 2\pi\hbar$
m	equivalent mass of the wave, i.e., mass of the particle
λ	length of the particle wave
f	frequency of the particle wave

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