Review of: "Horizon and curvature"

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Potential competing interests: No potential competing interests to declare.

Dear Alain,

I like quite more this version than the previous one. The addition of missing explanations, suggested changes and figures has improved the quality of presentation.

I detail below on a few typos and things that can be improved and also add some final comments:

- Figure 1 is not referenced in the text.
- Page 3, below Eq. (2.2): there is a missing space between 1.75 and m.
- Two lines below: "when If"; delete "If" or delete "when" and write "if" (without capital letter).
- Page 4, at the end of the first paragraph a dot is missing.
- Page 5, line below the first centered equation: write P and Oa.
- Statement of Prop. 4.1, what curve? I believe that putting that K is a closed convex domain of R^2, intead of R^3 would be better. And specifying that the curve you are referring to is \Gamma.
- Write $(0,0)$ and R in the proof of Prop. 4.1 (between dollars). One instance in page 5 and two instances in page 6.
- Page 6, there is an instance of "disc" that should be replaced by "disk" to maintain English coherence.
- I find the proof of Prop. 4.1 too complicated, there should be a way to shorten it. Also, we know from the very beginning that \Gamma is compact. This goes in relation to the sentence "Since we **now** know that \Gamma is compact..." at the end of page 6. And regarding this proposition, I have not looked for bibliography, but the result should be already known.
- I suggest deleting Remarks 4.2 and 4.3, the term "probably" is not serious; if it can be proved, OK, or cite some reference, but otherwise do not use the term "probably".
- Prop. 5.1 (statement): Specify where the domain D lives in, which should be R^2.
- Page 8, line 3: "k is an integer between -(N-1) and (N-1)". If the inequalities are not strict, maybe it is better to write $-(N-1) \leq k\leq N-1$.
- Figure 7: when the reader look at this figure for Prop. 5.1 without having read Prop. 5.2, the definition of $D"$ that appears in the caption has still not been introduced and can be a bit confusing.
- Figure 8: in the last equation there is a "prime" missing in the last f(x).
- Line above Eq. (6.3), remove the square from H.
- Line below Eq. (6.3): add dollars to the expressions $(x,f(x))$ and $(0,h)$
- Regarding the big O notation (Landau notation): I did not say that your expressions were incorrect, but for instance in

(6.7) one can specify more and put O(x^3), as this term does not actually include any term with x^2. With the usual Landau notation, O(x^n) refers to terms of degree n and greater; if there is no term of degree n and there is some of degree n+1, one may use O(x^{n+1}) instead of O(x^n), although both are correct.

- Remark 6.3: the point M is commonly written in the tangent plane as a subindex: $T_M(S)$.
- First sentence of Section 7: add dollars to $H(h)$.
- First paragraph of page 11: better if you use \left(and \right) in the interval (0, pi/2). Add a dot at the end of "the following proposition".
- Prop. 7.1: "incoming", I believe you mean "inward". I believe that there is a h missing in the definition of A(M). A figure illustrating this notation and also the notation in the proof would be helpful.
- Proof of Prop. 7.1: I don't understand the expresion "T + (M,P) := T", maybe it can be clarified. A line below, I guess that A should be T. Another line below: better "rectangled triangle". I got a bit lost in the sequence at the end of the proof, maybe with a figure it could be more understandable.
- Sec. 8.3: "Theorem 6.1" (with capital letter). What is C(k)?
- Sec. 8.4, second line from below: "has to be modified".
- Sec. 8.5: second line, write "Earth" (with capital letter). Include the minus sign of -40C between dollars to get the proper character. Add a final dot for this section 8.5.

It would be better if the figures of the whole paper are centered. You can use \centering inside the figure environment or adding there

\begin{center}...\end{center}

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I don't know the reason but I noticed that there are some missing figures in the PDF version (they appear only in the web version). In particular, the problems are with Figs. 3, 4 and 7.

Finally, my impression is that a deep bibliography search should be done because the kind of problems and results presented here seem quite simple and may have been studied before. Probably not in the context of looking at the "horizon" but as pure geometrical problems. For example, regarding Prop. 7.1, Remark 7.2 and the MAIN QUESTION, may I refer you to the paper "Jerónimo-Castro, Jesús; Ruiz-Hernández, Gabriel; Tabachnikov, Sergei. The equal tangents property. Adv. Geom. 14 (2014), no. 3, 447--453." In particular, see their Theorem 6. In your case, the set of external points to the convex body K such that all the tangent segments for these points to K have equal length is a parallel surface to the boundary of K (by condition 1), and so it contains K in its interior. Threfore K is a sphere. There are many other papers dealing with similar problems and characterizations of spheres/disks by Jesús Jerónimo-Castro, Sergei Tabachnikov and some collaborators. I may note that conditions like $\theta/\pi \notin \mathbb{Q}$ and similar are sometimes needed for some problems (having counterexamples otherwise, and so they are not that strange in some contexts). The answer to some of the problems you are interested in may appear in some of their papers as well.